Non-anticipative functional calculus and applications

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Outline

- Motivation: path-dependent derivatives.
- Adapted processes as non-anticipative flows.
- Functional calculus: horizontal and vertical derivatives.
- A pathwise change of variable formula for functionals.
- A functional Ito formula for semimartingales.
- Martingale representation formula I: regular case.
- A universal hedging formula for path-dependent options.
- Weak derivative. Integration by parts formula.
- Martingale representation formula II: general case.
- Numerical computation of hedging strategies.
- Martingales as solutions to functional equations.
- A universal pricing equation for path-dependent options.
- Example: Asian options.
Motivation: pricing and hedging of derivatives
Pathwise calculus for non-anticipative flows
Functional change of variable formulas
Functional Ito calculus
Martingale representation and hedging formulas
Extensions
Functional equations for martingales

References

- R. Cont & D Fournié (2009) Functional Ito formula and stochastic integral representation of martingales, arxiv/math.PR.
Mathematical models of financial markets model uncertainty by representing future evolution of prices in terms of a filtered scenario space \((\Omega, \mathcal{F}, \mathcal{F}_t)_{t \in [0, T]}\) and an \(\mathcal{F}_t\)-adapted price process \(S : [0, T] \times \Omega \to \mathbb{R}^d\) where

- \(\Omega\) represents the set of market scenarios
- \(\mathcal{F}\) represents the set of market events (which we can make sense of)
- \(\mathcal{F}_t\) represents the information revealed at date \(t\), and
- \(S(t, \omega) = (S^1(t, \omega), \ldots, S^d(t, \omega))\) represents the prices of securities at date \(t\) in scenario \(\omega\).

Specifying a probability measure \(\mathbb{P}\) gives a filtered probability space but there is often no consensus about the specification of \(\mathbb{P}\): model uncertainty.
Contingent claims

A *contingent claim* or derivative security on $S$ is given by a $(\mathcal{F}_T$-measurable) payoff $H : \Omega \mapsto \mathbb{R}$

- **Call option:** $H(\omega) = (S(T, \omega) - K)_+$
- **Asian option:** $H(\omega) = \left( \frac{1}{T} \int_0^T S(t, \omega) \ dt - K \right)_+$
- **Barrier option:** knockout call
  
  \[ H(\omega) = 1_{\max_{t \in [0, T]} S(t, \omega) < B} (S(T, \omega) - K)_+ \]

- **Variance swap:** with daily monitoring $t_k = k \Delta t$
  
  \[ H(\omega) = \sum_{k=1}^{N} \left| \ln\left( \frac{S(t_{k+1})}{S(t_k)} \right) \right|^2 - K \]

- **Option on realized variance:**
  
  \[ H(\omega) = \left( \sum_{k=1}^{N} \left| \ln\left( \frac{S(t_{k+1})}{S(t_k)} \right) \right|^2 - K \right)_+ \]
Some questions:

- **Pricing:** what is the fair value $F(t, \omega)$ at date $t$ of a contingent claim with payoff $H$?

- **Hedging:** can the option be hedged by trading in market instruments? What is the Profit/Loss (P&L) of a given hedging strategy?

- **Sensitivity analysis:** how do changes in market variables and parameters affect the value of the claim?

The main issue for financial applications is to give **computable** answers, not just existence/uniqueness results.
Diffusion models provide (computable) answers to all these questions, for non-path-dependent options, hence their popularity. If the underlying asset is given by a diffusion

\[ dS_t = \mu(t)S(t)dt + S(t)\sigma(t, S_t)dW(t) \quad \text{under } \mathbb{P} \]

then under some technical conditions on \( \mu, \sigma(., .) \),

- The fair value of an option with payoff \( H = h(S(T)) \) is \( f(t, S(t)) \) where \( f \) solves the **valuation PDE**

\[
\partial_t f + r(t)x\partial_x f + \frac{\sigma^2(t, x)x^2}{2}\partial_{xx}^2 f = r(t)f(t, x)
\]

with terminal condition \( f(T, x) = h(x) \).
Delta hedging: The option may be hedged by a self-financing strategy where the position in the underlying equal to the sensitivity ('delta') of the option:

\[
h(S(T)) = f(0, S(0)) + \int_0^T \frac{\partial}{\partial x} f(t, S(t)) \ dS(t) \\
+ \int_0^T [f(t, S(t)) - S(t) \frac{\partial}{\partial x} f(t, S(t))]r(t)dt \ \mathbb{P} - a.s.
\]

"Robustness": a hedging strategy \((\phi_t)\) computed in a (misspecified) Black-Scholes model with volatility \(\sigma_0\) leads to a Profit/Loss given by

\[
\int_0^T \left[ \frac{\sigma_0^2(t, S(t)) - \sigma^2(t, S(t))}{2} S(t)^2 e^{\int_t^T r(s)ds} \right] \frac{\Gamma(t)}{\partial^2_{xx} f(t, S(t))} dt
\]
Path-dependence:
Do these results generalize to path-dependent options and/or non-Markovian models?

Robustness to uncertainty:
These results only make use of $\mathbb{P}$ through its ’equivalence class’ i.e. its null sets. In actual situations there is no consensus on the level of volatility, expected return etc. How far can one go without actually specifying a probability measure $\mathbb{P}$?
Consider a path-dependent derivative whose value at time $t$ in market scenario $\omega \in D([0, T], \mathbb{R}^d)$ is given by $F_t(\omega)$ where $F_t : D([0, T], \mathbb{R}) \rightarrow \mathbb{R}$.

Practitioners often perform sensitivity analysis of derivatives by “bumping”/perturbating a variable, repricing the derivative and taking the difference.

For example, in the case of the “delta” of a path-dependent option, this amounts adding a jump/shift of size $\epsilon$ today to the current path $\omega$ and recomputing the price $F_t$ in the new path $\omega + \epsilon 1_{[t,T]}$.

$$F_t(\omega + \epsilon 1_{[t,T]}) - F_t(\omega) \over \epsilon$$
For non-path dependent options $F_t(\omega) = f(t, \omega(t))$ and this definition is justified by the Black-Scholes delta-hedging rule:

$$
\frac{F_t(\omega + \epsilon 1_{[t,T]}) - F_t(\omega)}{\epsilon} \xrightarrow[\epsilon \to 0]{} \frac{\partial f}{\partial S}(t, S)
$$

For a path-dependent option, the value at $t$ depends on the path $\omega$ on $[0, t]$: it is not just a function of the current price $\omega(t)$ but a \textit{functional} of $\omega \in D([0, t], \mathbb{R})$, where $D([0, t], \mathbb{R})$ is the set of (possibly discontinuous) paths which are right continuous with left limits.
Bruno Dupire (2009) formalized this notion and defines, for a functional $F : [0, T] \times D([0, T], \mathbb{R}) \mapsto \mathbb{R}$ defined on cadlag paths,

$$
\nabla_\omega F_t(\omega) = \lim_{\epsilon \to 0} \frac{F_t(\omega + \epsilon 1_{[t,T]}) - F_t(\omega)}{\epsilon}
$$

Dupire argues that this is the correct hedge ratio for path-dependent options: if the option price $F$ is twice differentiable in the functional sense and $F, \nabla_\omega F, \nabla^2_\omega F$ are continuous in supremum norm, then

$$
F_T = E[F_T] + \int_0^T \nabla_\omega F_t \cdot dS_t
$$
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Summary

Dupire’s arguments apply to integral functionals
\( F_t(\omega) = \int_0^t g(\omega(t)) \, dt \) but not to stochastic integrals or functionals involving quadratic variation.
We show that these ideas can be extended, in a mathematically rigorous fashion, to all square integrable functionals. To do so, we develop a non-anticipative pathwise calculus for such functionals defined on cadlag paths.
This leads to a non-anticipative calculus for path-dependent functionals of a semimartingale, which is (in a precise sense) a “non-anticipative” version of the Malliavin calculus.
As an application, we obtain a martingale representation formula for square-integrable martingales and a universal hedging formula for path-dependent contingent claims.
Consider a $\mathbb{R}^d$-valued \textit{cadlag} semimartingale $X$ on a filtered probability space $(\Omega, \mathcal{B}, \mathcal{B}_t, \mathbb{P})$.

