THE RISK PREMIUM AND THE ESSCHER TRANSFORM IN POWER MARKET MODELS

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PLAN OF THE TALK

● THE ELECTRICITY MARKET MODELS

● MEASURE CHANGE: THE ESSCHER TRANFORM FOR THE ADDITIVE PROCESSES

● THE RISK PREMIUM CALCULATIONS

● DISCUSSION OF THE RESULTS

● CONCLUDING REMARKS AND FUTURE PERSPECTIVES
STARTING POINT

• THE ELECTRICITY FORWARD PRICES: CONTANGO AND BACKWARDATION


• THE RISK PREMIUM SIGN IN ELECTRICITY MARKETS EXHIBITS A SIGN CHANGE

• WHICH RISK-NEUTRAL MEASURE SHOULD BE USED FOR PRICING?
THE ELECTRICITY MARKET MODELS

The electricity market models we are going to consider in this paper belong to a wide class of models based on additive processes, i.e. stochastic processes with Independent Increments (II from now on). They can be grouped in two subclasses, the geometric and the arithmetic models.

Lévy-Kintchine representation for II Processes:

\[ E [\exp (i\theta(s) - i\theta(t))] = \exp [\psi(s,t;\theta)], \quad (1) \]
\[
\psi(s, t; \theta) = i\theta [\gamma(t) - \gamma(s)] - \frac{1}{2} \theta^2 [C(t) - C(s)] + \\
+ \int_s^t \int_{\mathbb{R}} \left[ e^{i\theta z} - 1 - i\theta z 1_{|z| \leq 1} \right] l(dz, du)
\]

The quantities \(\gamma, C, l(dz, du)\) are the semimartingale characteristics of the II process.

Both classes of models we are going to discuss are described by Ornstein-Uhlenbeck processes driven by an II, eventually including one or more Wiener processes \(B_k\).
GEOMETRIC MODELS:

\[
\ln S(t) = \ln \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t),
\]

(3)

where, for \( i = 1, \ldots, m \),

\[
dX_i(t) = [\mu_i(t) - \alpha_i(t)X_i(t)] dt + \sum_{k=1}^{p} \sigma_{ik}(t) dB_k(t)
\]

(4)

and, for \( j = 1, \ldots, n \),

\[
dY_j(t) = [\delta_j(t) - \beta_j(t)Y_j(t)] dt + \eta_j(t) dI_j(t)
\]

(5)
By assuming that the initial conditions for \( X_i \) and \( Y_j \) are such that:

\[
\sum_{i=1}^{m} X_i(0) + \sum_{j=1}^{n} Y_j(0) = \ln S(0) - \ln \Lambda(0),
\]  

(6)

an explicit representation of \( S(t) \) is given by:

\[
S(t) = \Lambda(t) \exp \left[ \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t) \right]
\]  

(7)
where, for $i = 1, ..., m$,

$$X_i(t) = X_i(0)e^{-\alpha_i(t-u)} + \int_0^t \mu_i(u)e^{-\alpha_i(t-u)} du +$$

$$\sum_{k=1}^{p} \int_0^t \sigma_{ik}(u)e^{-\alpha_i(t-u)} dB_k(u)$$

and, for $j = 1, ..., n$,

$$Y_j(t) = Y_j(0)e^{-\beta_j(t-u)} + \int_0^t \delta_j(u)e^{-\beta_j(t-u)} du + \int_0^t \eta_j(u)e^{-\beta_j(t-u)} dI_j(u)$$
REMARK: many models proposed for electricity market belong to the class just introduced:

- Schwarz model is a particular case with $p = m = 1, n = 0$;

- Lucia and Schwarz belongs to this same class with $m = p = 2, n = 0$.

- Eberlein and Stahl model fits also into this class, by setting $m = n = p = 1$, choosing $\alpha(t) = 0$, and letting $I(t)$ be a Lévy process of GH type.

