CENTRAL IDEALS AND CARTAN INVARIANTS
OF SYMMETRIC ALGEBRAS

László Héthelyi
Department of Algebra
Budapest University of Technology and Economics
H-1521 Budapest
Műegyetem Rkp. 3-9
Hungary
hethelyi@math.bme.hu

and

Erzsébet Horváth
Department of Algebra
Budapest University of Technology and Economics
H-1521 Budapest
Műegyetem Rkp. 3-9
Hungary
he@math.bme.hu

and

Burkhard Külshammer
Mathematisches Institut
Friedrich-Schiller-Universität
07740 Jena
Germany
kuelshammer@uni-jena.de

and

John Murray
Mathematics Department
National University of Ireland
Maynooth, C. Kildare
Ireland
John.Murray@maths.may.ie

Abstract. In this paper, we investigate certain ideals in the center of a symmetric algebra $A$ over an algebraically closed field of characteristic $p > 0$. These ideals include the Higman ideal and the Reynolds ideal. They are closely related to the $p$-power map on $A$. We generalize some results concerning these ideals from group algebras to symmetric algebras, and we obtain some new results as well. In case $p = 2$, these ideals detect odd diagonal entries in the Cartan matrix of $A$. In a sequel to this paper, we will apply our results to group algebras.

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1. Introduction

Let $A$ be a symmetric algebra over an algebraically closed field $F$ of characteristic $p > 0$, with symmetrizing bilinear form $(\cdot | \cdot)$. In this paper we investigate the following chain of ideals of the center $ZA$ of $A$:

$$ZA \supseteq T_1A^\perp \supseteq T_2A^\perp \supseteq \ldots \supseteq RA \supseteq HA \supseteq Z_0A \supseteq 0;$$
here $Z_0A := \sum_B ZB$ where $B$ ranges over the set of blocks of $A$ which are simple $F$-algebras. Thus $Z_0A$ is a direct product of copies of $F$, one for each simple block $B$ of $A$. Furthermore, $HA$ denotes the Higman ideal of $A$, defined as the image of the trace map

$$\tau : A \longrightarrow A, \quad x \mapsto \sum_{i=1}^n b_i x a_i;$$

here $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are a pair of dual bases of $A$. Moreover, $RA$ is the Reynolds ideal of $A$, defined as the intersection of the socle $SA$ of $A$ and the center $ZA$ of $A$. The ideals $T_nA^\perp$ ($n \in \mathbb{N}$) were introduced in [6 IV]; they can be viewed as generalizations of the Reynolds ideal. In fact, $RA$ is their intersection. These ideals are defined in terms of the $p$-power $A \longrightarrow A, x \mapsto x^p$, and the bilinear form $(, | )$. The precise definition will be given below. Motivated by the special case of group algebras [8,9], we show that

$$Z_0A \subseteq (T_1A^\perp)^2 \subseteq HA,$$

so that $(T_1A^\perp)^2$ fits nicely into the chain of ideals above. When $p$ is odd then

$$(T_1A^\perp)^2 = Z_0A.$$

The case $p = 2$ behaves differently and turns out to have some interesting special features. We show that, in this case,

$$(T_1A^\perp)^3 = (T_1A^\perp)(T_2A^\perp) = Z_0A,$$

but that $(T_1A^\perp)^2 \neq Z_0A$ in general. We prove that, in case $p = 2$, the mysterious ideal $(T_1A^\perp)^2$ is a principal ideal of $ZA$. It is generated by the element $\zeta_1(1)^2$ where $\zeta_1 : ZA \longrightarrow ZA$ is a certain natural semilinear map related to the $p$-power map. The map $\zeta_1$ was first defined in [6 IV].

Moreover, in case $p = 2$, the dimension of $(T_1A^\perp)^2$ is the number of blocks $B$ of $A$ with the property that the Cartan matrix $C_B = (c_{ij})$ of $B$ contains an odd diagonal entry $c_{ii}$. A primitive idempotent $e$ in $A$ satisfies $e\zeta_1(1)^2 \neq 0$ if and only if the dimension of $eAe$ is odd.

At the end of the paper, we investigate the behaviour of the ideals $T_nA^\perp$ under Morita and derived equivalences, and we dualize some of the results obtained in the previous sections. In a sequel [2] to this paper, we will apply our results to group algebras of finite groups. We will see that a finite group $G$ contains a real conjugacy class of 2-defect zero if and only if the Cartan matrix of $G$ in characteristic 2 contains an odd diagonal entry. We will also prove a number of related facts.

2. The Reynolds ideal and its generalizations

In the following, let $F$ be an algebraically closed field of characteristic $p > 0$, and let $A$ be a symmetric $F$-algebra with symmetrizing bilinear form $(, | )$. Thus $A$ is a finite-dimensional associative unitary $F$-algebra, and $(, | )$ is a non-degenerate symmetric bilinear form on $A$ which is associative, in the sense that $(ab|c) = (a|bc)$ for $a, b, c \in A$. We denote the center of $A$ by $ZA$, the Jacobson radical of $A$ by $JA$, the socle of $A$ by $SA$ and the commutator subspace of $A$ by $KA$. Thus $KA$ is the $F$-subspace of $A$ spanned by all commutators $ab - ba$ ($a, b \in A$). For $n \in \mathbb{N}$,

$$T_nA := \{x \in A : x^{p^n} \in KA\}$$

is a $ZA$-submodule of $A$, so that

$$KA = T_0A \subseteq T_1A \subseteq T_2A \subseteq \ldots$$

and

$$\sum_{n=0}^\infty T_nA = JA + KA$$

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(cf. [7]). For any $F$-subspace $X$ of $A$, we set

$$X^\perp := \{y \in A : (x|y) = 0 \text{ for } x \in X\}.$$ 

Then

$$ZA = KA^\perp = T_0A^\perp \supseteq T_1A^\perp \supseteq T_2A^\perp \supseteq \ldots$$

is a chain of ideals of $ZA$ such that

$$\bigcap_{n=0}^\infty T_nA^\perp = SA \cap ZA.$$ 

We call $RA := SA \cap ZA$ the Reynolds ideal of $ZA$, in analogy to the terminology used for group algebras. For $n \in \mathbb{N}$ and $z \in ZA$, there is a unique element $\zeta_n(z) \in ZA$ such that

$$(\zeta_n(z)|x)^p_n = (z|x^p_n) \text{ for } x \in A.$$ 

This defines a map $\zeta_n = \zeta_n^A : ZA \rightarrow ZA$ with the following properties:

**Lemma 2.1.** Let $m, n \in \mathbb{N}$, and let $y, z \in ZA$. Then the following holds:

(i) $\zeta_n(y + z) = \zeta_n(y) + \zeta_n(z)$ and $\zeta_n(yz) = \zeta_n(y)p_n(z).$

(ii) $\zeta_n \circ \zeta_m = \zeta_{m+n}$.

