Introduction to Malliavin calculus and Applications to Finance

Part I

Giulia Di Nunno

Finance and Insurance, Stochastic Analysis and Practical Methods

Spring School
Marie Curie ITN - Jena 2009
Introduction

The mathematical theory now known as Malliavin calculus was first introduced by Paul Malliavin in 1978, as an infinite-dimensional integration by parts technique. The purpose of this calculus was to prove results about the smoothness of densities of solutions of stochastic differential equations driven by Brownian motion. For several years this was the only known application.

In 1984, Ocone obtained an explicit interpretation of the Clark representation formula in terms of the Malliavin derivative (Clark-Ocone formula). In 1991 Ocone and Karatzas applied this result to finance: They proved that the Clark-Ocone formula can be used to obtain explicit formulae for replicating portfolios of contingent claims in complete markets.

Since then Malliavin calculus has been applied in various domains within finance and outside of it. In the meanwhile the very potentials in applications created the need for an extension of the calculus to other types of noise than Brownian motion.
PROGRAM

PART I:

▶ Elements of Malliavin calculus for Brownian motion
▶ Clark-Ocone formula and hedging in complete markets
▶ Sensitivity analysis: the Greek delta

PART II:

▶ Lévy processes and Poisson random measures
▶ Elements of Malliavin calculus for compensated Poisson random measures
▶ Minimal variance hedging in incomplete markets

PART III:

▶ Information, anticipative calculus and stochastic control
▶ Optimal portfolio selection and information
PART I

1. Elements of Malliavin calculus for Brownian motion
   • Iterated Itô integrals and Hermite polynomials
   • Wiener-Itô chaos expansions
   • Skorohod integral
   • Malliavin derivative
   • Fundamental rules of calculus

2. Clark-Ocone formula and hedging in complete markets
   • Clark-Ocone formula
   • A Black-Scholes type market model
   • Hedging
   • Hedging of Markovian setting: \( \Delta \)-hedging

3. Sensitivity analysis: the Greeks
   • Computation of the Greeks: the \( \Delta \)

References
1. Elements of Malliavin calculus for Brownian motion

We choose to introduce the operators Malliavin derivative and Skorohod integral via chaos expansions. Other, basically equivalent, approach is to use directional derivatives on the Wiener space, see e.g. Da Prato (2007), Malliavin (1997), Nualart (2006), Sanz-Solé (2005).

Let $W(t) = W(\omega, t), \omega \in \Omega, t \in [0, T] (T > 0)$, be a Brownian motion on the complete probability space $(\Omega, \mathcal{F}, P)$ such that $W(0) = 0$ $P$-a.s. For any $t$, let $\mathcal{F}_t$ be the $\sigma$-algebra generated by $W(s), 0 \leq s \leq t$, augmented by all the $P$-zero measure events. The resulting (continuous) filtration is denoted

$$\mathbb{F} = \{\mathcal{F}_t, \ t \geq 0\}.$$
Iterated Itô integrals

Let $f$ be a deterministic function defined on

$$S_n = \{(t_1, \ldots, t_n) \in [0, T]^n : \ 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T\} \ (n \geq 1)$$

such that $\|f\|_{L^2(S_n)}^2 := \int_{S_n} f^2(t_1, \ldots, t_n) dt_1 \cdots dt_n < \infty$.

Definition

The $n$-fold iterated Itô integrals are given by:

$$J_n(f) := \int \int \cdots \int f(t_1, \ldots, t_n) dW(t_1)dW(t_2) \cdots dW(t_{n-1})dW(t_n).$$

We set $J_0(f) := f$, for $f \in \mathbb{R}$.

Directly from the properties of Itô integrals we have:

- $J_n(f) \in L^2(P)$, by the Itô isometry $\|J_n(f)\|_{L^2(P)}^2 = \|f\|_{L^2(S_n)}^2$.

- If $g \in L^2(S_m)$ and $f \in L^2(S_n)$ ($m < n$), then $E\left[ J_m(g)J_n(f) \right] = 0$. 
Let $f \in \tilde{L}^2([0, T]^n)$, i.e. $f$ is a symmetric square integrable functions.

**Definition**

We also called *n-fold iterated Itô integral* the random variable:

$$I_n(f) := \int_{[0, T]^n} f(t_1, \ldots, t_n)dW(t_1) \ldots dW(t_n) := n!J_n(f).$$

About symmetric functions:

- The function $f : [0, T]^n \to \mathbb{R}$ is *symmetric* if $f(t_{\sigma_1}, \ldots, t_{\sigma_n}) = f(t_1, \ldots, t_n)$ for all permutations $\sigma$ of $(1, 2, \ldots, n)$.
- If $f$ is a real function on $[0, T]^n$, then the *symmetrization* $\tilde{f}$ of $f$ is

$$\tilde{f}(t_1, \ldots, t_n) := \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \ldots, t_{\sigma_n}),$$

where the sum is taken over all permutations $\sigma$ of $(1, \ldots, n)$. Naturally, $\tilde{f} = f$ if and only if $f$ is symmetric.
- If $f \in \tilde{L}^2([0, T]^n)$, then $\|f\|^2_{L^2([0, T]^n)} = n!\|f\|^2_{L^2(S_n)}$. 