- Cartea and Figueroa model corresponds to the choice of $I(t)$ as a compound Poisson still with $m = n = p = 1$. 
ARITHMETIC MODELS:

\[ S(t) = \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t) \]  

(10)

where \( X_i(t), Y_j(t), i = 1, \ldots, m; j = 1, \ldots, n \) are given as before and the seasonality function satisfies the same conditions required for geometric models.
REMARK: Arithmetic models did not gain the same popularity as the geometric models in order to describe the commodity behavior in energy markets and this is due mainly to the possibility to obtain negative prices in this framework.

Benth, Kallsen and Meyer-Brandis Model (in which the probability to reach negative prices is zero).
This model is obtained by setting $m = 0$, and by interpreting the seasonality function $\Lambda(t)$ as a floor towards which the processes $Y_j(t)$ revert. Moreover, the II processes $I_j(t)$ are assumed to be pure jump increasing process. By letting $\delta_j = 0$ for $j = 1,...,n$, $Y_1(0) = S(0) - \Lambda(0)$ and $Y_j(0) = 0$ for $j = 2,...,n$, we obtain the following explicit representation of the spot price dynamics:

$$S(t) = \Lambda(t) + \sum_{j=1}^{n} Y_j(t)$$ (11)
INTEGRABILITY CONDITIONS:

Condition 1: (for Geometric Models) For each $j = 1, \ldots, n$ there exists a constant $c_j > 0$ such that:

$$\int_0^T \int_1^\infty [e^{c_jz} - 1] l_j(dz, du) < \infty$$  \hspace{1cm} (12)

Condition 2: (for Arithmetic Models) For each $j = 1, \ldots, n$ there exists a constant $c_j > 0$ such that:

$$\int_0^T \int_{|z| \geq 1} |z|^{c_j} l_j(dz, du) < \infty$$  \hspace{1cm} (13)
MEASURE CHANGE: THE ESSCHER TRANSFORM FOR ADDITIVE PROCESSES

Let’s define the Esscher transform for II processes. Let $\theta(t)$ be a $(p + n)$–dimensional vector of real-valued continuous functions on $[0, T]$:

$$
\theta(t) = \left( \hat{\theta}_1(t), ..., \hat{\theta}_p(t), \bar{\theta}_1(t), ..., \bar{\theta}_n(t) \right). \tag{14}
$$

Define the stochastic exponential by the following relation:

$$
Z^{\theta}(t) := \prod_{k=1}^{p} \tilde{Z}_{\theta}^k(t) \times \prod_{j=1}^{n} \tilde{Z}_{\theta}^j(t), \tag{15}
$$

where, for $k = 1, ..., p$,

$$
\tilde{Z}_{\theta}^k(t) = \exp \left[ \int_0^t \hat{\theta}_k(u) dB_k(u) - \frac{1}{2} \int_0^t \hat{\theta}_k^2(u) du \right] \tag{16}
$$
and, for $j = 1, \ldots, n,$

$$\tilde{Z}_j^\theta(t) = \exp \left[ \int_0^t \tilde{\theta}_j(u) dI_j(u) - \psi_j(0, t; \tilde{\theta}_j(\cdot)) \right]$$

(17)

where the functions $\psi_j$ are defined by:

$$\psi_j(t, s; g(\cdot)) := i \int_t^s g(u) d\gamma_j(u) - \frac{1}{2} \int_t^s g^2(u) dC_j(u)$$

$$+ \int_s^t \int_{\mathbb{R}} \left[ e^{ig(u)z} - 1 - ig(u)z \mathbf{1}_{|z| \leq 1} \right] l_j(dz, du)$$
If the following condition holds:

$$\sup_{0 \leq t \leq T} |\tilde{\theta}_j(t)| \leq c_j,$$

where $c_j$ is a constant granting that condition 1 is satisfied, it follows immediately by Ito formula that $\tilde{Z}_j^{\theta}(t)$ is a positive local martingale with expectation equal to one, so it is a martingale process. With a similar consideration, if $\hat{\theta}_k(u)$ is a continuous function, the Novikov condition holds, implying that $\hat{Z}_k^{\theta}(t)$ is a martingale as well.
Hence we can define an equivalent probability measure $Q^\theta$ such that $Z^\theta(t)$ is the density process of the Radon-Nikodym derivative $dQ^\theta/dP$:

$$dQ^\theta|_{\mathcal{F}_t} = Z^\theta(t) = \prod_{k=1}^p \tilde{Z}_k^\theta(t) \times \prod_{j=1}^n \tilde{Z}_j^\theta(t)$$