Denote $\mathcal{F}_t = \sigma(\mathcal{F}_t^X \cup \mathcal{N}(\mathbb{P}))$ the ($\mathbb{P}$-completed) natural filtration of $X$, assumed right-continuous: $\mathcal{F}_t = \mathcal{F}_{t+}$

$D([0, T], \mathbb{R}^d)$ space of cadlag (right continuous with left limits) functions.

$C_0([0, T], \mathbb{R}^d)$ space of continuous paths.

Example: any cadlag Feller process verifies these assumptions.
Functional notation

For a path \( \omega \in D([0, T], \mathbb{R}^d) \), denote by

- \( \omega(t) \in \mathbb{R}^d \) the value of \( \omega \) at \( t \)
- \( \omega_t = \omega|_{[0,t]} = (\omega(u), 0 \leq u \leq t) \in D([0, t], \mathbb{R}^d) \) the restriction of \( \omega \) to \([0, t] \).

We will also denote \( \omega_{t-} \) the function on \([0, t] \) given by

\[
\omega_{t-}(u) = \omega(u) \quad u < t \quad \omega_{t-}(t) = \omega(t-)
\]

For a process \( X \) we shall similarly denote

- \( X(t) \) its value and
- \( X_t = (X(u), 0 \leq u \leq t) \) its path on \([0, t] \).
Outline

1. Functional representation of stochastic processes.
2. Pathwise functional calculus for non-anticipative flows.
3. Ito formula for functionals of semimartingales.
5. Numerical computation of martingale representations
(Doob 1956): A process $Y$ adapted to $\mathcal{F}_t$ may be represented as a family of functionals

$$Y(t, .) : \Omega = D([0, t], \mathbb{R}^d) \mapsto \mathbb{R}$$

with the property that $Y(t, .)$ only depends on the path stopped at $t$: $Y(t, \omega) = Y(t, \omega(\cdot \wedge t))$ so

$$\omega|_{[0, t]} = \omega'|_{[0, t]} \Rightarrow Y(t, \omega) = Y(t, \omega')$$

So there exists a $\mathcal{F}_t$-measurable map $F_t : D([0, t], \mathbb{R}^d) \to \mathbb{R}$ such that

$$Y(t, \omega) = F_t(\omega_t) \quad \mathbb{P} - a.s. \quad \text{where} \quad \omega_t = \omega|_{[0, t]}$$
This motivates the following definition:

**Definition (Non-anticipative functional)**

A *non-anticipative flow* on \((D([0, T], \mathbb{R}^d), (\mathcal{F}_t)_{t \geq 0})\) is a family \(F = (F_t)_{t \in [0, T]}\) where

\[ F_t : D([0, t], \mathbb{R}^d) \rightarrow \mathbb{R} \]

is \(\mathcal{F}_t\)-measurable.

\(F = (F_t)_{t \in [0, T]}\) may be viewed as a functional on the vector bundle \(\bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d)\).
Any process $Y$ adapted to $\mathcal{F}_t = \mathcal{F}_t^X$ may be represented as

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}) = F_t(X_t)$$

where the functional $F_t : D([0, t], \mathbb{R}^d) \to \mathbb{R}$ represents the dependence of $Y(t)$ on the path of $X$ on $[0, t]$. $F = (F_t)_{t \in [0, T]}$ may then be viewed as a functional on the vector bundle $\gamma([0, T]) = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d)$. 
Horizontal extension of a path
Horizontal extension of a path

Let \( \omega \in D([0, T] \times \mathbb{R}^d) \), \( \omega_t \in D([0, t] \times \mathbb{R}^d) \) its restriction to \([0, t]\). For \( h \geq 0 \), the \textit{horizontal} extension \( \omega_{t,h} \in D([0, t+h], \mathbb{R}^d) \) of \( \omega_t \) to \([0, t+h]\) is defined as

\[
\omega_{t,h}(u) = \omega(u) \quad u \in [0, t] ; \quad \omega_{t,h}(u) = \omega(t) \quad u \in ]t, t+h] 
\]
A distance between paths on different time intervals

For \( T \geq t' = t + h \geq t \geq 0, \omega \in D([0, t], \mathbb{R}^d) \) and \( \omega' \in D([0, t + h], \mathbb{R}^d) \)

\[
d_{\infty}(\omega, \omega') = \sup_{u \in [0, t+h]} |\omega_{t,h}(u) - \omega'(u)| + h
\]

defines a metric on \( \Upsilon([0, T]) = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d) \).

On \( D([0, t], \mathbb{R}^d) \), \( d_{\infty} \) coincides with the supremum norm.
Continuity for non-anticipative flows

A non-anticipative flow $F = (F_t)_{t \in [0, T]}$ is said to be continuous at fixed times if for all $t \in [0, T]$, the functional

$$F_t : D([0, t], \mathbb{R}^d) \rightarrow \mathbb{R}$$

is continuous w.r.t. the supremum norm.

**Definition (Left-continuous flows)**

Define $\mathbb{C}^{0,0}_l$ as the set of non-anticipative flows $F = (F_t, t \in [0, T])$ which are continuous at fixed times and

$$\forall t \in [0, T], \quad \forall \epsilon > 0, \forall \omega \in D([0, t], \mathbb{R}^d),$$

$$\exists \eta > 0, \forall h \in [0, t], \quad \forall \omega' \in D([0, t-h], \mathbb{R}^d),$$

$$d_{\infty}(\omega, \omega') < \eta \Rightarrow |F_t(\omega) - F_{t-h}(\omega')| < \epsilon$$
Boundedness-preserving flows

We call a flow “boundedness preserving” if it is bounded on each bounded set of paths:

**Definition (Boundedness-preserving flows)**

Define \( \mathcal{B}([0, T]) \) as the set of non-anticipative flows \( F \) on \( \Upsilon([0, T]) \) such that for every compact subset \( K \) of \( \mathbb{R}^d \), every \( R > 0 \) and \( t_0 < T \)

\[
\exists C_{K,R,t_0} > 0, \quad \forall t \leq t_0, \quad \forall \omega \in D([0, t], K), \quad \sup_{s \in [0,t]} |v(s)| \leq R \Rightarrow |F_t(\omega)| \leq C_{K,R,t_0}
\]
Measurability and continuity

A non-anticipative flow $F = (F_t)$ applied to $X$ generates an $\mathcal{F}_t$–adapted process

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}) = F_t(X_t)$$

**Theorem**

Let $\omega \in D([0, T], \mathbb{R}^d)$. If $F \in \mathbb{C}^{0,0}$,

- the path $t \mapsto F_t(\omega_{t-})$ is left-continuous.
- $Y(t) = F_t(X_t)$ defines a predictable process.
Definition (Horizontal derivative)

We say that a non-anticipative flow \( F = (F_t)_{t \in [0,T]} \) is horizontally differentiable at \( \omega \in D([0, t], \mathbb{R}^d) \) if

\[
\mathcal{D}_t F (\omega) = \lim_{{h \to 0^+}} \frac{F_{t+h}(\omega_t, h) - F_t(\omega)}{h}
\]

exists

We will call \( \mathcal{D}_t F (\omega) \) the horizontal derivative \( \mathcal{D}_t F \) of \( F \) at \( \omega \).

By right continuity of \( (\mathcal{F}_t) \), \( \mathcal{D}_t F \) is \( \mathcal{F}_t \)-measurable:

\( \mathcal{D} F = (\mathcal{D}_t F)_{t \in [0,T]} \) defines a non-anticipative flow.