(iii) $\text{Im}(\zeta_n) = T_nA^\perp$.

(iv) $\zeta_n^A(z)e = \zeta_n^A(ze)$ for every idempotent $e$ in $A$.

**Proof.** (i), (ii) and (iii) are proved in [7, (44)-(47)].

(iv) Recall that $eAe$ is a symmetric $F$-algebra; a corresponding symmetric bilinear form is obtained by restricting $(\cdot | \cdot)$ to $eAe$. Note that $ez = eze \in eZAe \subseteq Z(eAe)$ and that, similarly, $\zeta_n^A(z)e \in Z(eAe)$. Moreover, for $x \in eAe$, we have

$$(\zeta_n^A(z)e|x)^p_n = (\zeta_n^A(z)|ex)^p_n = (\zeta_n^A(z)|x)^p_n = (z|x^p_n)$$

and the result follows.

We apply these properties in order to prove:

**Lemma 2.2.** Let $m, n \in \mathbb{N}$. Then

$$(T_mA^\perp)(T_nA^\perp) \subseteq \zeta_{m+n}((T_nA^\perp)^p_n(p^m-1)) \subseteq T_{m+n}A^\perp.$$ 

**Proof.** Let $y, z \in ZA$. Then Lemma 2.1 implies that

$$\zeta_m(y)\zeta_n(z) = \zeta_m(y\zeta_n(z)^p_n) = \zeta_m(\zeta_m(y)p_n(z)\zeta_n(z)^p_n-1)) = \zeta_m(y)p_n(z\zeta_n(z)^p_n(p^m-1))) \in \zeta_{m+n}((T_nA^\perp)^p_n(p^m-1)).$$

Thus the result follows from Lemma 2.1 (iii).

Let $B_1, \ldots, B_r$ denote the blocks of $A$, so that $A = B_1 \oplus \cdots \oplus B_r$. Each $B_i$ is itself a symmetric $F$-algebra. If a block $B_i$ is a simple $F$-algebra then $B_i \cong \text{Mat}(d_i, F)$ for a positive integer $d_i$, and thus $ZA \cong F$. We set

$$Zd_iA := \sum_i ZB_i.$$

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where the sum ranges over all \( i \in \{1, \ldots, r \} \) such that \( B_i \) is a simple \( F \)-algebra. Then \( Z_0A \) is an ideal of \( ZA \) and an \( F \)-algebra which is isomorphic to a direct sum of copies of \( F \). Its dimension is the number of simple blocks of \( A \). We exploit Lemma 2.2 in order to prove:

**Theorem 2.3.** (i) \((T_1A^\perp)^2 \subseteq RA\).
(ii) \((T_1A^+)(T_2A^+) = (T_1A^\perp)^3 = Z_0A\).
(iii) If \( p \) is odd then \((T_1A^\perp)^2 = Z_0A\).

**Proof.** (i) Lemma 2.2 implies

\[ (T_1A^\perp)^2 \subseteq \zeta_2((T_1A^\perp)^p_{(p-1)}) \subseteq \zeta_2((T_1A^\perp)^2). \]

Iteration yields

\[ (T_1A^\perp)^2 \subseteq \zeta_2(\zeta_2((T_1A^\perp)^2)) = \zeta_4((T_1A^\perp)^2) \subseteq \zeta_6((T_1A^\perp)^2) \subseteq \ldots. \]

Thus

\[ (T_1A^\perp)^2 \subseteq \bigcap_{n=0}^{\infty} \text{Im}(\zeta_{2n}) = \bigcap_{n=0}^{\infty} T_{2n}A^\perp = SA \cap ZA = RA, \]

by Lemma 2.1 (iii).

(ii) It is easy to see that \( T_nA = T_nB_1 \oplus \cdots \oplus T_nB_r \) and \( T_nA^\perp = T_nB_1^\perp \oplus \cdots \oplus T_nB_r^\perp \) for \( n \in \mathbb{N} \) where \( T_nB_i^\perp = \{ x \in B_i : \langle x, T_nB_i \rangle = 0 \} \) for \( i = 1, \ldots, r \). So we may assume that \( A \) itself is a block.

If \( A \) is simple then \( JA = 0 \), so \( T_nA = KA \) and \( T_nA^\perp = ZA \) for all \( n \in \mathbb{N} \). Hence

\[ ZA = (T_1A^\perp)(T_2A^\perp) = (T_1A^\perp)^3 \]

in this case.

Now suppose that \( A \) is non-simple. Then \( JA \neq 0 \). So \( ZA \neq RA \). It follows that \( JA + KA \neq KA \), whence \( JA \) is not contained in \( KA \). So \( T_1A \neq KA \). This means that \( T_1A^\perp \) is a proper ideal of \( ZA \). Since \( ZA \) is a local \( F \)-algebra this implies that \( T_1A^\perp \subseteq JA \subseteq ZA \). Thus we may conclude, using (i), that \((T_1A^\perp)^3 \subseteq (RA)(JA) = 0\). Hence Lemma 2.2 yields

\[ (T_1A^\perp)(T_2A^\perp) \subseteq \zeta_3((T_2A^\perp)^p_{(p-1)}) \subseteq \zeta_3((T_1A^\perp)^3) = \zeta_3(0) = 0. \]

(iii) Suppose that \( p \) is odd. As in the proof of (ii), we may assume that \( A \) is a block, and that \( A \) is non-simple. Then Lemma 2.2 and (ii) imply that

\[ (T_1A^\perp)^2 \subseteq \zeta_2((T_1A^\perp)^p_{(p-1)}) \subseteq \zeta_2((T_1A^\perp)^3) = \zeta_2(0) = 0, \]

and the result is proved.

