Consistency problems for term structure models driven by Wiener process and Poisson measures

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Introduction

• Stochastic partial differential equations (SPDEs)

\[
\frac{dr_t}{dt} = (A r_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt)
\]

in Hilbert spaces.

• Connection: SPDEs ↔ Term structure models.
  – HJMM equation.

• Consistency problems: \( r_t \in P \subset H, \ t \geq 0 \).
  – Positivity preserving models, finite dimensional realizations.
Zero Coupon Bonds

- Zero Coupon Bonds \( P(t, T) \).

- Financial assets paying the holder one unit of cash at \( T \).

Figure 1: Price process of a \( T \)-bond with date \( T = 10 \).
**Extended HJM Methodology**

- For every $T \in \mathbb{R}_+$ we have the forward rates

$$f(t, T) = f^*(0, T) + \int_0^t \alpha(s, T)ds + \int_0^t \sigma(s, T)dW_s$$

$$+ \int_0^t \int_E \gamma(s, x, T)(\mu(ds, dx) - F(dx)ds), \quad t \in [0, T].$$

- Implied bond market:

$$P(t, T) = \exp \left( - \int_t^T f(t, s)ds \right).$$
Literature on HJM models

- Forward rates for $T \in \mathbb{R}_+$ are given by

$$f(t, T) = f^*(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \sigma(s, T) dW_s$$

$$+ \int_0^t \int_{E} \gamma(s, x, T)(\mu(ds, dx) - F(dx)ds), \quad t \in [0, T].$$

- Heath, Jarrow, Morton (see [15]): Brownian motion.

- Björk, Kabanov, Runggaldier, Di Masi (see [1, 2]): Poisson measures.

- Eberlein et al. (see, e.g., [8]): Lévy processes. $\gamma(t, x, T) = x\sigma(t, T)$. 
Musiela parametrization

- **Musiela parametrization** of forward rates:

  \[ r_t(\xi) := f(t, t + \xi), \quad \xi \geq 0. \]

- Then \((r_t)_{t \geq 0}\) is one stochastic process with values in \(H\), that is

  \[ r : \Omega \times \mathbb{R}_+ \to H, \]

  where \(H\) is the space of forward curves \(r : \mathbb{R}_+ \to \mathbb{R}\).
Financial modeling

• Drift and volatilities depend on the current forward curve, that is:

\[ \alpha : H \rightarrow H, \]
\[ \sigma : H \rightarrow H, \]
\[ \gamma : H \times E \rightarrow H. \]

• We assume that \( \alpha = \alpha_{\text{HJM}} : H \rightarrow H \) is given by

\[
\alpha_{\text{HJM}}(h) := \sigma(h) \int_0^* \sigma(h)(\eta)d\eta - \int_E \gamma(h, x) \left( e^{-\int_0^* \gamma(h, x)(\eta)d\eta} - 1 \right) F(dx).
\]
The forward curve evolution \((r_t)_{t \geq 0}\) satisfies

\[
r_t = S_th_0 + \int_0^t S_{t-s} \alpha_{\text{HJM}}(r_s) ds + \int_0^t S_{t-s} \sigma(r_s) dW_s
\]

\[
+ \int_0^t \int_E S_{t-s} \gamma(r_s, x) (\mu(ds, dx) - F(dx) ds), \quad t \geq 0
\]

where \(h_0\) denotes the initial forward curve,

and \((S_t)_{t \geq 0}\) is the shift-semigroup \(S_th := h(t + \cdot)\) on \(H\).
The HJMM equation

• Thus, \((r_t)_{t \geq 0}\) is a mild solution of the SPDE

\[
dr_t = \left( \frac{d}{d\xi} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t \\
+ \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt),
\]

• where \(\frac{d}{d\xi}\) becomes the generator of the \(C_0\)-semigroup of shifts \((S_t)_{t \geq 0}\).

• Existence of a solution for the HJMM equation?
Overview

- Existence result for SPDEs \((Method of the moving frame)\).

- Application to the HJMM equation.

- Consistency problems:
  - Positivity preserving models.
  - Invariant manifolds.
Strongly continuous semigroups

- Let \((H, \| \cdot \|)\) be a separable Hilbert space.