If \( F_t(\omega) = f(t, \omega(t)) \) with \( f \in C^{1,1}([0, T] \times \mathbb{R}^d) \) then

\( \mathcal{D}_t F (\omega) = \partial_t f(t, \omega(t)) \).
Vertical perturbation of a path

**Figure:** For $e \in \mathbb{R}^d$, the **vertical** perturbation $\omega^e_t$ of $\omega_t$ is the cadlag path obtained by shifting the endpoint:

$$\omega^e_t(u) = \omega(u) \text{ for } u < t \text{ and } \omega^e_t(t) = \omega(t) + e.$$
Definition

A non-anticipative flow $F = (F_t)_{t \in [0,T]}$ is said to be *vertically differentiable* at $\omega \in D([0, t]), \mathbb{R}^d$ if

$$
\mathbb{R}^d \rightarrow \mathbb{R}
$$

$$
e \rightarrow F_t(\omega_t + e1_t)
$$

is differentiable at 0. Its gradient at 0 is called the *vertical derivative* of $F_t$ at $\omega$:

$$
\nabla_\omega F_t (\omega) = (\partial_i F_t(\omega), i = 1..d)
$$

where

$$
\partial_i F_t(\omega) = \lim_{h \rightarrow 0} \frac{F_t(\omega_t^h e_i) - F_t(\omega)}{h}
$$
Vertical derivative of a flow

- $\nabla_\omega F_t (\omega).e$ is simply a *directional* derivative in the direction of the indicator function $1_{\{t\}}e$.

- Note that to compute $\nabla_\omega F_t (\omega)$ we need to compute $F$ outside $C_0$: even if $\omega \in C_0$, $\omega^h_t \notin C_0$.

- $\nabla_\omega F_t (\omega)$ is 'local' in the sense that it is computed for $t$ fixed and involves perturbing the endpoint of paths ending at $t$.

- $\nabla_\omega F = (\nabla_\omega F_t)_{t\in[0,T]}$ is a non-anticipative flow.

This derivative was introduce by B. Dupire to compute the sensitivity of an option price to its underlying.
Spaces of differentiable flows

Definition (Spaces of differentiable functionals)

For $j, k \geq 1$ define $\mathbb{C}^{j,k}_b([0, T])$ as the set of functionals $F \in \mathbb{C}^0,0$ which are differentiable $j$ times horizontally and $k$ times vertically at all $\omega \in D([0, t], \mathbb{R}^d)$, $t < T$, with

- horizontal derivatives $D^m_t F$, $m \leq j$ continuous on $D([0, T])$ for each $t \in [0, T]$;
- left-continuous vertical derivatives: $\forall n \leq k, \nabla^n \omega F \in \mathbb{C}^0,0$.

$D^m_t F, \nabla^n \omega F \in \mathbb{B}([0, T])$.

We can have $F \in \mathbb{C}^{1,1}_b([0, T])$ while $F_t$ not Fréchet differentiable for any $t \in [0, T]$. 
Example (Cylindrical functionals)

For $g \in C^0(\mathbb{R}^d)$, $h \in C^k(\mathbb{R}^d)$ with $h(0) = 0$. Then

$$F_t(\omega) = h(\omega(t) - \omega(t_n-)) \quad 1_{t \geq t_n} \quad g(\omega(t_1-), \omega(t_2-), \ldots, \omega(t_n-))$$

is in $\mathbb{C}^{1,k}_b$ and

$$\mathcal{D}_t F(\omega) = 0, \quad \text{and} \quad \forall j = 1..k,$$

$$\nabla_\omega^j F_t(\omega) = h^{(j)}(\omega(t) - \omega(t_n-)) \quad 1_{t \geq t_n} g(\omega(t_1-), \omega(t_2-), \ldots, \omega(t_n-))$$
Examples of regular functionals

Example (Integral functionals)

For $g \in C_0(\mathbb{R}^d)$, $Y(t) = \int_0^t g(X(u))\rho(u)du = F_t(X_t)$ where

$$F_t(\omega) = \int_0^t g(\omega(u))\rho(u)du$$

$F \in \mathbb{C}^{1,\infty}_b$, with:

$$D_t F(\omega) = g(\omega(t))\rho(t) \quad \nabla^j_\omega F_t(\omega) = 0$$
Obstructions to regularity

Example (Jump of $x$ at the current time)

$$F_t(\omega) = \omega(t) - \omega(t-)$$ has regular pathwise derivatives:

$$\mathcal{D}_t F(\omega) = 0 \quad \nabla_{\omega} F_t(\omega) = 1$$

But $F \notin \mathbb{C}^{0,0}_r \cup \mathbb{C}_l^{0,0}$.

Example (Jump of $x$ at a fixed time)

$$F_t(\omega) = 1_{t \geq t_0}(\omega(t_0) - \omega(t_0-))$$

$F \in \mathbb{C}^{0,0}$ has horizontal and vertical derivatives at any order, but

$$\nabla_{\omega} F_t(\omega) = 1_{t = t_0}$$ fails to be left (or right) continuous.
Obstructions to regularity

Example (Maximum)

\[ F_t(\omega) = \sup_{s \leq t} \omega(s) \]

\[ F \in C^{0,0} \] but is not vertically differentiable on

\[ \{ \omega \in D([0, t], \mathbb{R}^d), \quad \omega(t) = \sup_{s \leq t} \omega(s) \}. \]
Given a process $Y$ (say, the value of a path dependent option)

If $F^1, F^2 \in \mathbb{C}^{1,1}$ coincide on continuous paths

$$\forall t < T, \quad \forall \omega \in C_0([0, t], \mathbb{R}^d),$$

$$F^1_t(\omega) = F^2_t(\omega)$$

then

$$\mathbb{P}(\forall t \in [0, T], F^1(X_t) = F^2(X_t)) = 1$$

Yet, $\nabla_\omega F$ depends on the values of $F$ computed at discontinuous paths...
Derivatives of flows defined on continuous paths

**Theorem**

If $F^1, F^2 \in \mathbb{C}^{1,1}$ coincide on continuous paths

$$\forall t < T, \ \forall \omega \in C_0([0, t], \mathbb{R}^d), F^1_t(\omega_t) = F^2_t(\omega_t)$$

then their pathwise derivatives also coincide:

$$\forall t < T, \ \forall \omega \in C_0([0, t], \mathbb{R}^d), \nabla \omega F^1_t(\omega_t) = \nabla \omega F^2_t(\omega_t), \quad D_t F^1_t(\omega_t) = D_t F^2_t(\omega_t)$$
Paths of finite quadratic variation
A pathwise change of variable formula for functionals
Ito-Föllmer integral

Quadratic variation for cadlag paths [Föllmer (1979)]

\( x \in D([0, T], \mathbb{R}) \) has finite quadratic variation along the sequence of partitions \( \pi_n = (0 = t_0^n < .. t_k^{n(n)} = T) \) if the discrete measures

\[
\xi^n = \sum_{i=0}^{k(n)-1} (x(t^n_{i+1}) - x(t^n_i))^2 \delta_{t^n_i}
\]

converge weakly to some Radon measure \( \xi \) on \([0, T] \):

\[
\forall f \in C^0_b(\mathbb{R}), \quad \sum_{i=0}^{k(n)-1} (x(t^n_{i+1}) - x(t^n_i))^2 f(t^n_i) \xrightarrow{n \to \infty} \int_0^T f \ d\xi
\]

In particular

\[
\sum_{i=0}^{k(n)-1} (x(t^n_{i+1}) - x(t^n_i))^2 \xrightarrow{n \to \infty} [x](t) := \xi([0, t])
\]
Denote $Q([0, T], \pi)$ the set of $x \in D([0, T], \mathbb{R}^d)$ with finite quadratic variation with respect to the partition $\pi = (\pi_n)_{n \geq 1}$. For $x \in Q([0, T], \pi)$, the increasing function

$$
\sum_{i=0}^{k(n)-1} (x(t_{i+1}^n) - x(t_i^n))^2 \xrightarrow{n \to \infty} [x](t) := \xi([0, t])
$$

is called the quadratic variation of $x$ along the partition $(\pi_n)$ and has Lebesgue decomposition

$$
[x](t) = [x]^c(t) + \sum_{0 < s \leq t} (\Delta x(s))^2
$$

where $[x]^c$ is the continuous part of $[x]$. 
Similarly \( x = (x^1, \ldots, x^d) \in D([0, T], \mathbb{R}^d) \) is said to have finite quadratic variation along \( \pi_n = (0 = t^n_0 < \ldots t^n_k(n) = T) \) if \( x^i, x^i + x^j \) have finite quadratic variation along \( \pi_n \) for \( i, j = 1 \ldots d \). Then

\[
\sum_{i=0}^{k(n)-1} t^n_i (x(t^n_{i+1}) - x(t^n_i))(x(t^n_{i+1}) - x(t^n_i)) \xrightarrow{n \to \infty} [x](t) \in Sym^+(d \times d)
\]

is an increasing function with values in \( Sym^+(d \times d) \) and

\[
[x]_{ij} = ([x^i + x^j] - [x^i] - [x^j])/2.
\]
Processes with finite quadratic variation

