American Foreign Exchange Options and some Continuity Estimates of the Optimal Exercise Boundary with respect to Volatility

Nasir Rehman

Allam Iqbal Open University Islamabad, Pakistan.
In the last fifty years we see the emergence of a new scientific discipline, Theory of finance. This theory deals with how financial markets work, how to make them more efficient and how they should be regulated. The theory of finance has become increasingly mathematical because the problems in finance are now driving research in mathematics.

This is a very wide subject. Here we have:

- Basic financial instruments
- Discrete-time finance
- Continuous-time finance
- Stochastic calculus applied to finance
- etc
Options

Options are fundamentally different from the contracts in the sense that in a contract both the parties are obliged to exercise the contract while in an option one party (the buyer of the option) has a choice to exercise the option but is not obliged to exercise. Secondly when two parties enter into a standard contract (forward, futures) usually there is no exchange of money. But when we buy an option we have to pay a certain amount (the price of the option). Options are basically of two types

- Call options and Put options

A call option gives its holder the right to buy a certain amount of a financial asset by a certain date (Maturity) for a certain price (Strike). Similarly the put option gives its holder the right to sell the same.
Options

Examples
When we look at exercise style the options are of many types:

■ European Options
■ Bermudan Options
■ American Options
■ Exotic Options (Path-dependent Options)
■ Asian Options
■ Look back Options
■ Shoot Options
■ Barrier Options
Options

European Option
It gives its holder the right to buy or sell a share of a stock at a predetermined price (strike price) and at a predetermined date (maturity date).

American Option
It gives its holder the right to buy or sell a share of a stock at a predetermined price (strike price) and at any time including the maturity date i.e. It can be exercised at any time. So it is more valuable and expensive as compared to the European option.
American Foreign Exchange Put Option
It gives its holder the right to sell one unit of a foreign currency at a predetermined price and at any time including the maturity date. Since it gives its holder greater rights than the European option, via the right of early exercise, potentially it has a higher value.
Now here is a basic question. If we want to buy an option whether it is European or American how much we should pay? What is the fair price of an option? This is called the **pricing problem** and it is of fundamental importance in mathematical finance. As long as the European options are concerned this question has a complete solution.i.e. we have the well known BSM(Black Scholes Merton) model. But for the case of American options uptill now there does not exist any explicit form solution to price them. Although there exist many many approximations.
Garman, Kohlhagen Model

BSM model was given in 1973. Ten years latter in 1983 Garman and Kohlhagen extended this model for the case of options defined on the foreign exchange rate. The exchange rate process is denoted by $Q_t, 0 \leq t \leq T$, (where $Q_t$ gives the units of domestic currency per unit of foreign currency at time $t$) This classical Garman-Kohlhagen model for the currency exchange assumes that the domestic and foreign currency risk-free interest rates are constant and the exchange rate follows a log-normal diffusion process.
The General Case

Here we consider the general case, when exchange rate evolves according to arbitrary one-dimensional diffusion process with local volatility that is the function of time and the current exchange rate and where the domestic and foreign currency risk-free interest rates may be arbitrary continuous functions of time. Some non-trivial problems we encounter in time-dependent case which include the continuity in time argument of the value function of the American put option and the regularity properties of the optimal exercise boundary.
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t), 0 \leq t \leq T$, a one-dimensional standard Brownian motion on it. Denote by $(\mathcal{F}_t)_{0 \leq t \leq T}$ the natural filtration of $(W_t), 0 \leq t \leq T$. We assume that the time horizon $T$ is finite. On the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P), 0 \leq t \leq T$, we consider a financial market with two currencies domestic and foreign with their corresponding interest rates $r^d(t)$ and $r^f(t), 0 \leq t \leq T$, being two arbitrary strictly positive continuous functions of time.
The exchange rate process \((Q_t, \mathcal{F}_t), 0 \leq t \leq T\), is the strong solution of the following stochastic differential equation (see for example Chapter 9, Section 9.3, Shreve)

\[
dQ_t = Q_t \left( r^d(t) - r^f(t) \right) dt + Q_t \cdot \sigma(t, Q_t) \cdot dW_t,
\]

\[
Q_0 > 0, 0 \leq t \leq T.
\]
We define the value of the American put option and characterize it in terms of the Snell envelope of the discounted payoff process. We shall state the American put option problem on foreign exchange following the general framework of chapter 2 in Karatzas and Shreve.