- Family \(S_t : H \to H, t \geq 0\) of bounded linear operators satisfying
  - \(S_0 = \text{Id}\),
  - \(S_{s+t} = S_s S_t\) for all \(s, t \geq 0\),
  - \(\lim_{t \to 0} S_th = h\) for all \(h \in H\).

- Infinitesimal generator \(A : D(A) \subset H \to H\) given by

\[
A h = \lim_{t \to 0} \frac{S_th - h}{t}.
\]
Stochastic partial differential equations

• Let coefficients be given:

\[ \alpha : H \to H, \]
\[ \sigma : H \to H, \]
\[ \gamma : H \times E \to H. \]

• (Semi-linear) Stochastic partial differential equation:

\[ dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt). \]

• Definition of a solution?
Strong Solutions

- **Strong solution:** We have \( r_t \in \mathcal{D}(A), \ t \geq 0 \) and

\[
    r_t = h_0 + \int_0^t (A r_s + \alpha(r_s)) \, ds + \int_0^t \sigma(r_s) \, dW_s
    + \int_0^t \int_E \gamma(r_{s-}, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\]

- Note that \( \mathcal{D}(A) \) is dense in \( H \).

- In general, this solution concept is too strict.
Weak Solutions

- **Weak solution:** For each \( \zeta \in \mathcal{D}(A^*) \) we have

\[
\langle \zeta, r_t \rangle = \langle \zeta, h_0 \rangle + \int_0^t \left( \langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) \rangle \right) ds + \int_0^t \langle \zeta, \sigma(r_s) \rangle dW_s \\
+ \int_0^t \int_E \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\]

- Note that \( \mathcal{D}(A^*) \) is dense in \( H \).
Mild Solutions

- **Mild solution**: We have the identity

\[
    r_t = S_th_0 + \int_0^t S_{t-s} \alpha(r_s) \, ds + \int_0^t S_{t-s} \sigma(r_s) \, dW_s + \int_0^t \int_E S_{t-s} \gamma(r_s, x) (\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\]

- "Variation of constants formula".
Relations between the solution concepts

• In general we have:

\[ \text{strong} \Rightarrow \text{weak} \Rightarrow \text{mild}. \]

• If $A$ is bounded (in particular if $\dim H < \infty$):

\[ \text{strong} \Leftrightarrow \text{weak} \Leftrightarrow \text{mild}. \]
SPDEs as time-transformed SDEs

• Assume there exist $\mathcal{H}$, $(U_t)_{t \in \mathbb{R}}$ and $\ell, \pi$ such that the diagram

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{U_t} & \mathcal{H} \\
\uparrow \ell & \quad & \downarrow \pi \\
H & \xrightarrow{S_t} & H
\end{array}
\]

commutes for every $t \in \mathbb{R}_+$.

• If $(S_t)_{t \geq 0}$ is pseudo-contractive, i.e. there exists $\omega \in \mathbb{R}$ such that

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0$$

then this assumption is fulfilled. \textit{(Sköokefalvi-Nagy Theorem)}
Assumptions

• Assume \( \int_E \| \gamma(0, x) \|^2 F(dx) < \infty \).

• Assume there exists \( L > 0 \) such that

\[
\| \alpha(h_1) - \alpha(h_2) \| \leq L \| h_1 - h_2 \|, \\
\| \sigma(h_1) - \sigma(h_2) \| \leq L \| h_1 - h_2 \|, \\
\left( \int_E \| \gamma(h_1, x) - \gamma(h_2, x) \|^2 F(dx) \right)^{1/2} \leq L \| h_1 - h_2 \|
\]

for all \( h_1, h_2 \in H \).
Existence Result

• Then, there exists a unique mild and weak solution for the SPDE

\[ dr_t = (Ar_t + \alpha(r_t))\,dt + \sigma(r_t)\,dW_t + \int_E \gamma(r_{t-}, x)(\mu\,dt, dx) - F(dx\,dt). \]

• Method of the moving frame:
  – SPDE on \( H \rightsquigarrow \) SDE on \( \mathcal{H} \), via \( r \mapsto S_{-t}r \) \( (\text{jump to the moving frame}) \),
  – Solve the transformed SDE on \( \mathcal{H} \) (Banach fixed point theorem),
  – Transform the solution by \( r \mapsto S_t r \) \( (\text{leave the moving frame}) \).
Space of forward curves

• Let $\beta > 0$ be arbitrary.