For a stochastic process $Z : [0, T] \times \Omega \mapsto \mathbb{R}^d$, define

$$\Omega_{\pi}(Z) = \{\omega \in \Omega, \ Z(., \omega) \in Q([0, T], \pi)\}.$$ 

- Wiener process: if $W$ is a Wiener process under $\mathbb{P}$, then for any sequence of partitions $\pi$ with $|\pi_n| \to 0$, the paths of $W$ lie in $Q([0, T], \pi)$ with probability 1:

$$\mathbb{P}(\Omega_{\pi}(W)) = 1 \quad \text{and} \quad \forall \omega \in \Omega_{\pi}(W), [W(., \omega)](t) = t.$$
Processes with finite quadratic variation

- Fractional Brownian motion: if $W^H$ is fractional Brownian motion with Hurst index $H \in (0.5, 1)$ then for any sequence of partitions $\pi$ with $|\pi_n| \to 0$, the paths of $W^H$ lie in $Q([0, T], \pi)$ with probability 1:
  \[
  \mathbb{P}(\Omega_\pi(W^H)) = 1 \quad \text{and} \quad \forall \omega \in \Omega_\pi(W^H), [W(., \omega)](t) = 0.
  \]
  (See e.g. Zähle 1998)

- Lévy process: if $L$ is a Lévy process with triplet $(b, A, \nu)$ then for any sequence of partitions $\pi$ with $|\pi_n| \to 0$, $\mathbb{P}(\Omega_\pi(L)) = 1$ and
  \[
  \forall \omega \in \Omega_\pi(L), [L(., \omega)](t) = tA + \sum_{s \in [0, t]} |X(s, \omega) - X(s-, \omega)|^2.
  \]
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Processes with finite quadratic variation

**Theorem (Föllmer (1981))**

Let $S$ be a semimartingale on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$, $T > 0$. There exists a sequence of (random) partitions $\pi = (\pi_n)_{n \geq 1}$ of $[0, T]$, composed of $\mathbb{F}$-stopping times, with $|\pi_n| \to 0$ a.s., such that the paths of $S$ lie in $Q([0, T], \pi)$ with probability 1:

$$\mathbb{P}(\{\omega \in \Omega, \ S(., \omega) \in Q([0, T], \pi(\omega))\}) = 1.$$  

Proof: take the dyadic partition $t^n_k = kT/2^n$, $k = 0..2^n$. Since $\sum (S(t^n_{k+1}) - S(t^n_k))^2 \to [S]_T$ in probability, there exists a subsequence $\pi_n$ such that $\sum \pi_n (S(t_i) - S(t_{i+1}))^2 \to [S]_T$ a.s.
Change of variable for functionals of cadlag paths

Let \( \omega \in D([0, T] \times \mathbb{R}^d) \) where \( \omega \) has finite quadratic variation along \((\pi_n)\) and

\[
\sup_{t \in [0, T] - \pi_n} |\omega(t) - \omega(t-)| \to 0
\]

Denote \( h^n_i = t^n_{i+1} - t^n_i \),

\[
\omega^n(t) = \sum_{i=0}^{k(n)-1} \omega(t_{i+1}-)1_{[t_i, t_{i+1})}(t) + \omega(T)1_{\{T\}}(t)
\]

Consider the ’Riemann sum’:

\[
\sum_{i=0}^{k(n)-1} \nabla_{\omega} F^n_{t_i}(\omega^n_{t_i^-}, \delta\omega(t^n_i))(\omega(t^n_{i+1}) - \omega(t^n_i)) \quad \delta\omega(t^n_i) = \omega(t^n_{i+1}) - \omega(t^n_i)
\]
A pathwise change of variable formula for functionals

**Theorem (R.C. & D Fournié (Jour. Func. Analysis 2010))**

For any $F \in \mathbb{C}^{1,2}_b([0, T])$, $\omega \in Q([0, T], \pi)$ the Föllmer integral

$$
\int_0^T \nabla_\omega F_t(\omega_{t-}) d^\pi \omega := \lim_{n \to \infty} \sum_{i=0}^{k(n)-1} \nabla_\omega F^n_t(\omega^n_{t_i-}) (\omega(t^n_{i+1}) - \omega(t^n_i))
$$

exists and $F_T(\omega_T) - F_0(\omega_0) = \int_0^T D_tF_t(\omega_{t-}) du$

$$+
\int_0^T \frac{1}{2} \text{tr} (t \nabla^2_\omega F_t(\omega_{t-}) d[\omega]^c(u)) + \int_0^T \nabla_\omega F_t(\omega_{t-}) d^\pi \omega
$$

$$+
\sum_{u \in ]0, T]} [F_u(\omega_u) - F_u(\omega_{u-}) - \nabla_\omega F_u(\omega_{u-}) \cdot \Delta \omega(u)]$$
Sketch of proof: continuous case

\( \omega \in C_0([0, T]) \) is (uniformly) continuous on \([0, T]\) so

\[
\eta_n = \sup_{0 \leq i \leq k(n) - 1, u \in [t^n_i, t^n_{i+1}]} \{|\omega(u) - \omega(t^n_i)| + |t^n_{i+1} - t^n_i|\} \to 0
\]

Consider the decomposition:

\[
F_{t^n_{i+1}}(\omega^n_{t^n_{i+1}^-}) - F_{t^n_i}(\omega^n_{t^n_i^-}) = F_{t^n_{i+1}}(\omega^n_{t^n_{i+1}^-}) - F_{t^n_i}(\omega^n_{t^n_i})
+ F_{t^n_i}(\omega^n_{t^n_i}) - F_{t^n_i}(\omega^n_{t^n_i^-})
\]

First term = \( \psi(h^n_i) - \psi(0) \) where \( \psi(u) = F_{t^n_i + u}(\omega^n_{t^n_i}, u) \).

Since \( F \in C^{1,2}([0, T]) \), \( \psi \) is right-differentiable so

\[
F_{t^n_{i+1}}(\omega^n_{t^n_i}, h^n_i) - F_{t^n_i}(\omega^n_{t^n_i}) = \int_0^{t^n_{i+1} - t^n_i} D_{t^n_i + u} F(\omega^n_{t^n_i}, u) du
\]
Sketch of proof: continuous case

The second term $= \phi(\delta \omega_i^n) - \phi(0)$, where: $\phi(u) = F_{t_i^n}(\omega_{t_i^n-}^n,u)$. Since $F \in \mathbb{C}^{1,2}([0, T])$, $\phi$ is $C^2$ and:

$$\phi'(u) = \nabla \omega F_{t_i^n}(\omega_{t_i^n-}^n,u) \quad \phi''(u) = \nabla^2 \omega F_{t_i^n}(\omega_{t_i^n-}^n,u)$$

Second order Taylor expansion of $\phi$ at $u = 0$:

$$F_{t_i^n}(\omega_{t_i^n}^n) - F_{t_i^n}(\omega_{t_i^n-}^n) = \nabla \omega F_{t_i^n}(\omega_{t_i^n-}^n) \delta \omega_i^n$$

$$+ \frac{1}{2} \text{tr} \left( \nabla^2 \omega F_{t_i^n}(\omega_{t_i^n-}^n) \left( \delta \omega_i^n \delta \omega_i^n \right) \right) + r_i^n$$

where $\delta \omega_i^n = \omega(t_{i+1}^n) - \omega(t_i^n)$ and

$$r_i^n \leq K \left| \delta \omega_i^n \right|^2 \sup_{t \in B(\omega(t_i^n), \eta_n)} \left| \nabla^2 \omega F_{t_i^n}(\omega_{t_i^n-}^n \omega(t) - \omega(t_i^n)) - \nabla^2 \omega F_{t_i^n}(\omega_{t_i^n-}^n) \right|$$
So finally we have the decomposition

\[ F_{t_{i+1}}^n(\omega_{t_{i+1}^-}^n) - F_{t_i}^n(\omega_{t_i^-}^n) = \int_{t_i}^{t_{i+1}} D_u F(\omega_{t_i+u-t_i}^n) \, du + \nabla \omega F_{t_i}^n(\omega_{t_i^-}^n) \delta \omega_i^n \]

\[ + \frac{1}{2} \text{tr} \left( \nabla^2 \omega F_{t_i}^n(\omega_{t_i^-}^n) \delta \omega_i^n \delta \omega_i^n \right) + r_i^n \]

