Investment-consumption problem in illiquid markets with regime switching

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Introduction

- An important aspect of market liquidity: restriction on trading times.
- We assume the investor can trade the risky asset only at random times, corresponding to the arrival of buy/sell orders.
- Empirical evidence for liquidity fluctuation: market with regime-switching liquidity/price dynamics.
- In this context, we study the problem of optimal investment/consumption over an infinite horizon.
Previous works

- Rogers, Zane (02); Matsumoto (06); Pham, Tankov (08): single-regime case. In all of these papers, the trading times are the jump times of a Poisson process with constant intensity $\lambda$.
- Diesinger, Kraft, Seifried (10); Ludkovski, Min (10): Two regimes, fully liquid (continuous trading) and fully illiquid (no trading).
Outline

1. The illiquid market model
2. HJB equations and characterization of the solution
3. Power utility functions and numerical results
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- Filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\).

- Liquidity regime cycles: Markov chain \((I_t)_{t \geq 0}\) with state space \(\mathbb{I}_d = \{1, \ldots, d\}\), and intensity matrix \(Q = (q_{ij})\). \(N^{ij}\) associated Poisson processes.

- Risky asset of price process \((S_t)_{t \geq 0}\).

- The investor is restricted to trading \(S\) only at random times \(\tau_n, n \geq 1\), jump times of a Cox process \((N_t)_{t \geq 0}\) with intensity \((\lambda I_t)\).
Hybrid regime-switching jump diffusion model

In the liquidity regime $l_t = i$, 

$$dS_t = S_t (b_i dt + \sigma_i dW_t),$$

where $W$ is a $({\mathcal F}_t)$-Brownian Motion, and $b_i \in \mathbb{R}$, $\sigma_i \geq 0$. At the times of transition from $l_{t-} = i$ to $l_t = j$, $\Delta S_t = -S_{t-} \gamma_{ij}$, where $\gamma_{ij} < 1$. Overall:

$$dS_t = S_{t-} \left( b_{l_{t-}} dt + \sigma_{l_{t-}} dW_t - \gamma_{l_{t-},l_t} dN^{l_{t-},l_t}_t \right).$$
Trading strategies

We consider an agent investing and consuming in this market. Trading strategy: pair \((c, \zeta)\), where

- \((c_t)\) nonnegative adapted process is the consumption process,
- \((\zeta_t)\) predictable is the investment strategy: at \(t = \tau_n\), the agent buys an amount \(\zeta_t\) in the risky asset.

\((X^c_t, \zeta, Y^c_t, \zeta)\) amounts invested in cash and in the risky asset.

\[
dX^c_t, \zeta = -c_t \, dt - \zeta_t \, dN_t.
\]

\[
dY^c_t, \zeta = Y_t \left( b_{t_1^t} \, dt + \sigma_{t_1^t} \, dW_t - \gamma_{t_1^t, l_t} \, dN_{t_1^t, l_t} \right) + \zeta_t \, dN_t,
\]
Admissible strategies

Total wealth $R_t := X_t + Y_t$.
Under initial conditions $(i, x, y)$, $(c, \zeta) \in A_i(x, y)$ (set of admissible strategies) if $R_{\tau_n} \geq 0$ a.s., $n \geq 1$.
This is equivalent to a no-short sale constraint:

$$X_t \geq 0 \ ; \ Y_t \geq 0,$$

or in terms of the controls:

$$-Y_t^- \leq \zeta_t \leq X_t^-, \quad \int_t^{\tau_{n+1}} c_s ds \leq X_t, \quad \tau_n \leq t < \tau_{n+1}, \ n \geq 1.$$
Optimal investment/consumption

\[ U \text{ increasing, concave function on } [0, \infty) \text{ with } U(0) = 0. \]
\[ \rho > 0 \text{ discount factor.} \]
\[ \text{Value functions :} \]

\[ v_i(x, y) = \sup_{(\zeta, c) \in A_i(x, y)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(c_t) dt \right], \quad i \in \mathbb{I}_d, \ (x, y) \in \mathbb{R}^2_+, \]

\[ \tilde{v}_i(r) = \sup_{x+y=r} v_i(x, y) \quad i \in \mathbb{I}_d, \ r \in \mathbb{R}_+. \]
Outline

1. The illiquid market model
2. HJB equations and characterization of the solution
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Assumptions:

- There exists $p \in (0, 1)$, $K_1 > 0$ s.t.
  \[ U(x) \leq K_1 x^p, \quad x \geq 0. \]