Theorem 2.3 extends [9, Theorem 9] from group algebras to symmetric algebras. We will later improve on part (i). But we first note the following consequence.

**Corollary 2.4.** Suppose that \( A \) is a block, and denote the central character of \( A \) by \( \omega : ZA \to F \). Moreover, let \( m, n \in \mathbb{N} \) with \( m \neq 0 \neq n \), and let \( x, y \in ZA \). Then

\[ \zeta_m(x)\zeta_n(y) = \omega(x)^{p^{-m}}\omega(y)^{p^{-n}}\zeta_m(1)\zeta_n(1). \]

In particular, we have

\[ (T_mA^\perp)(T_nA^\perp) = F\zeta_m(1)\zeta_n(1), \]

so that \( \text{dim}(T_mA^\perp)(T_nA^\perp) \leq 1 \).

**Proof.** Theorem 2.3 (i) implies that \( \zeta_m(x)^p \in RA \subseteq SA \). Thus

\[ \zeta_m(x)^p = \omega(y)\zeta_m(x)^p. \]
Similarly, we have \( xζ_n(1)p^m = ω(x)ζ_n(1)p^m \). So we conclude that
\[
ζ_m(x)ζ_n(y) = ζ_n(ζ_m(x)p^n y) = ζ_n(ω(y)ζ_m(x)p^n) = ω(y)p^m ζ_m(x)ζ_n(1)
\]
\[
= ω(y)p^m ζ_m(xζ_n(1)p^n) = ω(y)p^m ζ_n(ω(x)ζ_n(1)p^m) = ω(y)p^m ω(x)p^m ζ_m(1)ζ_n(1).
\]
The remaining assertions follow from Lemma 2.1 (iii).

We can generalize part of Corollary 2.4 in the following way.

**Proposition 2.5.** Let \( m, n \in \mathbb{N} \) with \( m \neq 0 \neq n \). Then
\[
(T_m A^⊥)(T_n A^⊥) = \mathbb{Z}A : ζ_m(1)ζ_n(1)
\]
is a principal ideal of \( \mathbb{Z}A \). If \( p \) is odd, or if \( m + n > 2 \), then the dimension of \( (T_m A^⊥)(T_n A^⊥) \) equals the number of simple blocks of \( A \) and in particular does not depend on \( m + n \).

**Proof.** It is easy to see that we may assume that \( A \) is a block. In this case the assertion follows from Corollary 2.4 and Theorem 2.3.

In the next two sections, we will handle the remaining case \( p = 2 \) and \( m = n = 1 \). Here we just illustrate this exceptional case by an example.

Let \( G \) be a finite group. Then the group algebra \( FG \) is a symmetric \( F \)-algebra; a symmetrizing bilinear form on \( FG \) satisfies
\[
(g|h) = \begin{cases} 1, & \text{if } gh = 1, \\ 0, & \text{otherwise}, \end{cases}
\]
for \( g, h \in G \). An element \( g \in G \) is called real if \( g \) is conjugate to its inverse \( g^{-1} \), and \( g \) is said to be of \( p \)-defect zero if \( |C_G(g)| \) is not divisible by \( p \). We denote the set of all real elements of \( 2 \)-defect zero in \( G \) by \( R_G \). For a subset \( X \) of \( G \), we set
\[
X^+ := \sum_{x \in X} x \in FG.
\]
It was proved in [8, Proposition 4.1] that \( R_G^⊥ = ζ_1(1)^2 \in (T_1 FG^⊥)^2 \), in case \( p = 2 \).

**Example 2.6.** Let \( p = 2 \), and suppose that \( G \) is the symmetric group \( S_4 \) of degree 4. Then \( FG \) has no simple blocks; in fact, \( FG \) has just one block, the principal one. Thus \( \mathbb{Z}_0 FG = 0 \). On the other hand, \( R_G \) is precisely the set of all 3-cycles in \( S_4 \). Thus \( 0 \neq R_G^⊥ \in (T_1 FG^⊥)^2 \). (In fact, \( (T_1 FG^⊥)^2 \) is one-dimensional, by Corollary 2.4.) This example shows that \( (T_1 A^⊥)^2 \neq \mathbb{Z}_0 A \), in general.

### 3. Odd Cartan invariants

Let \( F \) be an algebraically closed field of characteristic \( p = 2 \), and let \( A \) be a symmetric \( F \)-algebra with symmetrizing bilinear form \( (., .) \). In this section, we will prove some remarkable properties of the ideal \( (T_1 A^⊥)^2 \) of \( \mathbb{Z}A \). We start by recalling some known facts concerning symmetric bilinear forms over \( F \).

**Lemma 3.1.** Let \( V \) be a finite-dimensional vector space over \( F \), and let \( (., .) \) be a non-degenerate symmetric bilinear form on \( V \). Then either \( (., .) \) is symplectic (i.e. \( \langle v|v \rangle = 0 \) for every \( v \in V \)), or there exists an orthonormal basis \( v_1, \ldots, v_n \) of \( V \) (i.e. \( \langle v_i|v_j \rangle = δ_{ij} \) for \( i, j = 1, \ldots, n \)).

**Proof.** This can be found in [4, Hauptsatz V.3.5], for example.

If \( (., .) \) is symplectic then there exists a symplectic basis \( v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m} \) of \( V \), i.e.
\[
\langle v_i|v_{m+i} \rangle = \langle v_{m+i}|v_i \rangle = 1 \quad \text{for } i = 1, \ldots, m,
\]
\[
\langle v_i|v_j \rangle = 0 \quad \text{otherwise},
\]
Lemma 3.2.  

\[(\zeta_1(1)|\zeta_1(1)) = (\dim A) \cdot 1_F.\]

Proof. By Lemma 3.1, there exists an $F$-basis

\[a_1, \ldots, a_m, a_{m+1}, \ldots, a_{2m}, a_{2m+1}, \ldots, a_n\]

of $A$ such that

\[(a_i|a_{m+i}) = (a_{m+i}|a_i) = 1 \quad \text{for} \quad i = 1, \ldots, m,
(a_i|a_i) = 1 \quad \text{for} \quad i = 2m + 1, \ldots, n,
(a_i|a_j) = 0 \quad \text{otherwise},\]