Consider the discounted payoff process

\[ X_t = e^{-\int_0^t r(v) dv} (K - Q_t)^+, \quad 0 \leq t \leq T. \]
Value function of the American Put Option

Theorem 5.8, Chapter 2 of Karatzas and Shreve states that the value $V_t$ at time $t$ equals

$$V_t = e^{\int_0^t r^d(v)dv} \cdot Y_t, \quad (a.s.)$$

where the process $(Y_t, \mathcal{F}_t), 0 \leq t \leq T$ is the so-called Snell envelope of the process $(X_t, \mathcal{F}_t), 0 \leq t \leq T$, that is the minimal supermartingale majorizing the latter process.

The Snell envelope $(Y_t, \mathcal{F}_t), 0 \leq t \leq T$ has rightcontinuous paths and for arbitrary $(\mathcal{F}_t)_{0 \leq t \leq T}$—stopping time $\tau$ the following equality is valid:

$$Y_{\tau} = \text{ess sup}_{\tau \leq \rho \leq T} E(X_{\rho}/\mathcal{F}_\tau), \quad (a.s.)$$
Value function of the American Put Option

Therefore
\[ V_t = \text{ess sup}_{t \leq \tau \leq T} E \left( e^{-\int_t^\tau r^d(v)dv} \left( K - Q_\tau \right)^+ / \mathcal{F}_t \right) \quad (a.s.) \]

Let us introduce now American foreign exchange put value function
\[ \nu(t, x) = \sup_{t \leq \tau \leq T} E \left[ e^{-\int_t^\tau r^d(u)du} (K - Q_\tau(t, x))^+ \right], 0 \leq t \leq T, x \geq 0, \]
where the supremum is taken over all \((\mathcal{F}_u)_{0 \leq u \leq T}\) — stopping times \(\tau\) such that \(t \leq \tau \leq T\).

and where \(Q_\tau(t, x)\), is the unique strong solution of the following stochastic differential equation
\[ dQ_\tau(t, x) = Q_\tau(t, x) \cdot b_i(u)du + Q_\tau(t, x) \cdot \sigma(u, Q_\tau(t, x)) \cdot dW_u, \]
Lipschitz continuity of the value function

Lemma

If the pathwise monotonicity in $x$ holds for the solution of the stochastic differential equation then the value function $v(t, x)$ is Lipschitz continuous in $x$

$$|v(t, y) - v(t, x)| \leq |y - x|, \quad 0 \leq x, 0 \leq y.$$
We introduce the continuation set for the optimal stopping problem

\[ D^T = \{(t, x) : 0 \leq t < T, x > 0, v(t, x) > (K - x)^+ \}, \]

and its section \( D^T_t \) at time instant \( t \)

\[ D^T_t = \{x > 0 : (t, x) \in D^T\}, \quad 0 \leq t < T. \]

then the shape of this section is explained by the lemma:

**Lemma**

The section \( D^T_t \) has the following form

\[ D^T_t = (b^T(t), \infty), \quad 0 \leq t < T, \]

where \( b^T(t) \) satisfies \( 0 < c \leq b^T(t) < K, \quad 0 \leq t < T \), for some strictly positive constant \( c \).
Continuity of the American put option value function

Now we state the continuity of the American foreign exchange put value function which can be considered to be the key technical tool for the investigation of the regularity properties of the American put value process as well as of the corresponding optimal exercise boundary $b^T(t), 0 \leq t < T$.

**Proposition 1**

The value function $v(t, x)$ is continuous with respect to pair $(t, x)$ of its arguments. Moreover the following estimate does hold

$$|v(s, x) - v(t, y)| \leq c \cdot (1 + x) \cdot |t - s|^\frac{1}{2} + |x - y|,$$

where $s, t \in [0, T], |t - s| \leq 1, x, y \in [0, \infty)$ and $c$ is a constant dependent upon $\bar{r}^d, \bar{r}^f, \bar{\sigma}$ and $K$. 
Decomposition of the Discounted Payoff Process

We get a canonical decomposition of the discounted payoff process \( X_u(t, x), t \leq u \leq T \).

\[
X_u(t, x) = e^{-\int_t^u r^d(v) \, dv} (K - Q_u(t, x))^+, \quad t \leq u \leq T. 
\]

Applying Tanaka-Meyer formula to the function \( f(x) = (K - x)^+ \) and simplifying we get
\[
X_u(t, x) = M_u(t, x) + \frac{1}{2} \int_t^u e^{-\int_t^v r^d(s) \, ds} \, dL_0^0(K - Q)^+ 
+ \int_t^u e^{-\int_t^v r^d(s) \, ds} \cdot I_{(Q_v(t, x) < K)} \cdot [Q_v(t, x) \cdot r^f(v) - K \cdot r^d(v)] \, dv, \quad t \leq u \leq T.
\]
Upper bound for the optimal exercise boundary

The next proposition gives the upper bound for the boundary \( b^T(t), \ 0 \leq t < T \).