(19)

The following proposition describes how the characteristics of $B$ and $I$ change under the application of the Esscher transform:
**Proposition 1** With respect to the probability measure $Q^\theta$ the processes

\[ B^\theta_k(t) = B_k(t) - \int_0^t \tilde{\theta}_k(u) du \]  

(20)

are Brownian motions for $k = 1, \ldots, p$ and $0 \leq t \leq T$. Moreover, for each $j = 1, \ldots, n$, $I_j(t)$ is an Independent Increment process on $0 \leq t \leq T$ with drift:

\[ \gamma_j(t) + \int_0^t \int_{|z|<1} z \left[ e^{\tilde{\theta}_j(u)z} - 1 \right] l_j(dz, du) \]  

(21)

and predictable compensator measure $e^{\tilde{\theta}_j(t)}z l_j(dz, dt)$. 

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EMM FOR POWER MARKET MODELS:

MODEL INCOMPLETENESS

NO POSSIBILITY TO HEDGE WITH THE UNDERLYING
(the price process need not to be a martingale under the Risk-Neutral measure)
FORWARD PRICES

By using the Esscher transform $Q^\theta$ as a risk-neutral measure, it is possible to obtain the prices of forward contracts on electricity in a closed form. Let’s examine the geometric case first. We have the following proposition:

**Proposition 2** Suppose that $S(t)$ is the geometric spot price model and that condition 1 holds for $j = 1, ..., n$ with

$$\sup_{0 \leq t \leq T} |\eta_j(u)e^{-\beta(t-u)} + \tilde{\theta}_j(u)| \leq c_j. \quad (22)$$
Then the forward price $f(t, \tau)$ is given by:

$$f(t, \tau) = \Lambda(t) \Theta(t, \tau; \theta(\cdot)) \times \exp \left[ \sum_{i=1}^{m} \int_{t}^{\tau} \mu_i(u) e^{-\alpha_i(\tau-u)} du \right] \times$$

$$\exp \left\{ \sum_{j=1}^{n} \int_{t}^{\tau} \delta_j(u) e^{-\beta_j(\tau-u)} du \right\} \times$$

$$\times \exp \left\{ \sum_{i=1}^{m} e^{-\alpha_i(\tau-u)} X_i(t) + \sum_{j=1}^{n} e^{-\beta_j(\tau-u)} Y_j(t) \right\}$$

(23)
where $\Theta(t, \tau; \theta(\cdot))$ is defined as:

$$
\ln \Theta(t, \tau; \theta(\cdot)) = \sum_{j=1}^{n} [\psi_j(t, \tau; -i(\eta_j(\cdot)e^{-\beta(\tau-\cdot)}) + $

$$
+ \tilde{\theta}(\cdot))) - \psi_j(t, \tau; -i\tilde{\theta}(\cdot))]$

$$
+ \frac{1}{2} \sum_{k=1}^{p} \int_{t}^{\tau} \left( \sum_{i=1}^{m} \sigma_{ik}(u)e^{-\alpha_i(\tau-u)} \right)^2 du$

$$
+ \sum_{i=1}^{m} \sum_{k=1}^{p} \int_{t}^{\tau} \sigma_{ik}(u)\theta_k(u)e^{-\alpha_i(\tau-u)} du.
$$
For the Arithmetic models the price of forward contracts is determined according to the following proposition:

PROPOSITION Suppose the spot price dynamics is given by the arithmetic model and there exists \( \varepsilon > 0 \) such that condition 2 holds with, for \( j = 1, \ldots, n \),

\[
\sup_{0 \leq t \leq T} |\tilde{\theta}_j(u)| + \varepsilon \leq c_j. \tag{24}
\]
The forward price for $0 \leq t \leq \tau$ is then given by:

$$f(t, \tau) = \Lambda(t) + \Theta(t, \tau; \theta(\cdot)) + \sum_{i=1}^{m} \int_{t}^{\tau} \mu_i(u)e^{-\alpha_i(\tau-u)}du +$$

$$+ \sum_{j=1}^{n} \int_{t}^{\tau} \delta_j(u)e^{-\beta_j(\tau-u)}du \times + \sum_{i=1}^{m} e^{-\alpha_i(\tau-u)}X_i(t) + \sum_{j=1}^{n} e^{-\beta_j(\tau-u)}Y_j(t)$$

where:
\[
\Theta(t, \tau; \theta) = \sum_{i=1}^{m} \sum_{k=1}^{p} \int_{t}^{\tau} \sigma_{ik}(u) \tilde{\theta}_k(u) e^{-\alpha_i(\tau-u)} du + \\
+ \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u) e^{-\beta_j(\tau-u)} d\gamma_j(u) + \\
+ \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u) e^{-\beta_j(\tau-u)} z \left[ e^{\tilde{\theta}_j(u)z} - 1_{|z| \leq 1} \right] l_j(dz, du)
\]
As a special case of an arithmetic model let’s consider the dynamics proposed by Benth, Kallsen and Meyer-Brandis ensuring positive spot prices. In that case $m = 0$ (no diffusion part), $\delta_j = 0$ (zero mean-reversion level), and the support of the compensator measures $l_j$ are the positive real half-line (jumps are assumed to be positive). The forward prices in this model are then given by:
\[ f(t, \tau) = \Lambda(t) + \Theta(t, \tau; \theta(\cdot)) + \sum_{j=1}^{n} e^{-\beta_j(\tau-u)}Y_j(t), \quad (25) \]

where

\[ \Theta(t, \tau; \theta(\cdot)) = \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u)e^{-\beta_j(\tau-u)}d\gamma_j(u) + \quad (26) \]

\[ + \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u)e^{-\beta_j(\tau-u)}z \left[ e^{\tilde{\theta}_j(u)z} - 1 \right]_{|z| \leq 1} l_j(dz, du) \]
The last formula can be rewritten in terms of the new semimartingale characteristics of the spot process by simply moving the compensating term in the integrals with respect to $l_j$ to the drift integral $d\gamma_j$:

$$\Theta(t, \tau; \theta(\cdot)) = \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u) e^{-\beta_j(\tau-u)} d\tilde{\gamma}_j(u) + \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u) e^{-\beta_j(\tau-u)} z e^{\tilde{\theta}_j(u)} z l_j(dz, du).$$
THE RISK PREMIUM

The risk premium is defined as the difference between the expectation of the underlying prices calculated with respect to the risk-neutral measure $Q$ and the objective measure $P$ respectively:

$$R(t, \tau) := E^Q [S(\tau) | \mathcal{F}_t] - E^P [S(\tau) | \mathcal{F}_t]$$  \hspace{1cm} (28)
We can observe that the first term appearing in the definition just given is nothing more than the forward price calculated according to the risk neutral dynamics and the second the same forward price calculated with respect to the objective dynamics. In both the Geometric and Arithmetic classes of models a close relationship turns out to exist between the risk premium and the quantity $\Theta(t, \tau; \theta)$. 
For the Geometric models an explicit calculation of the risk premium provides the following result:

\[ R^G(t, \tau) = \Lambda(t) [\Theta(t, \tau; \theta) - 1] \times \exp \left[ \sum_{i=1}^{m} \int_t^{\tau} \mu_i(u) e^{-\alpha_i(\tau-u)} du \right] \times 
\]

\[ \times \exp \left[ \sum_{j=1}^{n} \int_t^{\tau} \delta_j(u) e^{-\beta_j(\tau-u)} du \right] \times 
\]