(and either $n = 2m$ or $m = 0$). Then the dual basis $b_1, \ldots, b_n$ of $a_1, \ldots, a_n$ is given by

\[a_{m+1}, \ldots, a_{2m}, a_1, \ldots, a_m, a_{2m+1}, \ldots, a_n.\]

Thus \((\zeta_1(1)|a_i)^2 = (1|a_i^2) = (a_i|a_i) = (a_i|a_i)^2\) for $i = 1, \ldots, n$, so

\[\zeta_1(1) = \sum_{i=1}^n (\zeta_1(1)|a_i)b_i = \sum_{i=1}^n (a_i|a_i)b_i = \sum_{i=2m+1}^n a_i\]

and

\[(\zeta_1(1)|\zeta_1(1)) = \sum_{i,j=2m+1}^n (a_i|a_j) = \sum_{i=2m+1}^n (a_i|a_i) = (n - 2m) \cdot 1_F = n \cdot 1_F = (\dim A) \cdot 1_F,\]

and the result is proved.

The next statement holds in arbitrary characteristic. It is essentially taken from [11, Corollary (1.1)].

Lemma 3.3. Let $e$ be a primitive idempotent in $A$, and let $r \in RA$. Then $er = 0$ if and only if $(e|r) = 0$.

Proof. If $er = 0$ then $0 = (er|1) = (e|r)$. Conversely, if $(e|r) = 0$ then

\[\langle eAer | e r \rangle = (eAe|r) = (Fe + J(eAe)|r) \subseteq F(e|r) + (JA \cdot r|1) = 0.\]

Thus $0 = ere = er$ since the restriction of $(\cdot | \cdot)$ to $eAe$ is non-degenerate.

Now we choose representatives $a_1 = e_1, \ldots, a_l = e_l$ for the conjugacy classes of primitive idempotents in $A$. (This means that $e_1, \ldots, e_l$ are representatives for the isomorphism classes of indecomposable projective left $A$-modules.) Moreover, we let $a_{l+1}, \ldots, a_n$ denote an $F$-basis of $JA + KA$. Then $a_1, \ldots, a_n$ form an $F$-basis of $A$.

Let $b_1, \ldots, b_n$ denote the dual basis of $a_1, \ldots, a_n$. Then $r_1 := b_1, \ldots, r_l := b_l$ are contained in $(JA + KA)^{\perp} = SA \cap ZA = RA$, so they form an $F$-basis of $RA$. Moreover, Lemma 3.3 implies that $e_i r_j = 0$ for $i \neq j$ and $e_i r_i \neq 0$ for $i = 1, \ldots, l$.

Lemma 3.4. With $e_1, \ldots, e_l$ as above, we have $\zeta_1(1)^2 = \sum_{i=1}^l (\dim e_i Ae_i) \cdot r_i$ and $e_i \zeta_1(1)^2 = (\dim e_i Ae_i) \cdot e_i r_i$ for $i = 1, \ldots, l$. 

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Proof. Lemma 2.1 (iii) and Theorem 2.3 (i) imply that $\zeta_1(1)^2 \in (T_1 A^\perp)^2 \subseteq RA$. By making use of Lemma 2.1 (iv) and Lemma 3.2, we obtain

$$\zeta_1(1)^2 = \sum_{i=1}^{l} (\zeta_1(1)^2 e_i) r_i = \sum_{i=1}^{l} (\zeta_1(1)e_i \zeta_1(1)e_i) r_i = \sum_{i=1}^{l} (\dim e_i Ae_i) \cdot r_i.$$

Since $e_i r_j = 0$ for $i \neq j$ the result follows.

The next theorem is the main result of this section.

**Theorem 3.5.** For $A$ a symmetric algebra over an algebraically closed field $F$ of characteristic 2 and for $e$ a primitive idempotent in $A$, the following assertions are equivalent:

1. $\dim e Ae$ is even.
2. $e \zeta_1(1)^2 = 0$.
3. $(e|\zeta_1(1)^2) = 0$.

*Proof.* We may assume that $e = e_i$ for some $i \in \{1, \ldots, l\}$. Then $e_i \zeta_1(1)^2 = (\dim e_i Ae_i) \cdot e_i r_i$ with $e_i r_i \neq 0$, by Lemma 3.4. This shows that (1) and (2) are equivalent. Since $\zeta_1(1)^2 \in RA$, Lemma 3.3 implies that (2) and (3) are equivalent.

The Cartan matrix $C := (c_{ij})_{i,j=1}^{l}$ of $A$ is defined by

$$c_{ij} := \dim e_i Ae_j \quad \text{for} \quad i, j = 1, \ldots, l.$$

Thus $C$ is a symmetric matrix with non-negative integer coefficients, the Cartan invariants of $A$. Hence Theorem 3.5 has the following consequence.

**Corollary 3.6.** With the notation for the Cartan matrix of $A$ as above, $\zeta_1(1)^2 \neq 0$ if and only if $c_{ii}$ is odd for some $i$. More precisely, for a block $B$ of $A$, we have $\zeta_1(1)^2 1_B \neq 0$ if and only if the Cartan matrix of $B$ contains an odd diagonal entry.

In order to illustrate Corollary 3.6 recall that, by Example 2.6, the group algebra $FG$, for $G = S_4$, satisfies $\zeta_1(1)^2 = R_G^2 \neq 0$. Thus the Cartan matrix of $FG$ contains an odd diagonal entry, by Corollary 3.6. Indeed, the Cartan matrix of $FG$ is

$$C := \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix},$$

as is well-known. More substantial examples will be presented in [2].

It may be of interest to note that the existence of odd diagonal Cartan invariants in characteristic 2 is invariant under derived equivalences (cf. [5]).

**Proposition 3.7.** Let $A'$ be a symmetric $F$-algebra which is derived equivalent to $A$. Then the Cartan matrix of $A'$ contains an odd diagonal entry if and only if the Cartan matrix of $A$ does.