- $\rho > k(p)$, where
  \[
  k(p) := \sup_{i \in \mathbb{I}, z \in [0,1]} pb_i z - \frac{\sigma_i^2}{2} p(1 - p)z^2 + \sum_{j \neq i} q_{ij}((1 - z \gamma_{ij})^p - 1).
  \]
Proposition

For some positive constant $C$, 

$$v_i(x, y) \leq C(x + y)^p, \quad (i, x, y) \in I_d \times \mathbb{R}_+ \times \mathbb{R}_+. \quad (1)$$

$v_i$ is concave in $(x, y)$, increasing in both variables, and continuous on $\mathbb{R}_+^2$.

$\tilde{v}_i$ is increasing, concave and continuous on $\mathbb{R}_+$. 
HJB System

The Hamilton-Jacobi-Bellman equation for this problem is:

$$\rho v_i - b_i y \frac{\partial v_i}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v_i}{\partial y^2} - \sup_{c \geq 0} \left[ U(c) - c \frac{\partial v_i}{\partial x} \right]$$

$$- \sum_{j \neq i} q_{ij} \left[ v_j \left( x, y(1 - \gamma_{ij}) \right) - v_i(x, y) \right]$$

$$- \lambda_i \left[ \sup_{-y \leq \zeta \leq x} v_i(x - \zeta, y + \zeta) - v_i(x, y) \right] = 0.$$  (2)

on $\mathbb{I}_d \times (0, \infty) \times \mathbb{R}_+$, together with the boundary conditions:

$$v_i(0, 0) = 0$$  (3)

$$v_i(0, y) = \mathbb{E}_i \left[ \sup_{0 \leq \zeta \leq y \frac{S_{\tau_1}}{S_0}} v_{I_{\tau_1}} \left( \zeta, y \frac{S_{\tau_1}}{S_0} - \zeta \right) \right]$$  (4)
Viscosity characterization of the value function

Theorem

The value function \( v \) is a viscosity solution to the HJB system \((2)\) and the boundary conditions \((3)\) and \((4)\).

It is the unique solution satisfying the growth condition

\[ v_i(x, y) \leq C(x + y)^p. \]
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Power utility functions

We consider the case $U(c) = \frac{c^p}{p}$, $0 < p < 1$.
Scaling property for the value function: $v_i(kx, ky) = k^p v_i(x, y)$.
Change of variables

$$r = x + y$$
$$z = \frac{y}{x + y}.$$

The value function for $(r, z)$ can be written as

$$u_i(r, z) = \frac{r^p}{p} \varphi_i(z)$$
The HJB equation for the $\varphi_i$ is the system of IODEs:

\[
(\rho - q_{ii} + \lambda_i - pb_i z + \frac{1}{2} p(1 - p)\sigma_i^2 z^2)\varphi_i - z(1 - z)(b_i - z(1 - p)\sigma_i^2)\varphi_i' \\
- \frac{1}{2}z^2(1 - z)^2\sigma_i^2 \varphi_i'' - (1 - p)(\varphi_i - \frac{z}{p}\varphi_i') - \frac{p}{1 - p}
\]

\[
- \sum_{j \neq i} q_{ij} \left[(1 - z\gamma_{ij})^p \varphi_j \left(\frac{z(1 - \gamma_{ij})}{1 - z\gamma_{ij}}\right)\right] - \lambda_i \sup_{\pi \in [0,1]} \varphi_i(\pi) = 0,
\]

together with the boundary conditions

\[
(\rho - q_{ii} + \lambda_i)\varphi_i(0) - (1 - p)\varphi_i(0)^{-\frac{p}{1 - p}} = \sum_{j \neq i} q_{ij} \varphi_j(0) + \lambda_i \sup_{\pi \in [0,1]} \varphi_i(\pi),
\]

\[
(\rho - q_{ii} + \lambda_i - pb_i + \frac{1}{2} p(1 - p)\sigma_i^2)\varphi_i(1) = \sum_{j \neq i} q_{ij} (1 - \gamma_{ij})^p \varphi_j(1) + \lambda_i \sup_{\pi \in [0,1]} \varphi_i(\pi).
\]
Regularity of the value function

Proposition

$(\varphi_i)_{i=1}^{\ldots,d}$ is concave and continuous on $[0, 1]$, $C^2$ on $(0, 1)$ and is a classical solution of (5) on $(0, 1)$, with boundary conditions (6) - (7).
Optimal Control