**Proposition**

For arbitrary \( t \) such that \( 0 \leq t < T \), we have the inequality

\[
b^T(t) \leq \min \left( \frac{r^d(t)}{r^f(t)} \cdot K, K \right).
\]
Two Important Inequalities

Lemma

Fix $t$, $0 \leq t < T$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| 1 - e^{-\int_s^u r^d(v)dv} - \frac{r^d(t)}{r^f(t)} \right| \leq \epsilon,$$

whenever the pair $(s, u)$ satisfies condition $t - \delta \leq s < u \leq t + \delta$.

Lemma

For arbitrary $t$, $0 \leq t < T$ and $C$, $0 \leq C < b^T(t)$ there exists $\delta > 0$ such that

$$C \left(1 - e^{-\int_s^u r^f(v)dv}\right) < K \left(1 - e^{-\int_s^u r^d(v)dv}\right),$$

for the pair $(s, u)$ satisfying the condition $t - \delta \leq s \leq u \leq t + \delta$. 
Comparison of the American put values

We state a result which consists in comparison of the value functions of the American puts with different strikes, maturities and volatilities. It is inspired by similar result of Achdou Lemma 3.6 concerning the comparison of the solutions of certain variational inequalities.
American Put Value Function

Formulation

For the formulation of this result we need to consider the solution \( \overline{Q}_u(s, x) \) of the stochastic differential equation with constant volatility \( \overline{\sigma} \)

\[
d\overline{Q}_u(s, x) = \overline{Q}_u(s, x)(r^d(u) - r^f(u))du + \overline{Q}_u(s, x) \cdot \overline{\sigma} \cdot dW_u, \overline{Q}_s(s, x) = x,
\]

and we introduce the value function of the corresponding optimal stopping problem for arbitrary time interval \([s, t] \), \( s \leq t \leq T \) and arbitrary strike price \( C \), \( 0 \leq C \leq K \),

\[
\overline{V}(s, x; t, C) = \sup_{s \leq \tau \leq t} E \left[ e^{-\int_s^\tau r^d(v)dv} \cdot (C - \overline{Q}_\tau(s, x))^+ \right], \quad 0 \leq s \leq t \leq T.
\]
Proposition 3

The following relationship is valid between American put values with different strikes, maturities and volatilities

\[ v(s, x) \leq \bar{v}(s, x; t, C) + (K - C), \quad 0 \leq s \leq t < T, \quad x \geq 0, \]

where \( C \) is arbitrary constant such that \( 0 \leq C \leq b^T(t) \), and \( b^T(t) \) is the value at instant \( t \) of the optimal exercise boundary \( b^T(u), 0 \leq u < T \), of the optimal stopping problem.
Regularity properties of the optimal exercise boundary

Proposition

The optimal exercise boundary $b^T(t)$, $0 \leq t < T$, is left continuous and right-hand limit exists at arbitrary time instant $t$, $0 \leq t < T$. 
Early exercise premium representation

Now we shall use these properties to establish the early exercise premium representation for the American foreign exchange put option value function.

**Theorem**

The following early exercise premium representation formula is valid for the value function $v(t, x), \; 0 \leq t \leq T, x \geq 0$ of the American foreign exchange put option:

$$v(t, x) = E\left[e^{-\int_t^T r^d(u)du} \left(K - Q_T(t, x)\right)^+\right] +$$

$$+ E\left[\int_t^T I(Q_u(t, x) \leq b^T(u)) \cdot e^{-\int_t^u r^d(v)dv} \left(K \cdot r^d(u) - Q_u(t, x) \cdot r^f(u)\right) du\right],$$

where the first term in the right hand side is the value function of the corresponding European put option and the second term is the so-called early exercise premium.
It is important to note that the expression under integral in the latter equality is actually nonnegative \((a.s.)\) as easily seen from the Proposition 2 and from the following inequality

\[
K \cdot r^d(u) - Q_u(t, x) \cdot r^f(u) \geq K \cdot r^d(u) - b^T(u) \cdot r^f(u) \geq 0
\]

provided that \(Q_u(t, x) \leq b^T(u)\).
Nonlinear integral equation for the optimal exercise boundary

Notations

To derive the nonlinear integral equation for the boundary $b^T(t), 0 \leq t < T$ we shall need to introduce some standard notations. Denote by

$$P(t, x; u, y) = P(Q_u(t, x) \leq y), \quad 0 \leq t \leq u \leq T, x \geq 0, y \geq 0,$$

$$R(t, x; u, y) = \int_0^y P(t, x; u, z)dz, \quad 0 \leq t \leq u \leq T, x \geq 0, y \geq 0,$$

where $P(t, x; u, y)$ is the transition probability function of the diffusion process $Q_u$ and $R(t, x; u, y)$ is the integral function of it.
Nonlinear integral equation for the optimal exercise boundary