*Proof.* It is known that the Cartan matrices $C := (c_{ij})_{i,j=1}^{l}$ of $A$ and $C' := (c'_{ij})_{i,j=1}^{l}$ of $A'$ have the same format, and that they are related by an equation

$$C' = Q \cdot C \cdot Q^\top,$$

where $Q := (q_{ij})_{i,j=1}^{l}$ is an integral matrix with determinant $\pm 1$ (cf. [5]). Thus

$$c'_{ii} = \sum_{j,k=1}^{l} q_{ij} q_{ik} c_{jk} \equiv \sum_{j=1}^{l} q_{ij}^2 c_{jj} \pmod{2}.$$
for \( i = 1, \ldots, l \). If \( c_i \) is odd then \( c_j \) has to be odd for some \( j \in \{1, \ldots, l\} \) (and conversely).

4. The Higman ideal

Let \( F \) be an algebraically closed field, and let \( A \) be a symmetric \( F \)-algebra with symmetrizing bilinear form \((|.|)\). Moreover, let \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) denote a pair of dual bases of \( A \). In the following, the \( F \)-linear map

\[
\tau : A \longrightarrow A, \quad x \longmapsto \sum_{i=1}^{n} b_i x a_i,
\]

will be of interest (cf. [3, §66]). We record the following properties of this \textit{trace map} \( \tau \):

\begin{lemma}
\text{(i)} \( \tau \) is independent of the choice of dual bases.
\text{(ii)} \( \tau \) is self-adjoint with respect to \((.|.|)\).
\text{(iii)} \( \text{Im}(\tau) \subseteq SA \cap ZA = RA \) and \( JA + KA \subseteq \text{Ker}(\tau) \).
\end{lemma}

\textbf{Proof.} (i) Let \( a_1', \ldots, a_n' \) and \( b_1', \ldots, b_n' \) be another pair of dual bases of \( A \). Then \( b_i' = \sum_{j=1}^{n} (a_i|b_j')b_j \) and \( a_i = \sum_{j=1}^{n} (a_i|b_j')a_j' \) for \( i = 1, \ldots, n \). Thus

\[
\sum_{i=1}^{n} b_i' x a_i' = \sum_{i,j=1}^{n} (a_i|b_j')b_j x a_i' = \sum_{j=1}^{n} b_j x \sum_{i=1}^{n} (a_i|b_j')a_i' = \sum_{j=1}^{n} b_j x a_j
\]

for \( x \in A \).

(ii) Let \( x, y \in A \). Then, by (i), we get

\[
(\tau(x)|y) = \sum_{i=1}^{n} (b_i x a_i|y) = \sum_{i=1}^{n} (x|a_i y b_i) = (x|\tau(y)).
\]

(iii) Let \( x, y \in A \). Then

\[
\tau(x) y = \sum_{i=1}^{n} b_i x a_i y = \sum_{i,j=1}^{n} b_i x (a_i y b_j) a_j = \sum_{i,j=1}^{n} (a_i y b_j) b_i x a_j = \sum_{j=1}^{n} y b_j x a_j = y \tau(x).
\]

Hence \( \text{Im}(\tau) \subseteq ZA \). In order to prove \( \text{Im}(\tau) \subseteq SA \), we choose \( a_1, \ldots, a_n \) appropriately. Indeed, we may assume that \( a_1 + JA, \ldots, a_r + JA \) form an \( F \)-basis of \( A/JA \), that \( a_{r+1} + (JA)^2, \ldots, a_s + (JA)^2 \) form an \( F \)-basis of \( (JA)/(JA)^2 \), that \( a_{s+1} + (JA)^3, \ldots, a_t + (JA)^3 \) form an \( F \)-basis of \( (JA)^2/(JA)^3 \), etc. Then \( b_1, \ldots, b_r \) are contained in \( (JA)^{1} \), \( b_1, \ldots, b_t \) are contained in \( (JA)^{1} \), etc.

Now let \( x \in A \) and \( y \in JA \). Then \( b_i x a_i y \in (JA)^{1} \cdot A \cdot (JA) = 0 \) for \( i = 1, \ldots, r \), \( b_i x a_i y \in ((JA)^{1})^{2} \cdot A \cdot (JA) \cdot (JA) = 0 \) for \( i = r+1, \ldots, s \), \( b_i x a_i y \in ((JA)^{1})^{2} \cdot A \cdot (JA)^3 \cdot (JA) = 0 \) for \( i = s+1, \ldots, t \), etc. We see that \( \tau(x) y = 0 \), so \( \text{Im}(\tau) \subseteq SA \).

Since \( \tau \) is self-adjoint (i.e. \( \tau^* = \tau \)) we conclude that

\[
\text{Ker}(\tau) = \text{Ker}(\tau^*) = \text{Im}(\tau)^{\perp} \supseteq (SA \cap ZA)^{\perp} = JA + KA.
\]

Thus \( HA := \text{Im}(\tau) \) is an ideal of \( ZA \) contained in \( RA \), called the \textit{Higman ideal} of \( ZA \). By Lemma 4.1, it is independent of the choice of dual bases. In the following, we write

\[
1_A = e_1 + \cdots + e_m
\]
with pairwise orthogonal primitive idempotents \(e_1, \ldots, e_m\) of \(A\).

**Lemma 4.2.** We have \((\tau(e_i)|e_j) = (\dim e_i Ae_j) \cdot 1_F\) for \(i, j = 1, \ldots, m\).

**Proof.** We consider the decomposition \(A = \bigoplus_{j=1}^m e_i Ae_j\). For \(i, j = 1, \ldots, m\), let \(X_{ij}\) be an \(F\)-basis of \(e_i Ae_j\). Then \(X := \bigcup_{j=1}^m X_{ij}\) is an \(F\)-basis of \(A\). We denote the dual basis of \(X\) by \(X^*\). For \(x \in X\), there is a unique \(x^* \in X^*\) such that \((x|x^*) = 1\). Then the map \(X \rightarrow X^*, x \mapsto x^*\), is a bijection. Moreover, for \(i, j = 1, \ldots, m\), \(X_{ij}^* := \{x^* : x \in X_{ij}\}\) is an \(F\)-basis of \(e_i Ae_j\). Thus

\[
\tau(e_i)e_j = e_j \tau(e_i)e_j = \sum_{x \in X} e_j x^* e_i xe_j = \sum_{x \in X_{ij}} e_j x^* e_i xe_j = \sum_{x \in X_{ij}} x^* x
\]

and

\[
(\tau(e_i)|e_j) = (\tau(e_i)e_j|1) = \sum_{x \in X_{ij}} (x^* x|1) = \sum_{x \in X_{ij}} (x^*|x) = (|X_{ij}| \cdot 1_F = (\dim e_i Ae_j) \cdot 1_F,
\]

so the result is proved.