Define

\[ c^*(i, z) = \begin{cases} 
\left( \varphi_i(z) - \frac{z}{p} \varphi_i'(z) \right)^{\frac{-1}{1-p}} 
& \text{when } 0 < z < 1 \\
\left( \varphi_i(0) \right)^{\frac{-1}{1-p}} 
& \text{when } z = 0 \\
0 
& \text{when } z = 1
\end{cases} \]

\[ \pi^*(i) = \arg \sup_{\pi \in [0,1]} \varphi_i(\pi). \]

Proposition

Given initial conditions \( i, r, z \), there exists an admissible control \((\hat{c}, \hat{\zeta})\) such that :

\[ \hat{c}_t = R_t - c^*(I_t, Z_t), \]

\[ \hat{\zeta}_t = R_t (\pi^*(I_t) - Z_t), \]

and this control is optimal.
Numerical analysis : iterative method

Recall the HJB system:

\[(\rho - q_{ii} + \lambda_i)v_i - b_i y \frac{\partial v_i}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v_i}{\partial y^2} - \sup_{c \geq 0} \left[ U(c) - c \frac{\partial v_i}{\partial x} \right] = \sum_{j \neq i} q_{ij} v_j \left( x, y(1 - \gamma_{ij}) \right) + \lambda_i \sup_{-y \leq \zeta \leq x} v_i(x - \zeta, y + \zeta)\]
Numerical analysis : iterative method

We solve it by iteration:

- $v^0 = 0$,
- Given $v^n$,

\[
(\rho - q_{ii} + \lambda_i)v_{i}^{n+1} - b_i y \frac{\partial v_{i}^{n+1}}{\partial y} - \frac{1}{2} \sigma_i^2 y^2 \frac{\partial^2 v_{i}^{n+1}}{\partial y^2} - \sup_{c \geq 0} \left[ U(c) - c \frac{\partial v_{i}^{n+1}}{\partial x} \right] = \sum_{j \neq i} q_{ij} v_j^n \left( x, y(1 - \gamma_{ij}) \right) + \lambda_i \sup_{-y \leq \zeta \leq x} v_i^n(x - \zeta, y + \zeta) \tag{8}
\]

with boundary conditions

\[
v_{i}^{n+1}(0, 0) = 0, \tag{9}
\]

\[
v_{i}^{n+1}(0, y) = \mathbb{E}_i \left[ \sup_{0 \leq \zeta \leq y \frac{S_{T_1}}{S_0}} v_{l_{T_1}}^{n} \left( \zeta, y \frac{S_{T_1}}{S_0} - \zeta \right) \right] \tag{10}
\]

\[
v_{i}^{n+1}(x, y) \leq C(x + y)^{p} \tag{11}
\]
Stochastic control representation of $v^n$

Consider the stopping times

\[
\begin{align*}
\theta_0 &= 0, \\
\theta_{n+1} &= \inf \left\{ t > \theta_n \text{ s.t. } \Delta N_t \neq 0 \text{ or } \Delta N_{t^-}^l \neq 0 \right\},
\end{align*}
\]

i.e. $\theta_n$ is the $n$-th time where we have either a change of regime or a trading time.

**Proposition**

*Given* $v^n_0, \ldots, v^n_n$, *(8)-(11)* *admits a unique viscosity solution.* *Furthermore, we have for* $n \geq 0$,*

\[
v^n_i(x, y) = \sup_{(\zeta, c) \in A_i(x, y)} \mathbb{E} \left[ \int_0^{\theta_n} e^{-\rho t} U(c_t) dt \right].
\]
Convergence result

Proposition

$v^n \rightharpoonup v$.

*For some $C \geq 0$, $0 < \delta < 1$,*

$$v_i(x, y) - v^n_i(x, y) \leq C(x + y)^p \delta^n.$$
Going back to $U(c) = \frac{c^p}{p} : v_i^n(x, y) = \frac{r^p}{p} \varphi_i^n(z)$.