As \( n \to \infty \):

\[ \int_{t_i}^{t_{i+1}} D_u F(\omega_{t_i+u-t_i}^n) \, du \to \int_0^T D_u F(\omega_u) \, du \] by dominated convergence.

\[ r_i^n \leq \epsilon_i^n \delta \omega_i^n \] where \( C \geq |\epsilon_i^n| \to 0 \) so \( \sum_i r_i^n \to 0 \).
A diagonal lemma

Lemma (R.C. and D Fournié, 2010)

Let \((\mu_n)_{n \geq 1}\) be a sequence of Radon measures on \([0, T]\) converging weakly to a Radon measure \(\mu\) with no atoms, and \((f_n)_{n \geq 1}, f\) be left-continuous functions on \([0, T]\) with

\[\forall t \in [0, T], \lim_{n} f_n(t) = f(t) \quad \forall t \in [0, T], \|f_n(t)\| \leq K\]

then

\[\int_{s}^{t} f_n d\mu_n \xrightarrow{n \to \infty} \int_{s}^{t} f d\mu\]

So the sum of third terms converges to

\[\int_{0}^{T} \frac{1}{2} \text{tr} \left( t \nabla_{\omega}^2 F_t(\omega_u).d[\omega](u) \right) = \int_{0}^{T} \frac{1}{2} \text{tr} \left( t \nabla_{\omega}^2 F_t(\omega_u).d[\omega](u) \right)\]
The Föllmer integral

Since all other terms converge, we conclude that the limit

$$\int_0^T \nabla_\omega F_t(\omega_{t-})d^\pi_\omega := \lim_{n \to \infty} \sum_{i=0}^{k(n)-1} \nabla_\omega F_{t_i}^{n_i}(\omega_{t_i}^{n_i},\Delta \omega(t_i^{n})) \omega(t_{i+1})^{n_i} - \omega(t_i^{n_i}))$$

exists pathwise.

Such integrals were defined in [Föllmer 1981] for integrands of the form $f'(X(t))$ where $f \in C^2(\mathbb{R}^d)$.

The construction depends a priori on the sequence $\pi$ of partitions. But, for semimartingales and, more generally, Dirichlet processes, one obtains a limit object which is a.s. independent of the choice of $\pi$. 
Ito formula for functionals of semimartingales

Applied to a semimartingale, these results lead to a functional extension of the Ito formula:

**Theorem (Functional Ito formula (R.C.& Fournié, 2009))**

Let $X$ be a **continuous** semimartingale and $F \in \mathbb{C}_b^{1,2}([0, T])$. For any $t \in [0, T]$, 

$$F_t(X_t) - F_0(X_0) = \int_0^t D_u F(X_u) du + \int_0^t \nabla \omega F_u(X_u).dX(u) + \int_0^t \frac{1}{2} \text{tr} \left( t \nabla^2 \omega F_u(X_u) d[X](u) \right) \text{ a.s.}$$

In particular, $Y(t) = F_t(X_t)$ is a semimartingale.
Let $X$ be a cadlag semimartingale and denote for $t > 0$,
\[
X_t(u) = X(u)1_{[0,t)} + X(t-)1_t.
\]

**Theorem (Functionals of cadlag semimartingales)**

For any $F \in C_b^{1,2}([0, T])$, $t \in [0, T]$, \hfill 

\[
F_t(X_t) - F_0(X_0) = \int_0^t \nabla \omega F_u(X_u).dX(u) + \\
\int_0^t \mathcal{D}_u F(X_u)du + \int_0^t \frac{1}{2} \text{tr} \left( t\nabla^2 \omega F_u(X_u) d[X](u) \right) + \\
\sum_{u \in [0,T]} [F_u(X_u) - F_u(X_u-) - \nabla \omega F_u(X_u-).\Delta X(u)] a.s.
\]

In particular, $Y(t) = F_t(X_t)$ is a semimartingale.

If \( F_t(X_t) = f(t, X(t)) \) where \( f \in C^{1,2}([0, T] \times \mathbb{R}^d) \) this reduces to the standard Ito formula.

For \( F \in \mathbb{C}^{1,2} \), \( Y(t) = F_t(X_t) \) can be reconstructed from the ‘second-order jet’ \((F, D F, \nabla \omega F, \nabla^2 \omega F)\) of \( F \) along the path of \( X \).

If \( X \) has continuous paths then \( Y = F(X) \) depends on \( F \) and its derivatives only via their values on continuous paths: \( Y \) can be reconstructed from the second-order jet of \( F \) on \( \Upsilon_c = \bigcup_{t \in [0, T]} C_0([0, t], \mathbb{R}^d) \subset \Upsilon \).
Sketch of proof

Consider first a cadlag, “simple predictable” process:

\[ X(t) = \sum_{k=1}^{n} 1_{[t_k, t_{k+1}]}(t) \phi_k \quad \phi_k \ F_{t_k} \ - \ \text{measurable} \]

Each path of \( X \) is a sequence of horizontal and vertical moves:

\[ X_{t_{k+1}} = (X_{t_k}, h_k)^{\phi_{k+1} - \phi_k} \quad h_k = t_{k+1} - t_k \]

\[ F_{t_{k+1}}(X_{t_{k+1}}) - F_{t_k}(X_{t_k}) = \]

\[ F_{t_{k+1}}(X_{t_{k+1}}) - F_{t_{k+1}}(X_{t_k}, h_k) + \quad \text{vertical move} \]

\[ F_{t_{k+1}}(X_{t_k}, h_k) - F_{t_k}(X_{t_k}) \quad \text{horizontal move} \]
Sketch of proof

Horizontal step: fundamental theorem of calculus for
\[ \phi(h) = F_{t_k+h}(X_{t_k},h) \]

\[ F_{t_{k+1}}(X_{t_k},h_k) - F_{t_k}(X_{t_k}) = \phi(h_k) - \phi(0) = \int_{t_k}^{t_{k+1}} \mathcal{D}_t F(X_t) dt \]

Vertical step: apply Ito formula to \( \psi(u) = F_{t_{k+1}}(X_{t_k},h_k) \)

\[ F_{t_{k+1}}(X_{t_{k+1}}) - F_{t_{k+1}}(X_{t_k},h_k) = \psi(X(t_{k+1}) - X(t_k)) - \psi(0) \]

\[ = \int_{t_k}^{t_{k+1}} \nabla \omega F_{t_{k+1}}(X_t).dX + \frac{1}{2} \text{tr}(\nabla^2 \omega F_{t_{k+1}}(X_t)d[X]) \]
Functional Ito formula: continuous case

Functional Ito formula: cadlag case

Functional Ito formula: proof

Functional of Dirichlet processes

Sketch of proof

General case: approximate $X$ by a sequence of simple predictable processes $nX$ with $nX(0) = X(0)$:

$$F_T(nX_T) - F_0(X_0) = \int_0^T D_tF(nX_t) dt + \int_0^T \nabla_\omega F(nX_t).dX$$

$$+ \frac{1}{2} \int_0^T \text{tr}[t\nabla_\omega^2 F(nX_t)] d[X]$$

The $C^{1,2}_b$ assumption on $F$ implies that all derivatives involved in the expression are left continuous in $d_\infty$ metric, which allows to control their convergence as $n \to \infty$ using dominated convergence theorem for stochastic integrals.
Similar formulas are obtained for a 'Dirichlet process' (=semimartingale + process with zero quadratic variation). Example:

**Theorem (Functional change of variable formula for fBm)**

Let $W^H$ be a fractional Brownian motion with $H \in (0.5, 1)$. Then for any $F \in \mathbb{C}^{1,2}(\mathbb{R})$, $t \in [0, T]$

$$F_t(W^H_t) - F_0(0) = \int_0^t D_u F(W^H_u) du + \int_0^t \nabla \omega F_u(W^H_u) dW^H(u)$$

where $\int_0^t \nabla \omega F_u(W^H_u) dW^H(u)$ is the pathwise integral

$$\int_0^T \nabla \omega F_t(\omega_{t-}) d\omega := \lim_{n \to \infty} \sum_{i=0}^{k(n)-1} \nabla \omega F_{t^n_i}(W^H_{t^n_i})(W^H(t_{i+1}^n) - W^H(t_i^n))$$
Definition (Vertical derivative of a process)

Define $\mathcal{C}^{1,2}_b(X)$ the set of processes $Y$ which admit a representation in $\mathcal{C}^{1,2}_b$:

$$\mathcal{C}^{1,2}_b(X) = \{ Y, \exists F \in \mathcal{C}^{1,2}_b([0, T]), \ Y(t) = F_t(X_t) \ \text{a.s.}\}$$

If $\det(A) > 0$ a.s. then for $Y \in \mathcal{C}^{1,2}_b(X)$, the predictable process:

$$\nabla_X Y(t) = \nabla_\omega F_t(X_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathcal{C}^{1,2}_b$. We call $\nabla_X Y$ the **vertical derivative** of $Y$ with respect to $X$. 
Vertical derivative for Brownian functionals

In particular when $X$ is a standard Brownian motion, $A = I_d$:

**Definition**

Let $W$ be a standard $d$-dimensional Brownian motion. For any $Y \in C^{1,2}_b(W)$ with representation $Y(t) = F_t(W_t, t)$, the predictable process

$$\nabla_W Y(t) = \nabla_\omega F_t(W_t, t)$$

is uniquely defined up to an evanescent set, independently of the choice of the representation $F \in C^{1,2}_b$. 

Rama Cont
Consider now the case where \( X(t) = \int_0^t \sigma \, dW \) where \( \sigma \) is \( \mathcal{F}_t \)-adapted with

\[
\det(\sigma(t)) > 0 \text{ a.s.,} \quad E \left( \int_0^T \sigma^2(t) \, dt \right) < \infty
\]

\( X \) is then a square integrable martingale, with the predictable representation property: for any \( \mathcal{F}_T \)-measurable variable \( H = H(X(t), t \in [0, T]) = H(X_T) \) with \( E[|H|^2] < \infty \) there exists a predictable process \( \phi \) with

\[
H = E[H] + \int_0^T \phi(t) \, dX(t)
\]

This theorem is not constructive: various methods have been proposed for computing \( \phi \) (Clark, Haussmann, Ocone, Jacod-Meleard-Protter, Picard, Fitzsimmons) using various assumptions on \( H \).
Theorem

Consider an $\mathcal{F}_T$-measurable functional $H = H(X(t), t \in [0, T]) = H(X_T)$ with $E[|H|^2] < \infty$ and define the martingale $Y(t) = E[H|\mathcal{F}_t]$. If $Y \in C^{1,2}_b(X)$ then

$$Y(T) = E[Y(T)] + \int_0^T \nabla_X Y(t) dX(t)$$

$$= E[H] + \int_0^T \nabla_X Y(t) \sigma(t) dW(t)$$

This is a non-anticipative version of Clark’s formula.
Consider now a (discounted) asset price process
\[ S(t) = \int_0^t \sigma(t) \, dW(t) \] with \( \det(\sigma(t)) > 0 \) a.s and
\[ E^Q \left( \int_0^T \sigma^2(t) \, dt \right) < \infty. \]
Let \( H = H(S(t), t \in [0, T]) \) with \( E[|H|^2] < \infty \) be a path-dependent payoff. The price at date \( t \) is then \( Y(t) = E[H|\mathcal{F}_t]. \)

**Theorem (Hedging formula)**

If \( Y \in C_b^{1,2}(S) \) then
\[
H = E^Q[H] + \int_0^T \nabla_S Y(t) dS(t) \quad Q - a.s.
\]

The hedging strategy for \( H \) is given by the vertical derivative of the option price with respect to \( S \).
So the hedging strategy for $H$ may be computed \textbf{pathwise} as

$$\phi(t) = \nabla_X Y(t, X_t(\omega)) = \lim_{h \to 0} \frac{Y(t, X_t^h(\omega)) - Y(t, X_t(\omega))}{h}$$

where

- $Y(t, X_t(\omega))$ is the option price at date $t$ in the scenario $\omega$.
- $Y(t, X_t^h(\omega))$ is the option price at date $t$ in the scenario obtained from $\omega$ by moving up the current price ("bumping" the price) by $h$.

So, the usual "bump and recompute" sensitivity actually gives.. the hedge ratio!
Pathwise computation of hedge ratios

Consider for example the case where $X$ is a (component of a ) multivariate diffusion. Then we can use a numerical scheme (ex: Euler scheme) to simulate $X$.

Let $nX$ be the solution of a $n$-step Euler scheme and $\hat{Y}_n$ a Monte Carlo estimator of $Y$ obtained using $nX$.

- Compute the Monte Carlo estimator $\hat{Y}_n(t, nX_t(\omega))$
- Bump the endpoint by $h$.
- Recompute the Monte Carlo estimator $\hat{Y}_n(t, nX^h_t(\omega))$ (with the same simulated paths)
- Approximate the hedging strategy by

$$\hat{\phi}_n(t, \omega) := \frac{\hat{Y}_n(t, nX^h_t(\omega)) - \hat{Y}_n(t, nX_t(\omega))}{h}$$
Numerical simulation of hedge ratios

\[ \hat{\phi}_n(t, \omega) \sim \frac{\hat{Y}_n(t, nX^h_t(\omega)) - \hat{Y}_n(t, nX_t(\omega))}{h} \]

For a general \( C^{1,2}_b(S) \) path-dependent claim, with a few regularity assumptions

\[ \forall 1/2 > \epsilon > 0, \; n^{1/2 - \epsilon} |\hat{\phi}_n(t) - \phi(t)| \to 0 \quad \mathbb{P} - a.s. \]

This rate is attained for \( h = cn^{-1/4 + \epsilon/2} \)

By exploiting the structure further (Asian options, lookback options, ...) one can greatly improve this rate.
A non-anticipative integration by parts formula

\[ \mathcal{I}^2(X) = \{ \int_0^T \phi \, dX \mid \phi \text{ is } \mathcal{F}_t \text{-adapted}, \ E[\int_0^T \| \phi(t) \|^2 \, d[X](t)] < \infty \} \]

**Theorem**

Let \( Y \in C^{1,2}_b(X) \) be a \((\mathbb{P}, (\mathcal{F}_t))\)-martingale with \( Y(0) = 0 \) and \( \phi \) an \( \mathcal{F}_t \)-adapted process with \( E[\int_0^T \| \phi(t) \|^2 \, d[X](t)] < \infty \). Then

\[
E \left( Y(T) \int_0^T \phi \, dX \right) = E \left( \int_0^T \nabla_X Y \cdot \phi \, d[X] \right)
\]

This allows to extend the functional Ito formula to the closure of \( C^{1,2}_b(X) \cap \mathcal{I}^2(X) \) wrt to the norm

\[
E \| Y(T) \|^2 = E \left[ \int_0^T \| \nabla_X Y(t) \|^2 \, d[X](t) \right]
\]
Motivation: pricing and hedging of derivatives
Pathwise calculus for non-anticipative flows
Functional change of variable formulas
Functional Ito calculus
Martingale representation and hedging formulas

Extensions
Functional equations for martingales

Regular functionals are dense among square-integrable functionals

\[ \mathcal{L}^2(X) = \{ \phi \mathcal{F}_t - \text{adapted}, \ E[\int_0^T \| \phi(t) \|^2 d[X](t)] < \infty \} \]
\[ \mathcal{I}^2(X) = \{ \int_0^T \phi dX, \ \phi \in \mathcal{L}^2(X) \} \]