We may assume that \(e_1, \ldots, e_m\) are numbered in such a way that \(a_1 := e_1, \ldots, a_l := e_l\) represent the conjugacy classes of primitive idempotents in \(A\). We choose an \(F\)-basis \(a_1, \ldots, a_n\) of \(JA + KA\), so that \(a_1, \ldots, a_n\) form an \(F\)-basis of \(A\). We denote the dual basis of \(a_1, \ldots, a_n\) by \(b_1, \ldots, b_n\). As above, \(r_1 := b_1, \ldots, r_l := b_l\) form an \(F\)-basis of \(RA = SA \cap ZA\).

**Lemma 4.3.** We have \(\tau(e_i) = \sum_{j=1}^l (\dim e_i Ae_j) \cdot r_j\) for \(i = 1, \ldots, l\).

**Proof.** Let \(i \in \{1, \ldots, l\}\). Then \(\tau(e_i) \in HA \subseteq RA\), so

\[
\tau(e_i) = \sum_{j=1}^l (\tau(e_i)|e_j)r_j = \sum_{j=1}^l (\dim e_i Ae_j) \cdot r_j
\]

by Lemma 4.2.

In the following, suppose that \(\text{char } F = \rho > 0\). We know from Theorem 2.3 that \((T_1A^-)2 \subseteq RA\). We are going to show that, more precisely, \((T_1A^-)2 \subseteq HA\). In the proof, we will make use of the following fact.

**Lemma 4.4.** Let \(C = (c_{ij})\) be a symmetric \(n \times n\)-matrix with coefficients in the field \(\mathbb{F}_2\) with two elements. Then its main diagonal \(c := (c_{11}, c_{22}, \ldots, c_{nn})\), considered as a vector in \(\mathbb{F}_2^n\), is a linear combination of the rows of \(C\).

**Proof.** Arguing by induction on \(n\), we may assume that \(n > 1\). If \(c = 0\) then there is nothing to prove. So we may assume that \(c_{ii} = 1\) for some \(i \in \{1, \ldots, l\}\). Permuting the rows and columns of \(C\), if necessary, we may assume that \(c_{11} = 1\). We now perform elementary row operations on \(C\). For \(k = 2, \ldots, n\), we subtract the first row, multiplied by \(c_{k1}\), from the \(k\)-th row. The resulting matrix \(C'\) has the entries

\[
0, c_{k2} - c_{k1}c_{12}, \ldots, c_{kn} - c_{k1}c_{1n}
\]

in its \(k\)-th row and the entries

\[
c_{1k}, c_{2k} - c_{21}c_{1k}, \ldots, c_{nk} - c_{n1}c_{1k}
\]

in its \(k\)-th column. We now remove the first row and the first column from \(C'\) and end up with a symmetric \((n-1) \times (n-1)\)-matrix \(D\) with diagonal entries

\[
c_{kk} - c_{k1}c_{1k} = c_{kk} - c_{1k}^2 = c_{kk} - c_{1k}^2 \quad (k = 2, \ldots, n).
\]
On the other hand, if we subtract the first row of $C$ from $c$ then we obtain the vector
\[ c' := (0, c_{22} - c_{12}, \ldots, c_{nn} - c_{1n}). \]
Thus the vector $d := (c_{22} - c_{12}, \ldots, c_{nn} - c_{1n})$ coincides with the main diagonal of $D$. By induction, $d$ is a linear combination of the rows of $D$, so $c$ is a linear combination of the rows of $C$.

As Gary McGuire kindly pointed out to us, a different proof of Lemma 4.4 can be found in [1, Proposition 4.6.2]. We apply Lemma 4.4 in the proof of the following result which is a refinement of Theorem 2.3 (i). The special case of group algebras was first proved in [8, Lemma 5.1].

**Theorem 4.5.** We always have $(T_1A^\perp)^2 \subseteq HA$.

**Proof.** If $p$ is odd then, by Theorem 2.3 (iii), we have
\[ (T_1A^\perp)^2 \subseteq Z_0A = \sum B ZB = \sum B HB \subseteq HA \]
where $B$ ranges over the simple blocks of $A$; in fact, if $B = \text{Mat}(d, F)$ for a positive integer $d$ then $HB = ZB$.

Thus we may assume that $p = 2$. Then Lemma 2.2 gives us elements $\alpha_1, \ldots, \alpha_l$ in the prime field of $F$ such that
\[ \sum_{i,j} (\dim e_i Ae_j) \cdot \alpha_j = (\dim e_i Ae_i) \cdot 1_F \text{ for } i = 1, \ldots, l. \]
Thus Lemma 3.4 and Lemma 4.3 imply that
\[ \zeta_1(1)^2 = \sum_{i=1}^{l} (\dim e_i Ae_i) \cdot r_i = \sum_{i,j=1}^{l} (\dim e_i Ae_j) \cdot \alpha_j r_i = \sum_{j=1}^{l} \alpha_j \tau(e_j) \in HA. \]
Hence Proposition 2.5 implies that $(T_1A^\perp)^2 = ZA \cdot \zeta_1(1)^2 \subseteq HA$.

5. Morita invariance

Let $F$ be an algebraically closed field of characteristic $p > 0$, and let $A$ be a symmetric $F$-algebra. In this section we investigate the behaviour of the ideals $T_nA^\perp$ of $ZA$ under Morita equivalences. These results will be used in [2].

**Proposition 5.1.** Let $e$ be an idempotent in $A$ such that $eAe = A$. Then the map
\[ f : ZA \longrightarrow Z(eAe), \quad z \longmapsto ez = ze, \]
is an isomorphism of $F$-algebras mapping $T_nA^\perp$ onto $T_n(eAe)^\perp$, for $n \in \mathbb{N}$.