\[ \times \exp \left[ \sum_{i=1}^{m} e^{-\alpha_i(\tau-u)} X_i(t) + \sum_{j=1}^{n} e^{-\beta_j(\tau-u)} Y_j(t) \right] \]
Since we want to investigate the sign change of the risk premium, we can equivalently look at the sign of the quantity

\[ \ln E^Q [S(\tau) | \mathcal{F}_t] - \ln E^P [S(\tau) | \mathcal{F}_t] \]

which is simpler to evaluate;

\[ \ln E^Q [S(\tau) | \mathcal{F}_t] - \ln E^P [S(\tau) | \mathcal{F}_t] = \ln \Theta(t, \tau; \theta) \] (30)

For simplicity reasons we shall now focus on the case \( m = n = 1 \).
\[ \ln \Theta(t, \tau; \theta(\cdot)) = \int_t^\tau \eta(u) e^{-\beta(\tau - u)} d\gamma(u) + \]

\[ + \int_t^\tau \int_{\mathbb{R}} \left[ e^{\tilde{\theta}(u) z} \left( \eta(u) e^{-\beta(\tau - u)} - 1 \right) - \eta(u) e^{-\beta(\tau - u)} z 1_{|z| \leq 1} \right] l(dz, du) + \]

\[ + \frac{1}{2} \int_t^\tau \left( \sigma(u) e^{-\alpha_i(\tau - u)} \right)^2 du + \int_t^\tau \sigma(u) \tilde{\theta}(u) e^{-\alpha_i(\tau - u)} du. \]
It is apparent that the last expression can change sign according to the choice of the functions $\eta(t), \sigma(t)$ appearing in the model and the function $\theta(t)$ parametrizing the Esscher transform. If we simplify further the model considered removing the diffusion component, we obtain:

$$\ln \Theta(t, \tau; \theta(\cdot)) = \int_t^\tau \eta(u) e^{-\beta(\tau-u)} d\gamma(u) +$$

$$+ \int_t^\tau \int_{\mathbb{R}} \left[ e^{\bar{\theta}(u)} z \left( e^{\eta(u)} e^{-\beta(\tau-u)} - 1 \right) - \eta(u) e^{-\beta(\tau-u)} z 1_{|z| \leq 1} \right] l(dz, du),$$

where the sign change can be detected more easily with an explicit dependence on $\bar{\theta}(t)$. 

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For the Arithmetic class of models the calculation of the risk premium provides the following expression:

\[ R^A(t, \tau) = \Theta(t, \tau; \theta) = \sum_{i=1}^{m} \sum_{k=1}^{p} \int_{t}^{\tau} \sigma_{ik}(u) \tilde{\theta}_k(u) e^{-\alpha_i(\tau-u)} du + \]

\[ + \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u) e^{-\beta_j(\tau-u)} d\gamma_j(u) + \]

\[ + \sum_{j=1}^{n} \int_{t}^{\tau} \eta_j(u) e^{-\beta_j(\tau-u)} \left[ e^{\tilde{\theta}_j(u)z} - 1 \right]_{|z| \leq 1} l_j(dz, du) \]

(32)
The same consideration on the sign of this quantity and the choice of the functions $\eta(t), \sigma(t)$ appearing in the model and the function $\theta(t)$ parametrizing the Esscher transform can be done. If we look at the special case proposed by Benth, Kall sen and Meyer-Brandis the risk premium assumes a more readable form:

$$\Theta(t, \tau; \theta) = \int_t^\tau \int_0^\infty \eta(u) e^{-\beta(\tau-u)} z \left[ e^{\theta(u)z} - 1 \right] l(dzdu)$$  \hspace{1cm} (33)
DISCUSSION OF THE RESULTS