• Let $H_\beta$ be the space of absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ with
  \[
  \|h\|_\beta := \left( |h(0)|^2 + \int_{\mathbb{R}_+} |h'(\xi)|^2 e^{\beta \xi} d\xi \right)^{1/2} < \infty.
  \]

• $(H_\beta, \| \cdot \|_\beta)$ is a separable Hilbert space.

• The shift semigroup $(S_t)_{t \geq 0}$, $S_t h = h(t + \cdot)$ is strongly continuous on $H_\beta$. 
Extension to a group

Let $\mathcal{H}_\beta$ be the space of absolutely continuous functions $h : \mathbb{R} \to \mathbb{R}$ with

$$\|h\|_\beta := \left( |h(0)|^2 + \int_{\mathbb{R}} |h'(\xi)|^2 e^{\beta |\xi|} d\xi \right)^{1/2} < \infty.$$ 

$(\mathcal{H}_\beta, \| \cdot \|_\beta)$ is also a separable Hilbert space.

The shift group $(U_t)_{t \in \mathbb{R}}, U_t h = h(t + \cdot)$ is strongly continuous on $\mathcal{H}_\beta$. 
Diagram commutes

• Then the diagram

\[
\begin{array}{ccc}
\mathcal{H}_\beta & \xrightarrow{U_t} & \mathcal{H}_\beta \\
\uparrow \ell & & \downarrow \pi \\
H_\beta & \xrightarrow{S_t} & H_\beta
\end{array}
\]

commutes for every \( t \in \mathbb{R}_+ \),

• where \( \ell : H_\beta \to \mathcal{H}_\beta \) and \( \pi : \mathcal{H}_\beta \to H_\beta \) are given by

\[
\ell(h)(\xi) = \begin{cases} 
h(0), & \xi < 0 \\
h(\xi), & \xi \geq 0
\end{cases}
\text{ and } \pi(h) = h|_{\mathbb{R}_+}.
\]
Solution of the HJMM equation

- Recall the HJM drift condition:

\[ \alpha_{\text{HJM}}(h) = \sigma(h) \int_{0}^{\bullet} \sigma(h)(\eta)d\eta - \int_{E} \gamma(h, x) \left(e^{-\int_{0}^{\bullet} \gamma(h,x)(\eta)d\eta} - 1\right) F(dx). \]

- We assume that \( \sigma: H_{\beta} \to H_{\beta}, \)
  \( \gamma: H_{\beta} \times E \to H_{\beta} \)

are Lipschitz and *bounded*.

- Then \( \alpha_{\text{HJM}}: H_{\beta} \to H_{\beta} \) is also Lipschitz.
**Existence Result for the HJMM equation**

- Hence, there exists a unique mild and weak solution for the HJMM equation

\[
dr_t = \left( \frac{d}{d\xi} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx) dt).
\]

- The result applies in particular to Lévy term structure models

\[
dr_t = \left( \frac{d}{d\xi} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sum_{j=1}^{d} \sigma_j(r_{t-}) dX_t^j
\]

with Lévy processes \( X^1, \ldots, X^d \).
Consistency problems

- **Given:** An SPDE on a Hilbert space $H$:

$$dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt),$$

- and a subset $P \subset H$.

- **Stochastic Invariance:** For each starting point $r_0 \in P$ we have

$$r_t \in P, \quad t \geq 0.$$
Examples of invariant sets

• Let $H_\beta$ be the space of forward curves and $P$ be the convex cone

$$P = \bigcap_{\xi \in \mathbb{R}_+} \{ h \in H_\beta : h(\xi) \geq 0 \}.$$  

(Positivity preserving models)

• Let $\dim H = \infty$ (e.g. $H = H_\beta$) and $P$ be a finite dimensional submanifold

$$P = \mathcal{M}.$$  

(Finite dimensional realizations)
Positivity preserving equations

• Let $H$ be a function space and $P = \bigcap_{\xi \in \mathbb{R}_+} \{ h \in H : h(\xi) \geq 0 \}$.