Lemma

Let \( D(X) = \mathcal{C}^{1,2}_b(X) \cap \mathcal{I}^2(X) \) be the set of processes which are regular functionals of \( X \). \( \{ \nabla_X Y, \ Y \in D(X) \} \) is dense in \( \mathcal{L}^2(X) \) and the closure in \( \mathcal{I}^2(X) \) of \( D(X) \) is the set of all square-integrable stochastic integrals with respect to \( X \):

\[ \mathcal{W}^{1,2}(X) = \{ \int_0^T \phi dX, \ E \int_0^T \| \phi \|^2 d[X] < \infty \}. \]
Theorem (Weak derivative on $\mathcal{W}^{1,2}(X)$)

The vertical derivative $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$. Its closure defines a bijective isometry

$$\nabla_X : \mathcal{W}^{1,2}(X) \mapsto \mathcal{L}^2(X)$$

$$\int_0^T \phi . dX \mapsto \phi$$

characterized by the following integration by parts formula: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y$ is the unique element of $\mathcal{L}^2(X)$ such that

$$\forall Z \in D(X), \quad E[Y(T)Z(T)] = E \left[ \int_0^T \nabla_X Y(t) \nabla_X Z(t) d[X](t) \right].$$

In particular, $\nabla_X$ is the adjoint of the Ito stochastic integral.
A general martingale representation formula

\[ X(t) = \int_0^t \sigma_.dW_. \sigma_. \mathcal{F}_t \text{-adapted, det}(\sigma(t)) > 0 \text{ a.s,} \]
\[ E(\int_0^T \sigma^2(t)dt) < \infty. \]

**Theorem**

For any square-integrable \( \mathcal{F}_t \)-martingale \( Y \),

(i) \( \exists F^n \in C_{b}^{1,2} \), \( E\|F^n_T(X_T) - Y(T)\|^2 \to n \to \infty 0. \)

(ii) The vertical derivative \( \nabla_X Y \) of \( Y \) with respect to \( X \),

\[ \nabla_X Y(t) := \lim_{n \to \infty} \nabla_\omega F^n_t(X_t) \]

is uniquely determined outside an evanescent set, independently of the approximating sequence \( (F^n)_{n \geq 1} \).

(iii) \( Y(T) = E[H] + \int_0^T \nabla_X Y(t)dX(t) \) \( \mathbb{P} - \text{a.s.} \)
General hedging formula for path-dependent options

This allows to extend the hedging formula to all square integrable claims:

**Theorem (Hedging formula)**

Let $H = H(S(t), t \in [0, T])$ with $E[|H|^2] < \infty$ be a ($\mathcal{F}_T$-measurable) path-dependent payoff. Define price at date $t$ by $Y(t) = E^Q[H | \mathcal{F}_t]$. Then

$$H = E^Q[H] + \int_0^T \nabla S Y(t) dS(t) \quad Q-a.s.$$  

The hedging strategy for $H$ is given by the (weak) vertical derivative of the option price with respect to $S$. 

Rama Cont

Functional Ito calculus
Computation of the weak derivative

For $D(X) = C^1_{b,2}(X) \cap \mathcal{I}^2(X)$, the weak derivative may be computed \textbf{pathwise}: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y = \lim_n \nabla_X Y^n$ where $Y^n \in D(X)$ is an approximating sequence with

$$E\|Y^n(T) - Y(T)\|^2 \overset{n \to \infty}{\to} 0$$

An example of $\hat{Y}^n$ is the Euler scheme for $X$.

$$\nabla_X Y(t, X_t(\omega)) = \lim_{n \to \infty} \lim_{h \to 0} \frac{\hat{Y}^n(t, X_t^h(\omega)) - \hat{Y}^n(t, X_t(\omega))}{h}$$

In practice one may compute the estimator

$$\nabla_X Y(t, X_t(\omega)) = \frac{\hat{Y}^n(t, X_t^h(n)(\omega)) - \hat{Y}^n(t, X_t(\omega))}{h(n)}$$

where $h(n) \sim cn^{-\alpha}$ is chosen according to the "smoothness" of $Y$. 

Rama Cont
Malliavin derivative

Consider the case where $X = \mathcal{W}$. For a functional $F : C_0([0, T], \mathbb{R}^d) \to \mathbb{R}$ which possess a derivative $D_\mathcal{H}F$ in the direction of absolutely continuous functions

$$\forall h \in \mathcal{H}^1([0, T], \mathbb{R}^d), < D_\mathcal{H}F(\omega), h >_{L^2} = \lim_{\epsilon \to 0} \frac{F(\omega + \epsilon h) - F(\omega)}{\epsilon}$$

where $\mathcal{H}^1([0, T], \mathbb{R}^d) = \{ f \in L^2([0, T], \mathbb{R}^d), f(0) = 0, \nabla f \in L^2([0, T], \mathbb{R}^d) \}$, one can define the Malliavin derivative $\mathbb{D}F$ of $F$ as

$$\mathbb{D}_t F(\omega) = \frac{\partial}{\partial t} D_\mathcal{H}F(\omega)$$

$\mathbb{D}F$ may be defined for $F \in D^{1,2}([0, T]) = \{ G, D_\mathcal{H}G \in \mathcal{H}^1([0, T], \mathbb{R}^d) \}$.

In general $\mathbb{D}_t F$ is an anticipative functional: $\mathbb{D}F \in L^2([0, T] \times \Omega)$.
Clark-Haussman-Ocone formula

Note that the Malliavin derivative maps a $\mathcal{F}_T$-measurable functional into an (anticipative) process

$$\mathbb{D} : \mathbb{D}^{1,2}([0, T]) \mapsto L^2([0, T] \times \Omega)$$

$$F \mapsto (\mathbb{D}_t F)_{t \in [0, T]}$$

An important by-product of Malliavin calculus is the Clark-Haussmann-Ocone formula: for $F \in \mathbb{D}^{1,2}([0, T])$,

$$F = E[F] + \int_0^T PE[\mathbb{D}_t F | \mathcal{F}_t] dW(t)$$

See e.g. Watanabe (1987) Nualart (2009)
Consider the case where $X = W$. Then for $Y \in \mathcal{W}^{1,2}(W)$

$$Y(T) = E[Y(T)] + \int_0^T \nabla_W Y(t) dW(t)$$

If $H = Y(T)$ is Malliavin-differentiable e.g. $H = Y(T) \in \mathbf{D}^{1,2}$ then the Clark-Haussmann-Ocone formula implies

$$Y(T) = E[Y(T)] + \int_0^T pE[\mathbb{D}_t H | \mathcal{F}_t] dW(t)$$
Theorem (Intertwining formula)

Let $Y$ be a $(\mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ square-integrable martingale with $Y(T) = H \in D^{1,2}$. Then

$$E[D_t H | \mathcal{F}_t] = (\nabla_W Y)(t) \quad dt \times d\mathbb{P} \quad \text{a.e.}$$

i.e. the conditional expectation operator intertwines $\nabla_W$ and $D$:

$$E[D_t H | \mathcal{F}_t] = \nabla_W (E[H | \mathcal{F}_t]) \quad dt \times d\mathbb{P} \quad \text{a.e.}$$
In the Markovian case, the Clark-Ocone formula has been expressed [Nualart, Peng & Pardoux] in terms of the ‘diagonal’ Malliavin derivative $\mathbb{D}_t[Y(t)]$. In this case one may show similarly that $\nabla_W Y$ is a version of $\mathbb{D}_t[Y(t)]$. 
Relation with Malliavin derivative

The following diagram is commutative, in the sense of $dt \times d\mathbb{P}$ almost everywhere equality:

$$
\begin{align*}
\mathcal{W}^{1,2}(\mathcal{W}) & \xrightarrow{\nabla_\mathcal{W}} \mathcal{L}^2(\mathcal{W}) \\
\uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} & \xrightarrow{\mathbb{D}} \uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} \\
\mathbb{D}^{1,2} & \xrightarrow{\mathbb{D}} L^2([0,T] \times \Omega)
\end{align*}
$$

Note however that $\nabla_X$ may be constructed for any Ito process $X$ and its construction does not involve Gaussian properties of $X$. 
Consider now a semimartingale $X$ solution of a functional SDE:

$$dX(t) = b_t(X_t)dt + \sigma_t(X_t)dW(t)$$

where $b, \sigma$ are non-anticipative flows on $\Omega$ with values in $\mathbb{R}^d$-valued (resp. $\mathbb{R}^d$) whose coordinates are in $\mathbb{C}^{0,0}_l$ and are Lipschitz-continuous w.r.t sup-norm.