Consider the simple setting of $m = n = p = 1$ with a constant measure change, that is, $\theta = (\hat{\theta}, \tilde{\theta})$, and suppose that all parameters in the spot model is constant. Suppose that the II process $I$ is a compound Poisson process with a time-dependent jump intensity $\lambda(t)$, mimicking a seasonally varying spike process. The log-moment generating function becomes:

$$\phi(t, \tau; \tilde{\theta}) = \int_t^\tau \left( e^{\phi_J(\tilde{\theta})} - 1 \right) \lambda(s) \, ds,$$  (34)
where $\phi_J$ is the log-moment generating function of the jump size distribution of $I$ and $\lambda$ its jump frequency. Finally, we assume $\eta = 1$. Now, from the representation above we find

$$\ln \Theta_g(\theta) - \ln \Theta_g(0) = \int_0^{\tau-t} \left[ e^{\phi_J(\tilde{\theta} + e^{-\beta s})} - e^{\phi_J(e^{-\beta s})} - e^{\tilde{\theta}} + 1 \right] \lambda(\tau-s) \, ds$$

$$+ \frac{\sigma\tilde{\theta}}{\alpha} (1 - e^{-\alpha(\tau-t)}) = \lambda \int_0^{\tau-t} \mathbb{E}\left[ \left( e^{\tilde{\theta}J} - 1 \right) \left( e^{e^{-\beta s}J} - 1 \right) \right] \lambda(\tau-s) \, ds$$

$$+ \frac{\sigma\tilde{\theta}}{\alpha} (1 - e^{-\alpha(\tau-t)}).$$
Now, observe that whenever $J > 0$, $\exp(e^{-\beta sJ}) - 1$ is positive, and non-positive otherwise. Further, if $\tilde{\theta} > 0$, then $\exp(\tilde{\theta}J) - 1$ is positive whenever $J$ is, and non-positive otherwise. Hence, for $\tilde{\theta} > 0$, we will have that the expectation inside the integral will be strictly positive, and thus gives a strictly positive contribution. Since the jump intensity is non-negative, the integral term will give a positive contribution to the risk premium. If $\hat{\theta}$ is positive as well, we will have a positive risk premium. When $\tilde{\theta} < 0$, we get that $(\exp(\tilde{\theta}J) - 1)(\exp(e^{-\beta sJ}) - 1)$ is negative whatever $J$ is, and therefore we get a negative contribution. Hence, for negative parameters $\tilde{\theta}, \hat{\theta}$ we get a negative risk premium.
We may explain the results above by looking at the level change induced by the Esscher transform. For simplicity, let $\lambda(t) = \lambda$, a constant. Note that

$$\Phi(t, \tilde{\theta}) = \frac{\partial}{\partial \tilde{\theta}} \phi(t, \tau; \tilde{\theta}) = \lambda \int_{\mathbb{R}} z e^{\tilde{\theta} z} P_J(dz) \, t,$$

and thus the level change for the process $Y$ when going from $P$ to $Q^\theta$ is given by

$$\Phi(t, \tilde{\theta}) - \Phi(t, 0) = \lambda \int_{\mathbb{R}} z (e^{\tilde{\theta} z} - 1) P_J(dz) \, t,$$

where $P_J$ is the distribution of $J$. We observe that when $\tilde{\theta} > 0$, $z(\exp(\tilde{\theta} z) - 1)$ is positive for all $z$. On the other hand, if $\tilde{\theta} < 0$, then it is negative for all $z$. Hence, choosing $\tilde{\theta}$ positive yields an upward shift of the mean-reversion level of the jump process $Y$, while a negative choice of $\tilde{\theta}$ pushes it down. We see this reflected in the risk premium being positive or negative, respectively.
The intuition and experience from the market propose a long-term negative risk premium and a short term positive premium. One could obtain this by choosing \( \tilde{\theta} > 0 \) and \( \hat{\theta} < 0 \). The shape of the risk premium is determined by the factor \( \Theta_g(t, \tau; \theta) - \Theta_g(t, \tau; 0) \), which becomes

\[
\Theta_g(\theta) - \Theta_g(0) = \exp \int_0^{\tau-t} \left[ \exp \phi_J(\tilde{\theta} + e^{-\beta s}) - \exp \phi_J(\tilde{\theta}) \right] \lambda(\tau-s) \, ds \]

\[
+ \frac{\sigma \tilde{\theta}}{\alpha} (1 - e^{-\alpha(\tau-t)}) - \exp \int_0^{\tau-t} \left[ \exp \phi_J(e^{-\beta s}) - 1 \right] \lambda(\tau-s) \, ds.
\]