**Proposition**

The optimal exercise boundary $b^T(t), 0 \leq t < T$, satisfies the following nonlinear integral equation:

$$K - b^T(t) = e^{-\int_t^T r^d(u)du} \cdot R(t, b^T(t); T, K) + \int_t^T e^{-\int_u^T r^d(v)dv} \left[ (Kr^d(u) - b^T(u) \cdot r^f(u)) \cdot P(t, b^T(t); u, b^T(u)) + r^f(u) \cdot R(t, b^T(t); u, b^T(u)) \right] du.$$
Consider the volatilities $\sigma_0, \sigma_1$ and $\sigma_2$ which satisfy:

$$0 < \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \bar{\sigma}.$$ 

We know from the Karatzas and Shreve that the optimal exercise boundaries $b_{\sigma_0}(t), b_{\sigma_1}(t), b_{\sigma_2}(t), 0 \leq t < T$ are nondecreasing continuous functions of time. The remarkable fact holds true (Ekstrom, El-Karoui et al. Hobson) that increasing the volatility, the value function of the American option with convex payoff also increases, therefore we have

$$v_0(t, x) \leq v_1(t, x) \leq v_2(t, x), \quad 0 \leq t \leq T, x \geq 0.$$
Continuity Estimates

Optimal Exercise Boundary

We recall the definition of the optimal exercise boundaries

\[ b_{\sigma_i}(t) = \inf\{x \geq 0 : v_i(t, x) > (K-x)^+\}, \quad 0 \leq t < T, \ i = 0, 1, 2. \]

From this definition and previous inequality it is evident that

\[ b_{\sigma_0}(t) \geq b_{\sigma_1}(t) \geq b_{\sigma_2}(t), \quad 0 \leq t < T. \]
Continuity Estimates

**Optimal Exercise Boundary**

The last part of this talk is devoted to estimate the distance between the optimal exercise boundaries $b_{\sigma_2}(t)$ and $b_{\sigma_1}(t)$ in terms of the distance between the volatilities $\sigma_2$ and $\sigma_1$. For this we need some preliminary estimates regarding the optimal exercise boundary and the value function of the American foreign exchange put option. Some other properties of the optimal exercise boundary can be found in the works of Lamberton and Villeneuve. Since volatility is a major ingredient in option pricing, so these continuity estimates are important from practical viewpoint too.
Continuity Estimates

**Optimal Exercise Boundary**

Let us introduce the critical level

$$x_0 = \min \left( \frac{r^d}{r^f} \cdot K, K \right).$$

**Lemma**

Let $\sigma_0 > 0$ be arbitrary positive constant, then for the optimal exercise boundary $b_{\sigma_0}(t), 0 \leq t < T$, the following inequality is valid

$$b_{\sigma_0}(t) < x_0, \quad 0 \leq t < T.$$
Now we state the continuity estimate for the value functions $v_1(t, x)$ and $v_2(t, x)$ with respect to volatilities $\sigma_1$ and $\sigma_2$ respectively. For a more general result you can see Achdou paper.

**Lemma**

For the difference of American put value functions $v_2(t, x)$ and $v_1(t, x), 0 \leq t \leq T, x \geq 0$, the following estimate does hold

$$| v_2(t, x) - v_1(t, x) | \leq c_1 \cdot x \cdot | \sigma_2 - \sigma_1 |, \quad 0 \leq t \leq T, \ x \geq 0,$$

where the constant $c_1$ depends only on $\bar{r}, \bar{\sigma}$ and $T$. 
Continuity Estimates

Theorem

For the optimal exercise boundaries $b_{\sigma_2}(t)$ and $b_{\sigma_1}(t)$, $0 \leq t < T$ of the corresponding American put option problem, the following estimates are valid:

If $0 \leq r^f \leq r^d$, then

$$\left( r^d \cdot K - r^d \cdot b_{\sigma_0}(t) \right)^{1/2} \cdot |b_{\sigma_2}(t) - b_{\sigma_1}(t)| \leq c \cdot K^{3/2} \cdot |\sigma_2 - \sigma_1|^{1/2}$$

If $r^f > r^d > 0$, then

$$\left( r^d \cdot K - r^f \cdot b_{\sigma_0}(t) \right)^{1/2} \cdot |b_{\sigma_2}(t) - b_{\sigma_1}(t)| \leq c \cdot K^{3/2} \cdot |\sigma_2 - \sigma_1|^{1/2}$$
References

THANK YOU
FOR YOUR INTEREST