• The SPDE is positivity preserving iff for all $\xi \in \mathbb{R}_+$ we have

$$\int_E |\gamma(h, x)(\xi)| F(dx) < \infty, \quad \text{if } h(\xi) = 0$$

$$\alpha(h)(\xi) - \int_E \gamma(h, x)(\xi) \geq 0, \quad \text{if } h(\xi) = 0$$

$$\sigma(h)(\xi) = 0, \quad \text{if } h(\xi) = 0$$

$$h + \gamma(h, x) \in P, \quad h \in P.$$
Positivity preserving term structure models

- Let $H_{\beta}$ be the forward curve space and

$$P = \bigcap_{\xi \in \mathbb{R}_+} \{h \in H_{\beta} : h(\xi) \geq 0\}. \quad (Positive \ forward \ curves)$$

- The HJMM equation is positivity preserving iff for all $\xi \in \mathbb{R}_+$ we have

$$\sigma(h)(\xi) = 0, \quad if \ h(\xi) = 0$$

$$\gamma(h, x)(\xi) = 0 \quad for \ F\text{-almost all } x \in E, \quad if \ h(\xi) = 0$$

$$h + \gamma(h, x) \in P, \quad h \in P.$$
Submanifolds with boundary

- Let $\mathcal{M} \subset H$ be a finite dimensional $C^2$-submanifold with boundary of $H$.
- In particular, we think of affine parametrizations

$$\phi(t, y) = \psi(t) + \sum_{i=1}^{m} y_i h_i, \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m.$$ 

- Then $\mathcal{M}$ is a submanifold with boundary

$$\partial \mathcal{M} = \left\{ \psi(0) + \sum_{i=1}^{m} y_i h_i : y \in \mathbb{R}^m \right\}.$$
Consistency conditions

- The manifold $\mathcal{M}$ belongs to the domain:

$$\mathcal{M} \subset \mathcal{D}(A). \quad (1)$$

- The volatility $\sigma$ must be tangential:

$$\sigma(h) \in \begin{cases} T_h \mathcal{M}, & h \in \mathcal{M} \setminus \partial \mathcal{M} \\ T_h \partial \mathcal{M}, & h \in \partial \mathcal{M}. \end{cases} \quad (2)$$

- The manifold $\mathcal{M}$ must capture the jumps:

$$h + \gamma(h, x) \in \mathcal{M}, \quad (h, x) \in \mathcal{M} \times \text{supp}(F). \quad (3)$$
Consistency conditions at the boundary

- Small jumps must be of finite variation, unless parallel to $\partial \mathcal{M}$:

\[
\int_{J(h)} |\langle e_1, \phi^{-1}(h + \gamma(h, x)) \rangle| F(dx) < \infty, \quad h \in \partial \mathcal{M}.
\]  

- The map $\phi : V \subset \mathbb{R}^m_+ \rightarrow U \cap \mathcal{M}$ denotes a parametrization around $h$.

- The set $J(h)$ is defined as

\[
J(h) := \{x \in E : h + \gamma(h, x) \in U\}, \quad h \in \mathcal{M}.
\]

- *Independent* of the choice of the parametrization.
Consistency condition for the drift

- The first natural guess is the condition

\[ Ah + \alpha(h) - \frac{1}{2} D\sigma(h)\sigma(h) - \int_E \gamma(h, x) F(dx) \]

\[ \in \begin{cases} T_h \mathcal{M}, & h \in \mathcal{M} \setminus \partial \mathcal{M} \\ (T_h \mathcal{M})_+, & h \in \partial \mathcal{M}. \end{cases} \]

- **Problem:** In general, the integral diverges:

\[ \int_E \|\gamma(h, x)\| F(dx) = \infty. \]
Appropriate consistency condition for the drift

- Compensated drift must be tangential:
  \[
  \rho_U(h) := Ah + \alpha(h) - \frac{1}{2} D\sigma(h)\sigma(h) - \int_{E \setminus J(h)} \gamma(h, x) F(dx) \\
  - \int_{J(h)} \left( \gamma(h, x) - D\phi(y)(\phi^{-1}(h + \gamma(h, x)) - y) \right) F(dx) \in T_h\mathcal{M}, \quad h \in \mathcal{M}.
  \]

- First coordinate of the tangent space for boundary points:
  \[
  \lim_{U \downarrow \{h\}} \langle e_1, D\phi^{-1}(y) \rho_U(h) \rangle \geq 0, \quad h \in \partial\mathcal{M}.
  \]

- Hence, \( \rho_U(h) \) is not too far from \((T_h\mathcal{M})_+\) for \( h \in \partial\mathcal{M} \).
Consequences for invariant manifolds

- The manifold $\mathcal{M}$ is invariant iff we have (1)–(6).