The reason for a positive risk premium in the short end is explained by Geman and Vasicek as a result of the consumers/retailers in the market being willing to hedge the spike risk.
Consider now $Y$ modeling the spikes in electricity prices. Naturally, $J > 0$, that is, a price process which may jump only upwards. On the other hand, $\beta$ should be reasonably large so as to kill off the spikes quickly. It is straightforwardly seen that for $\tilde{\theta} > 0$, we’ll have:

$$\int_{t}^{\tau} [\exp \phi J(\tilde{\theta} + e^{-\beta s}) - \exp \phi J(\tilde{\theta})] \lambda(\tau - s) \, ds >$$

$$> \int_{0}^{\tau-t} [\exp \phi J(e^{-\beta s}) - 1] \lambda(\tau - s) \, ds.$$
If $\hat{\theta} = 0$ and $\lambda(t)$ constant, we observe that the risk premium increases fastly from zero up to a constant positive level as a function of time to maturity. However, if $\hat{\theta} < 0$, there will be a competing term in the expression for the risk premium which will force it down and eventually to a fixed level. If $\hat{\theta}$ is big enough, the risk premium may be pushed below zero to a negative level in the long end of the forward curve (that is, for large $\tau$, times of maturity). How fast the influence of $\hat{\theta}$ comes in is depending on the speed of mean-reversion $\alpha$, which one would expect to be slower than $\beta$. Hence, we should get a small “bump-shaped” positive premium in the short end, whereas it becomes negative in the long end of the market. This depends also on the choice of parameters.
Let us discuss how a time-dependent jump intensity may influence the sign. We consider a toy example of a market where the spikes occur during a specific period of the year, in January say, while we do not observe any spikes in the remaining part of the year. This means that $\lambda(t)$ is zero except for the month of January, where we suppose it is constant for the sake of simplicity. In the Nord Pool market, the spikes are most frequently observed in January and February, being the coldest period of the year and thus with the highest demand for heating. Let us assume that we are in July, looking at the forward curve one year ahead. Since the jump intensity is zero all the way up to January, the shape of the risk premium will only be given by
\[ \Theta_g(t, \tau; \theta) - \Theta_g(t, \tau; 0) = \exp\left(\frac{\sigma \hat{\theta}}{\alpha} (1 - e^{-\alpha(\tau-t)})\right) - 1, \]

for contracts maturing before January. Hence, the risk premium will, for \( \hat{\theta} < 0 \), be downward sloping towards the asymptote \( \sigma \hat{\theta}/\alpha \) as \( \tau - t \) approaches January. Then, for contracts maturing in January, we will gradually get more and more contribution from the jump term. Next, for contracts maturing after January, the influence from the jump term will decay slowly, with time to maturity. The reason is that the integral will integrate the constant intensity over January, however, the function

\[ s \mapsto \phi_J(\tilde{\theta} + e^{-\beta s}) - \phi_J(\tilde{\theta}) \]