Consider the topological support of the law of $X$ in $(C_0([0, T], \mathbb{R}^d), \|\cdot\|_\infty)$:

$$\text{supp}(X) = \{\omega, \text{any neighborhood } V \text{ of } \omega, \mathbb{P}(X \in V) > 0\}$$
A functional Kolmogorov equation for martingales

Theorem

Let $F \in \mathbb{C}^{1,2}_b$. Then $Y(t) = F_t(X_t)$ is a local martingale if and only if $F$ satisfies

$$\mathcal{D}_t F(\omega) + b_t(\omega)\nabla_\omega F_t(\omega) + \frac{1}{2} \text{tr}\left[\nabla^2_\omega F(\omega)\sigma_t^t\sigma_t(\omega)\right] = 0,$$

for $\omega \in \text{supp}(X)$.

We call such functionals $X$—harmonic functionals.
In particular when $X = W$ is a d-dimensional Wiener process, we obtain a characterization of ‘regular’ Brownian local martingales:

**Theorem**

Let $F \in \mathbb{C}^{1,2}_b$. Then $Y(t) = F_t(W_t)$ is a local martingale on $[0, T]$ if and only if

$$\forall t \in [0, T], \quad \omega \in C_0([0, T], \mathbb{R}^d),$$

$$\mathcal{D}_t F(\omega_t) + \frac{1}{2} \text{tr} \left( \nabla^2 \omega F(\omega_t) \right) = 0.$$
Theorem (Uniqueness of solutions)

Let $h$ be a continuous functional on $(C_0([0, T]), \| \cdot \|_\infty)$. Any solution $F \in C_{b, 2}^{1, 2}$ of the functional equation (1), verifying

$$\forall \omega \in C_0([0, T]),$$

$$D_tF(\omega) + b_t(\omega)\nabla_\omega F_t(\omega) + \frac{1}{2} \text{tr}[\nabla^2_\omega F(\omega)\sigma_t^t\sigma_t(\omega)] = 0$$

$$F_T(\omega) = h(\omega), \quad E[ \sup_{t \in [0, T]} |F_t(X_t)|] < \infty$$

is uniquely defined on the topological support $\text{supp}(X)$ of $(X)$: if $F^1, F^2 \in C_{b, 2}^{1, 2}([0, T])$ are two solutions then

$$\forall \omega \in \text{supp}(X), \quad \forall t \in [0, T], \quad F^1_t(\omega) = F^2_t(\omega).$$
Consider now a $\mathbb{R}^d$-valued Lévy process $L$ with characteristic triplet $(b, A, \nu)$

Denote $\text{supp}(L)$ the \textit{topological support} of the law of $L$ in $(D([0, T], \mathbb{R}^d), \|\cdot\|_\infty)$:

$$\text{supp}(L) = \{\omega, \text{any neighborhood } V \text{ of } \omega, \mathbb{P}(L \in V) > 0\}$$

The topological support may be characterized in terms of $(b, A, \nu)$

see e.g. Th Simon (2000)
Harmonic functionals of a Lévy process

Theorem (Harmonic functionals of a Lévy process)

Let $F \in \mathbb{C}^{1,2}_b$. Then $Y(t) = F_t(L_t)$ is a local martingale if and only if $F$ satisfies the following functional equation:

$$
\int_{\mathbb{R}^d} \left( F_t(\omega_{t-} + z 1_{\{t\}}) - F_t(\omega_{t-}) - 1_{|z| \leq 1} z \cdot \nabla_{\omega} F_t(X_{t-}) \right) \nu(dz) + D_t F(\omega_t) + b \cdot \nabla_{\omega} F_t(\omega_t) + \frac{1}{2} \text{tr} [\nabla^2_{\omega} F(\omega_t) A] = 0
$$

for $\omega \in \text{supp}(X)$. 
A universal pricing equation

Consider now a (discounted) asset price with dynamics
\[ S(t) = \int_0^t \sigma(u, S_u) \, dW(u) \]
under a pricing measure \( \mathbb{Q} \).

**Theorem (Pricing equation for path-dependent options)**

Consider a path-dependent payoff \( H(S(t), t \in [0, T]) \). If

\[ \exists F \in \mathbb{C}_b^{1,2}, \quad F_t(S_t) = E^\mathbb{Q}[H | \mathcal{F}_t] \]

then \( F \) is the unique solution of the pricing equation

\[ \mathcal{D}_t F(\omega) + \frac{1}{2} \nabla^2 \sigma_t(\omega)^2 \sigma_t(\omega) = 0, \]

for \( \omega \in \text{supp}(X) \) with the terminal condition

\[ F_T(\omega) = H(\omega) . \]
Universal pricing equation

\[ D_t F(\omega) + \frac{1}{2} \nabla^2 F_t(\omega) \sigma_t(\omega)^2 = 0, \]
\[ \forall \omega \in \text{supp}(X), \quad F_T(\omega) = H(\omega) \]

This equation implies all known PDEs for path-dependent options: barrier, Asian, lookback,...but also leads to new pricing equations. This equation shows the $\theta - \Gamma$ tradeoff still holds for path-dependent derivatives provided $\theta$ and $\Gamma$ are properly defined in terms of horizontal and vertical derivatives.
Example: Asian options

Consider an option with payoff $H(\omega) = h(\int_0^T \omega(s)ds)$ where $h : \mathbb{R} \to \mathbb{R}$ is measurable and has linear growth.

$$F_t(\omega_t) = f(t, \int_0^t \omega(s)ds, \omega(t))$$

is a $C^{1,2}$ solution of the functional pricing equation if and only if $f \in C^{1,1,2}([0, T]\times]0, \infty[\times]0, \infty[)$ is continuous on $[0, T]\times]0, \infty[\times]0, \infty[$ and solves

$$\partial_t f(t, a, x) + r(t)x\partial_x f(t, a, x) + x\partial_a f(t, a, x) + \frac{1}{2}x^2\sigma^2(t, x)\partial_{xx} f(t, a, x) = r(t)f(t, a, x)$$

with terminal condition: $f(T, a, x) = h(a)$
In particular this equation extends the Black-Scholes Theta-Gamma tradeoff relation for call options to path-dependent options:

\[ D_t F(\omega) + r(t) \nabla_\omega F_t(\omega) + \frac{1}{2} \nabla^2_\omega F_t(\omega) \sigma_t(\omega)^2 = 0, \]

where the time sensitivity (\( \theta(t) \)) is computed as a horizontal derivative and the Gamma is computed as the second vertical derivative of the price.
Summary

- The definition of the vertical derivative restores the relation between the first-order sensitivity to the underlying and its hedging strategy:  
  \[ \nabla_S Y(t) = \text{sensitivity of price to underlying} = \text{hedging portfolio}, \]
  just like for European options in the Black-Scholes model.

- The horizontal derivative is the correct definition of time decay of a path-dependent options:  
  \[ \theta(t) = D_t F(t, S_t). \]
  In particular it verifies the \( \theta - \Gamma \) tradeoff relation, just like for European options in the Black-Scholes model.

- The \( \theta - \Gamma \) tradeoff relation gives a unified derivation of all known pricing PDEs for path-dependent options: Asian, barrier, weighted variance swap,...
The result can be extended to functionals depending on quadratic variation and to discontinuous processes: $Y$ and $X$ can both have jumps.

The result can be *localized* using stopping times: important for applying to functionals involving stopped processes/exit times.

Pathwise maximum principle for non-Markovian control problems.

$\theta - \Gamma$ tradeoff for path-dependent derivatives.
References