- Existence of *strong* solutions, because $\mathcal{M} \subset \mathcal{D}(A)$.

- The operator $A$ is continuous on $\mathcal{M}$.

- For $(h, x) \in \mathcal{M} \times E$ with $\gamma(h, x) = 0$ we have

$$D\gamma(h, x)v = \begin{cases} T_h\mathcal{M}, & h \in \mathcal{M} \setminus \partial\mathcal{M} \\ (T_h\mathcal{M})_+, & h \in \partial\mathcal{M} \end{cases}$$

in direction $v \in E$ with $\{x + tv \mid t \in [0, \epsilon]\} \subset \text{supp}(F)$. 
The Lévy case

- Let $\mu = \mu^{(X_1, X_2)}$, where $\int_{-1}^{1} |x| F_1(dx) = \infty$ and $\int_{-1}^{1} |x| F_2(dx) < \infty$.

- Taylor expansion around $x = 0$:

  $$\gamma(h, x) = \delta_1(h)x_1 + \delta_2(h)x_2 + \Delta(h, x).$$

  $F$-integrable

- Define the vector field $\beta : D(A) \rightarrow H$ as

  $$\beta(h) := Ah + \alpha(h) - \frac{1}{2} D\sigma(h)\sigma(h) - \int_{\mathbb{R}^2} (\delta_2(h)x_2 + \Delta(h, x)) F(dx).$$
Invariance Result

- The manifold $\mathcal{M}$ is invariant iff we have

\[
\mathcal{M} \subset \mathcal{D}(A),
\]

\[
\sigma(h) \in \begin{cases} 
T_h \mathcal{M}, & h \in \mathcal{M} \setminus \partial \mathcal{M} \\
T_h \partial \mathcal{M}, & h \in \partial \mathcal{M}
\end{cases}
\]

\[
h + \gamma(h, x) \in \mathcal{M}, \quad (h, x) \in \mathcal{M} \times \text{supp}(F)
\]

\[
\delta_1(h) \in T_h \partial \mathcal{M}, \quad h \in \partial \mathcal{M}.
\]

\[
\beta(h) \in \begin{cases} 
T_h \mathcal{M}, & h \in \mathcal{M} \setminus \partial \mathcal{M} \\
(TM)_{+}, & h \in \partial \mathcal{M}
\end{cases}
\]

- In this case, $A$ and $\beta$ are continuous on $\mathcal{M}$ and $r$ is a strong solution.
Stochastic invariance of closed, convex sets

- Stochastic invariance for SDEs

\[ dZ_t = \alpha(Z_t)dt + \sigma(Z_t)dW_t + \int_E \gamma(Z_{t-}, x)(\mu(dt, dx) - F(dx)dt). \]

- Suppose $\mathcal{M}$ is a closed, convex set in $\mathbb{R}^m$. Then

\[ \mathcal{M} = \bigcap_{z \in \partial \mathcal{M}} \{ y \in \mathbb{R}^m : \langle \eta_z, y \rangle \geq b_z \} \]

is an intersection of half spaces.
Invariance Result

• The closed, convex set $\mathcal{M}$ is invariant iff we have

$$\int_{E} |\langle \eta_z, \gamma(z, x) \rangle| F(dx) < \infty, \quad z \in \partial \mathcal{M}$$

$$\langle \eta_z, \alpha(z) - \frac{1}{2} D\sigma(z)\sigma(z) \rangle - \int_{E} \langle \eta_z, \gamma(z, x) \rangle F(dx) \geq 0, \quad z \in \partial \mathcal{M}$$

$$\langle \eta_z, \sigma(z) \rangle = 0, \quad z \in \partial \mathcal{M}$$

$$z + \gamma(z, x) \in \mathcal{M}, \quad (z, x) \in \mathcal{M} \times \text{supp}(F).$$

• The vector $\eta_z \in \mathbb{R}^m$ denotes the inward pointing normal vector of $T_z \mathcal{M}$. 
Conclusion

- Forward curve evolutions as solutions of an SPDE (the HJMM equation)

\[ dr_t = \left( \frac{d}{d\xi} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt). \]

- Consistency problems.

- Current research:
  - Existence of invariant manifolds.
  - Application to the HJMM equation.
References


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