$\varphi_i^{n+1}$ solves the following BVP:

$$(\rho - q_{ii} + \lambda_i - pb_i z + \frac{1}{2} p(1 - p) \sigma_i^2 z^2) \varphi_i^{n+1} - z(1 - z)(b_i - z(1 - p) \sigma_i^2)(\varphi_i^{n+1})'$$

$$- \frac{1}{2} z^2 (1 - z)^2 \sigma_i^2 (\varphi_i^{n+1})'' - (1 - p)(\varphi_i^{n+1} - \frac{z}{p} (\varphi_i^{n+1})') - \frac{1}{1 - p}$$

$$= \sum_{j \neq i} q_{ij} \left[ (1 - z \gamma_{ij})^P \varphi_j^n \left( \frac{z(1 - \gamma_{ij})}{1 - z \gamma_{ij}} \right) \right] + \lambda_i \sup_{\pi \in [0,1]} \varphi_i^n(\pi),$$

$$(\rho - q_{ii} + \lambda_i) \varphi_i^{n+1}(0) - (1 - p) \varphi_i^{n+1}(0) - \frac{1}{1 - p} = \sum_{j \neq i} q_{ij} \varphi_j^n(0) + \lambda_i \sup_{\pi \in [0,1]} \varphi_i^n(\pi),$$

$$(\rho - q_{ii} + \lambda_i - pb_i + \frac{1}{2} p(1 - p) \sigma_i^2) \varphi_i^{n+1}(1) = \sum_{j \neq i} q_{ij} (1 - \gamma_{ij})^P \varphi_j^n(1) + \lambda_i \sup_{\pi \in [0,1]} \varphi_i^n(\pi).$$
Numerical Results : Single-regime case

\( \rho = 0.2, \ b = 0.4, \ \sigma = 1. \)

Cost of liquidity \( P(r) : \tilde{\nu}(r + P(r)) = \nu_M(r). \)

Comparison with (Pham, Tankov 08) : similar model, the difference is that the investor only observes the stock price at the trading times, so that the consumption process is piecewise-deterministic.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Discrete observation</th>
<th>Continuous observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.275</td>
<td>0.153</td>
</tr>
<tr>
<td>5</td>
<td>0.121</td>
<td>0.016</td>
</tr>
<tr>
<td>40</td>
<td>0.054</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table: Cost of liquidity \( P(1) \) as a function of \( \lambda \).
Value function $\varphi(z)$ for different values of $\lambda$

$\lambda = 1$ (red) $\lambda = 5$ (blue) $\lambda = 3$ (green) $\lambda = 10$ (purple) Merton ($z^*$)
Optimal consumption rate $c^*(z)$ for different values of $\lambda$

![Graph showing the optimal consumption rate for different values of \( \lambda \)]
Two regimes

Parameters:

\[ b_1 = 0.4, \quad \sigma_1 = 1, \]
\[ b_2 = 0.1, \quad \sigma_2 = 0.7, \]
\[ \gamma_{12} = \gamma_{21} = 0, \]
\[ q_{12} = q_{21} = 1. \]

Regime 1 is "better" than regime 2: \( \frac{b_1}{\sigma_1} > \frac{b_2}{\sigma_2} \).
The illiquid market model HJB equations and characterization of the solution

Power utility functions and numerical results

\( \varphi_1(z) \)

\( \varphi_2(z) \)

\[(\lambda_1, \lambda_2) = (1, 1) \]
\[(\lambda_1, \lambda_2) = (10, 1) \]
\[(\lambda_1, \lambda_2) = (1, 10) \]
\[(\lambda_1, \lambda_2) = (10, 10) \]

Merton
Numerical results: cost of liquidity

<table>
<thead>
<tr>
<th>λ</th>
<th>$P_1(1)$</th>
<th>$P_2(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.153</td>
<td>0.019</td>
</tr>
<tr>
<td>10</td>
<td>0.004</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table: Cost of liquidity $P_i(1)$ for the single-regime case.

<table>
<thead>
<tr>
<th>$(\lambda_1, \lambda_2)$</th>
<th>$P_1(1)$</th>
<th>$P_2(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>0.111</td>
<td>0.089</td>
</tr>
<tr>
<td>(10,1)</td>
<td>0.039</td>
<td>0.034</td>
</tr>
<tr>
<td>(1,10)</td>
<td>0.051</td>
<td>0.041</td>
</tr>
<tr>
<td>(10,10)</td>
<td>0.009</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Table: Cost of liquidity $P_i(1)$ as a function of $(\lambda_1, \lambda_2)$. 
Conclusion

Future work: incomplete observation, the agent only observes the stock price at the trading times and must infer the liquidity regime. \( \nu(\vec{\pi}, x, y), \) where \( \pi_i = \mathbb{P}(l_0 = i). \)