Iterated Itô integrals and Hermite polynomials

The **Hermite polynomials** \( h_n(x), x \in \mathbb{R}, n = 0, 1, 2, \ldots \) are defined by

\[
h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} \left( e^{-\frac{1}{2}x^2} \right), \quad n = 0, 1, 2, \ldots,
\]

Recall that the family of Hermite polynomials constitute an orthogonal basis for \( L^2(\mathbb{R}, \mu(dx)) \) if \( \mu(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \) (see e.g. Schoutens (2000)).

Note that:

\[
n! \int_{0}^{T} dW(t_1) \int_{0}^{t_1} \cdots \int_{0}^{t_{n-1}} g(t_1)g(t_2) \cdots g(t_n) dW(t_1) \cdots dW(t_n) = \|g\|^n h_n \left( \frac{\theta}{\|g\|} \right),
\]

where \( \|g\| = \|g\|_{L^2([0, T])} \) and \( \theta = \int_{0}^{T} g(t)dW(t) \).
Example

Let \( g \equiv 1 \) and \( n = 3 \), then we get

\[
6 \int_0^T \int_0^{t_3} \int_0^{t_2} 1 \, dW(t_1)dW(t_2)dW(t_3) = T^{3/2} h_3 \left( \frac{W(T)}{T^{1/2}} \right)
\]

\[
= W^3(T) - 3TW(T).
\]

In fact the first Hermite polynomials are:

\[
\begin{align*}
    h_0(x) & = 1, \\
    h_1(x) & = x, \\
    h_2(x) & = x^2 - 1, \\
    h_3(x) & = x^3 - 3x, \\
    h_4(x) & = x^4 - 6x^2 + 3, \ldots
\end{align*}
\]
The computation of the iterated Itô integrals is based on:

**Proposition**

If $\xi_1, \xi_2, \ldots$ are orthonormal functions in $L^2([0, T])$, we have that

\[
I_n(\xi_1 \otimes^{\alpha_1} \hat{\otimes} \cdots \hat{\otimes} \xi_m \otimes^{\alpha_m}) = \prod_{k=1}^{m} h_{\alpha_k} \left( \int_0^T \xi_k(t) W(t) \right),
\]

with $\alpha_1 + \cdots + \alpha_m = n$ and $\alpha_k \in \{0, 1, 2, \ldots\}$ for all $k$.

Recall that the *tensor product* $f \otimes g$ of two functions $f, g$ is defined by

\[
(f \otimes g)(x_1, x_2) = f(x_1)g(x_2)
\]

and the *symmetrized tensor product* $f \hat{\otimes} g$ is the symmetrization of $f \otimes g$. 

Wiener-Itô chaos expansions

**Theorem**
Let $\xi$ be an $\mathcal{F}_T$-measurable random variable in $L^2(P)$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \tilde{L}^2([0, T]^n)$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n),$$

where the convergence is in $L^2(P)$. Moreover, since

$$\|I_n(f_n)\|_{L^2(P)}^2 = n! \|f_n\|_{L^2([0, T]^n)}^2,$$

we have the isometry

$$\|\xi\|_{L^2(P)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2([0, T]^n)}^2.$$

**Example.** The chaos expansion of $\xi = \exp\{W(T) - \frac{1}{2} T\}$ is given by

$$\xi = \sum_{n=0}^{\infty} \frac{t^{n/2}}{n!} h_n\left(\frac{W(T)}{\sqrt{T}}\right).$$
Skorohod integral

Let $u(\omega, t), \omega \in \Omega, t \in [0, T]$, be a measurable stochastic process such that, for all $t \in [0, T], u(t)$ is a $\mathcal{F}_T$-measurable random variable and $E[u^2(t)] < \infty$.
Then for each $t \in [0, T]$ we can apply the Wiener-Itô chaos expansion to the random variable $u(t) := u(\omega, t), \omega \in \Omega$:

$$u(t) = \sum_{n=0}^{\infty} \lambda_n(f_{n,t}) \quad (f_{n,t} \in \tilde{L}^2([0, T]^n)).$$

The functions $f_{n,t}, n = 1, 2, \ldots$, depend on $t \in [0, T]$ as parameter. We can define $f_n(t_1, \ldots, t_n, t_{n+1}) := f_{n,t}(t_1, \ldots, t_n)$ as a function of $n + 1$ variables.
Its symmetrization $\tilde{f}_n$ is then given by

$$\tilde{f}_n(t_1, \ldots, t_{n+1}) = \frac{1}{n+1} \left[ f_n(t_1, \ldots, t_{n+1}) + f_n(t_2, \ldots, t_{n+1}, t_1) + \cdots + f_n(t_1, \ldots, t_{n-1}, t_{n+1}, t_n) \right]$$
Definition
Let $u(t), t \in [0, T]$, be a measurable stochastic process such that, for all $t$, the random variable $u(t)$ is $\mathcal{F}_T$-measurable and $E\left[\int_0^T u^2(t)dt\right] < \infty$. Let its Wiener-Itô chaos expansion be