is decreasing.
Thus, the farther the maturity is away from January, the smaller function we integrate against, yielding a decreasing contribution. By appropriately scaling the market price of risks, we can see a risk premium which is decreasing up till January, and then increasing to something positive, before it decreases again to a negative level, possibly higher than the previous. We may also have a risk premium which is decreasing till January, then increasing before it decreases again, without crossing to positivity.
Let us now suppose that we are in the beginning of January. Then the picture will be as follows. We will now get full influence of the jump term for the contracts maturing in January, and we will immediately see a positive premium for appropriate choice of the market price of risk. For later maturing contracts, the influence of the jumps in January will become smaller, and eventually we will cross zero and have a negative premium in the long end. In fact, depending on the speeds of mean-reversion, this crossing to negative premia may happen very quick.
From these considerations, we see that for a seasonally dependent spike intensity, we can have a negative risk premium structure which is downward sloping with small bumps along the curve in periods where the spike risk is high. In such periods, one may even get a positive premium. If present time corresponds with a high intensity period for spikes, we may have a positive premium in the short end, and negative in the long. The risk premium curve will be further scaled by the current level of the spike $Y$ and base component $X$. Running over the year, the risk premium structure can go from negative all over, to positive in the short end, and negative in the long, appropriately scaled by the current spike level $Y$ and base component $X$. Note that the spike component normally contributes in the short end since $\beta$ is usually fast, while $X$, the base component has a longer range of influence since it is associated with a slow mean-reversion $\alpha$. 
We illustrate the situation discussed above with a numerical example. Measuring time $t$ in days, we assume that $\beta = 0.3466$. This corresponds to a fast mean-reversion yielding a half-life of 2 days. Further, we consider exponentially distributed jump sizes with a mean equal to 0.5. To mimic seasonally occurring jumps, we suppose that the spikes only occur in January, where the frequency is 5, that is, on average 5 spikes during January. For the rest of the year we let the frequency be zero. The base component has a speed of mean reversion $\alpha = 0.099$, which is a half-life of 7 days, and the volatility is supposed to be $\sigma = 0.0158$, that is, 30% annually.
In the example, we calculate the contribution to the risk premium coming from the term $\Theta_g(t, \tau; \theta) - \Theta_g(t, \tau; 0)$, which we recall is the factor deciding the sign of the risk premium. We assume that the spike risk is positive, and the base component has a negative premium, here given as $\tilde{\theta} = 0.95$ and $\hat{\theta} = -4$, resp. Calculating this for various times over the year, for a time-to-maturity ranging up to 220 days, we obtain the curves in Figs. and . In the example, we have standardized each month to be 30 days long.
Middle of January

Time to maturity
In Fig. 1 we have plotted the curves seen from the first day in July, October, December and January. For the three first curves, they are all dropping downwards from a negative value, however, having a bump when the time-to-maturity is crossing over the month of January. We observe a negative premium overall, since the contribution from the base component is the strongest. When we consider the curve from January 1, the picture is changing. In the short end of the curve, we get a positive contribution, before becoming negative in the long end. This is a reflection of a strong contribution from the spike component which now pushes the premium above zero, reflecting the aversion to spike risk. Note that by increasing the spike size distribution and/or the frequency, we may even obtain positive premium in the bump along the curve.
Looking at Fig. 2 this is even more distinct. By considering the curve seen from January 15, we get an even more positive contribution, however, lasting shorter since only half of the spike period is taken into account. The kink in the curve around 15 days to maturity is a reflection of the spike period being constrained to only January. The risk premium is obtained by scaling the curves by positive factors which size is given by the mean-reversion discounted states of $X$ and $Y$. Hence, if we get a spike $Y(t)$, then the influence on the risk premium is relatively small if we are in July compared to January. Getting a spike in January, will immediately scale up the positive part in the short end. An increasing value of $X(t)$, on the other hand, will scale up a bigger part of the curve since the mean-reversion is slower. The seasonality function will also play an important role by an overall seasonal scaling of
the curve. The example clearly demonstrates that seasonally occurring jumps are responsible for bumps in the risk premium and a positive premium in the short end, as long as we are in a period of high spike intensity. By applying the Esscher transform, we can obtain a risk premium which goes from being negative downward sloping, to having a positive short end.
**MAIN CONCLUSION** By using the Esscher Transform as a risk-neutral measure in order to price Forward contracts in Electricity markets the sign change of the risk premium empirically observed can be detected and ”forecasted”.
FUTURES PERSPECTIVES

- CALIBRATION OF THE PARAMETER FUNCTIONS
- COMPARISON WITH SIGN CHANGE OBSERVED
- RISK PREMIUM CALCULATIONS IN MORE GENERAL MODELS
- AMBIT PROCESSES?