**Proof.** Certainly $f$ is a homomorphism of $F$-algebras. Let $z \in ZA$ such that $0 = f(z) = ez$. Then $0 = AeA = AeAe = Az$, so that $z = 0$. Thus $f$ is injective. Since $eAe = A$ the $F$-algebras $A$ and $eAe$ are Morita equivalent; in particular, their centers are isomorphic. Hence $f$ is an isomorphism of $F$-algebras. Lemma 2.1 (iv) implies that $f \circ \zeta_n^A = \zeta_n^{eAe} \circ f$, so
\[ f(T_nA^\perp) = f(\zeta_n^A(ZA)) = \zeta_n^{eAe}(f(ZA)) = \zeta_n^{eAe}(Z(eAe)) = T_n(eAe)^\perp \]
by Lemma 2.1 (iii).

We mention two consequences of Proposition 5.1.
Corollary 5.2. Let $d$ be a positive integer, and let $A_d$ denote the symmetric $F$-algebra $\text{Mat}(d, A)$. Then the map

$$h : ZA \rightarrow ZA_d, \quad z \mapsto z1_d,$$

is an isomorphism of $F$-algebras mapping $T_nA^\perp$ onto $(T_nA_d)^\perp$, for $n \in \mathbb{N}$.

Proof. We denote the matrix units of $A_d$ by $e_{ij}$ ($i, j = 1, \ldots, d$). Then the map

$$f : A \rightarrow e_{11}A_d e_{11}, \quad a \mapsto ae_{11},$$

is an isomorphism of $F$-algebras. This implies that $f(ZA) = Z(e_{11}A_d e_{11})$ and $f(T_nA^\perp) = T_n(e_{11}A_d e_{11})^\perp$ for $n \in \mathbb{N}$. On the other hand, Proposition 5.1 implies that the map

$$g : ZA_d \rightarrow Z(e_{11}A_d e_{11}), \quad z \mapsto ze_{11} = e_{11}z,$$

is an isomorphism of $F$-algebras such that $g((T_nA_d)^\perp) = T_n(e_{11}A_d e_{11})^\perp$ for $n \in \mathbb{N}$. Now observe that $h$ is an isomorphism of $F$-algebras such that $g \circ h$ is the restriction of $f$ to $ZA$. Thus $h(T_nA^\perp) = (T_nA_d)^\perp$ for $n \in \mathbb{N}$.

Corollary 5.3. Let $B$ be a symmetric $F$-algebra which is Morita equivalent to $A$. Then there is an isomorphism of $F$-algebras $ZA \rightarrow ZB$ mapping $T_nA^\perp$ onto $T_nB^\perp$, for $n \in \mathbb{N}$.

Proof. Let $e$ be an idempotent in $A$ such that $eAe$ is a basic algebra of $A$, and let $f$ be an idempotent in $B$ such that $fBf$ is a basic algebra of $B$. Then $AeA = A$ and $BfB = B$. Moreover, $eAe$ and $fBf$ are isomorphic since $A$ and $B$ are Morita equivalent. Thus Proposition 5.1 yields a chain of isomorphisms

$$ZA \rightarrow Z(eAe) \rightarrow Z(fBf) \rightarrow ZB$$

mapping $T_nA^\perp$ onto $T_nB^\perp$, for $n \in \mathbb{N}$.

It would be interesting to know whether Corollary 5.3 extends to symmetric $F$-algebras which are derived equivalent (cf. [5]).

Question 5.4. Suppose that $A$ and $B$ are derived equivalent symmetric $F$-algebras. Is there an isomorphism of $F$-algebras $ZA \rightarrow ZB$ mapping $T_nA^\perp$ onto $T_nB^\perp$, for $n \in \mathbb{N}$?

6. Some dual results

Let $F$ be an algebraically closed field of characteristic $p > 0$, and let $A$ be a symmetric $F$-algebra. For $n \in \mathbb{N},$

$$T_nZA := \{z \in ZA : z^{p^n} = 0\}$$

is an ideal of $ZA$. In this way we obtain an ascending chain of ideals

$$0 = T_0ZA \subseteq T_1ZA \subseteq T_2ZA \subseteq \ldots \subseteq JZA \subseteq ZA$$

of $ZA$ such that

$$\sum_{n=0}^{\infty} T_nZA = JZA.$$

This ascending chain of ideals turns out to be related to the descending chain of ideals

$$ZA = T_0A^\perp \supseteq T_1A^\perp \supseteq T_2A^\perp \ldots \supseteq RA \supseteq 0$$

of $ZA$ considered before.
Proposition 6.1. Let \( n \in \mathbb{N} \). Then \((T_nA^\perp)(T_nZA) = 0\).

Proof. Let \( y \in ZA \) and \( z \in T_nZA \), so that \( z^{p^n} = 0 \). Then Lemma 2.1 (i) implies that
\[
\zeta_n(y)z = \zeta_n(yz^{p^n}) = \zeta_n(y0) = \zeta_n(0) = 0.
\]
Hence \((T_nA^\perp)(T_nZA) = (\text{Im } \zeta_n)(T_nZA) = 0\), by Lemma 2.1 (iii).

The result above is essentially [9, Proposition 4]. We conclude that
\[
T_nZA \subseteq \{ z \in ZA : z(T_nA^\perp) = 0 \} \subseteq \{ z \in ZA : z\zeta_n(1) = 0 \}.
\]

In [2], we will see that these inclusions are proper in general, even for group algebras of finite groups. If \( n \) is sufficiently large then \( T_nZA = JZA \) and \( T_nA^\perp = RA \), and certainly
\[
JZA = \{ z \in ZA : z \cdot RA = 0 \}.
\]

Also, if \( n \) is large and \( A = FG \) for a finite group \( G \) then \( \zeta_n(1) = G^+_p \) where \( G_p \) denotes the set of \( p \)-elements in \( G \) (cf. [7, (48)]), and it is known that
\[
JZFG = \{ z \in ZFG : zG^+_p = 0 \}
\]
(cf. [7, (59)]). However, it is easy to construct an example of a symmetric \( F \)-algebra \( A \) such that
\[
JZA \neq \{ z \in ZA : z\zeta_n(1) = 0 \}
\]
for all sufficiently large \( n \).

For \( n \in \mathbb{N} \), the ideal \( T_nZA \) of \( ZA \) is related to a semilinear map \( \kappa_n : A/KA \rightarrow A/KA \) first constructed in [6 IV]; \( \kappa_n \) is defined in such a way that
\[
(z^{p^n}|x) = (z|\kappa_n(x))^{p^n} \quad \text{for} \quad z \in ZA \quad \text{and} \quad x \in A/KA;
\]
here we set \((z|a+KA) := (z|a)\) for \( z \in ZA \) and \( a \in A \). Also, we set \((a+KA)^{p^n} := a^{p^n} + KA\) for \( a \in A \). We recall the following properties of \( \kappa_n \) (cf. [7, (50) - (53)]).

Lemma 6.2. Let \( m, n \in \mathbb{N} \), let \( x, y \in A/KA \), and let \( z \in ZA \). Then the following holds:
(i) \( \kappa_n(x+y) = \kappa_n(x) + \kappa_n(y) \), \( z\kappa_n(x) = \kappa_n(z^{p^n}x) \) and \( \kappa_n(z^{p^n}x) = \zeta_n(z)x \).
(ii) \( \kappa_m \circ \kappa_n = \kappa_{m+n} \).
(iii) \( \text{Im}(\kappa_n) = T_nZ/A^\perp/KA \).

Our next result is a dual version of Theorem 2.3. For simplicity, we concentrate on the case where \( A \) is a non-simple block. (If \( A \) is a simple block then \( T_1ZA = 0 \), so \( T_1Z/A^\perp = A \). Moreover, we have \( T_2A^\perp = T_1A^\perp = ZA \) in this case.)

Proposition 6.3. Suppose that \( A \) is a non-simple block. Then the following holds:
(i) \( (T_1A^\perp)(T_1ZA^\perp) \subseteq KA \) for \( p \neq 2 \).
(ii) \( (T_2A^\perp)(T_1ZA^\perp) \subseteq KA \) and \( (T_1A^\perp)(T_2ZA^\perp) \subseteq KA \) for \( p = 2 \).
(iii) \( (T_1A^\perp)(T_2ZA^\perp) \subseteq JZA^\perp \) for \( p = 2 \). Moreover, in this case we have \( (T_1A^\perp)(T_1ZA^\perp) \subseteq KA \) if and only if \( \zeta_1(1)^2 = 0 \).

Proof. (i) Let \( y \in ZA \) and \( x \in A/KA \). Then \( \zeta_1(y)\kappa_1(x) = \kappa_1(\zeta_1(y)^p x) = 0 \) since \( \zeta_1(y)^p \in (T_1A^\perp)^p = 0 \) by Theorem 2.3 (iii). Thus
\[
(T_1A^\perp)(T_1ZA^\perp/KA) = (\text{Im } \zeta_1)(\text{Im } \kappa_1) = 0,
\]
and (i) is proved.
Similarly, we have $\zeta_2(y)\kappa_1(x) = \kappa_1(\zeta_2(y)^2x) = 0$ since $\zeta_2(y)^2 \in (T_2A^\perp)^2 = 0$, by Theorem 2.3 (ii). Thus

$$\langle T_2A^\perp\rangle(T_1ZA^\perp/KA) = (\text{Im } \zeta_2)(\text{Im } \kappa_1) = 0.$$  

Similarly, we have $\zeta_1(y)\kappa_2(x) = \kappa_2(\zeta_1(y)^2x) = 0$ since $\zeta_1(y)^2 \in (T_1A^\perp)^3 = 0$ by Theorem 2.3 (ii). Thus

$$\langle T_1A^\perp\rangle(T_2ZA^\perp/KA) = (\text{Im } \zeta_1)(\text{Im } \kappa_2) = 0,$$

and (ii) follows.

(iii) Again, let $x, y$ be as in (i). Then

$$\zeta_1(y)\kappa_1(x) = \kappa_1(\zeta_1(y)^2x) = \kappa_1(\zeta_1(y)\kappa_1(yx^2)) \in \kappa_1((\text{Im } \zeta_1)(\text{Im } \kappa_1)).$$

Iteration yields

$$(\text{Im } \zeta_1)(\text{Im } \kappa_1) \subseteq \kappa_1((\text{Im } \zeta_1)(\text{Im } \kappa_1)) \subseteq \kappa_1(\kappa_1((\text{Im } \zeta_1)(\text{Im } \kappa_1))) = \kappa_2((\text{Im } \zeta_1)(\text{Im } \kappa_1)) \subseteq \ldots .$$

Thus

$$\langle T_1A^\perp\rangle(T_1ZA^\perp/KA) = (\text{Im } \zeta_1)(\text{Im } \kappa_1) \subseteq \bigcap_{n=0}^{\infty} \text{Im}(\kappa_n) = \bigcap_{n=0}^{\infty} T_nZA^\perp/KA = JZA^\perp/KA,$$

and the first assertion of (iii) is proved. Now note that $(T_1A^\perp)(T_1ZA^\perp) \subseteq KA$ if and only if

$$0 = (\langle T_1A^\perp\rangle T_1ZA^\perp)(ZA) = (T_1A^\perp)(T_1ZA^\perp)$$

if and only if $T_1A^\perp \subseteq T_1ZA$ if and only if $z^2 = 0$ for all $z \in T_1A^\perp$. But $(T_1A^\perp)^2 = F\zeta_1(1)^2$ by Corollary 2.4, so $z^2 = 0$ for all $z \in T_1A^\perp$ if and only if $\zeta_1(1)^2 = 0$.

Note that, in the situation of Proposition 6.3 (iii), we have $\zeta_1(1)^2 = 0$ if and only if all diagonal Cartan invariants of $A$ are even, by Lemma 3.4. Also, we have

$$\dim(T_1A^\perp)(T_1ZA^\perp) + KA/KA \leq 1.$$  

There is the following dual of Proposition 6.1.

**Proposition 6.4.** Let $n \in \mathbb{N}$. Then $(T_nZA)(T_nZA^\perp) \subseteq KA$.

**Proof.** Let $z \in T_nZA$ and $x \in A/KA$. Then

$$z\kappa_n(x) = \kappa_n(z^{p^n}x) = \kappa_n(0x) = 0.$$  

Thus $(T_nZA)(T_nZA^\perp/KA) = (T_nZA)(\text{Im } \kappa_n) = 0$, and the result follows.

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References

8. J.C. Murray, Blocks of defect zero and products of elements of order $p$, J. Algebra 214 (1999), 385-399