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)) \quad (f_n(\cdot, t) \in \tilde{L}^2([0, T]^n)).$$

Then we define the Skorohod integral of $u$ by

$$\delta(u) := \int_0^T u(t)\delta W(t) := \sum_{n=0}^{\infty} l_{n+1}(\tilde{f}_n),$$

when convergent in $L^2(P)$ (here $\tilde{f}_n$ is the symmetrization of $f_n(\cdot, t)$).

Moreover,

$$\|\delta(u)\|^2_{L^2(P)} = \sum_{n=0}^{\infty} (n + 1)! \|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty.$$
Example Let us compute $\int_0^T W(T)\delta W(t)$.

The Wiener-Itô chaos expansion of the integrand is given by

$$u(t) = W(T) = \int_0^T 1\,dW(s) = I_1(1), \quad t \in [0, T],$$

i.e. for all $t$, $f_{0,t} \equiv 0$, $f_{1,t} \equiv 1$ and $f_{n,t} \equiv 0$ for all $n \geq 2$. Hence

$$\delta(u) = I_2(\tilde{f}_1) = I_2(1) = 2 \int_0^T \int_0^{t_2} dW(t_1)dW(t_2) = W^2(T) - T.$$

Note that, even if the integrand does not depend on $t$, we have

$$\int_0^T W(T)\delta W(t) \neq W(T) \int_0^T \delta W(t).$$
Some basic properties of the Skorohod integral

- The Skorohod integral is a linear operator
- \( E[\delta(u)] = 0 \)
- In general, if \( G \) is an \( \mathcal{F}_T \)-measurable random variable such that \( u, Gu \in \text{Dom}(\delta) \), we have that
  \[
  \int_0^T Gu(t) \delta W(t) \neq G \int_0^T u(t) \delta W(t).
  \]

**Theorem: Skorohod integral as extension of the Itô integral**

Let \( u(t), t \in [0, T] \), be a measurable \( \mathbb{F} \)-adapted stochastic process such that \( E\left[ \int_0^T u^2(t)dt \right] < \infty \).

Then \( u \) is both Itô and Skorohod integrable and

\[
\int_0^T u(t) \delta W(t) = \int_0^T u(t) dW(t).
\]
Malliavin derivative

**Definition**

Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable and have chaos expansion

$$F = \sum_{n=0}^{\infty} l_n(f_n)$$

where $f_n \in \tilde{L}^2([0, T]^n)$.

(i) We say that $F \in \mathbb{D}_{1,2}$, if $\|F\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=1}^{\infty} nn! \|f_n\|_{L^2([0, T]^n)}^2 < \infty$.

(ii) For any $F \in \mathbb{D}_{1,2}$, we define the *Malliavin derivative* $D_t F$ of $F$ at time $t$ as the expansion

$$D_t F := \sum_{n=1}^{\infty} n l_{n-1}(f_n(\cdot, t)), \quad t \in [0, T],$$

where $l_{n-1}(f_n(\cdot, t))$ is the $(n-1)$-fold iterated integral of $f_n(t_1, \ldots, t_{n-1}, t)$ with respect to the first $n-1$ variables $t_1, \ldots, t_{n-1}$ and we leave $t_n = t$ as parameter.

Note that $\|D.F\|_{L^2(P \times \lambda)}^2 = \|F\|_{\mathbb{D}_{1,2}}^2 < \infty$, thus the derivative $D_t F$ is well-defined as an element of $L^2(P \times \lambda)$. 
Theorem: Closability.
Suppose $F \in L^2(P)$ and $F_k \in D_{1,2}$, $k = 1, 2, \ldots$, such that

(i) $F_k \longrightarrow F$, $k \to \infty$, in $L^2(P)$

(ii) $\{D_tF_k\}_{k=1}^{\infty}$ converges in $L^2(P \times \lambda)$. 

Then $F \in D_{1,2}$ and $D_tF_k \longrightarrow D_tF$, $k \to \infty$, in $L^2(P \times \lambda)$.

Proof. Let $F = \sum_{n=0}^{\infty} l_n(f_n)$ and $F_k = \sum_{n=0}^{\infty} l_n(f^{(k)}_n)$, $k = 1, 2, \ldots$

Then by (i)

$$f^{(k)}_n \longrightarrow f_n, \quad k \to \infty, \quad \text{in } L^2(\lambda^n) \quad \forall n.$$ 

By (ii) we have

$$\sum_{n=1}^{\infty} nn! \|f^{(k)}_n - f^{(j)}_n\|_{L^2(\lambda^n)}^2 = \|D_tF_k - D_tF_j\|_{L^2(P \times \lambda)}^2 \longrightarrow 0, \quad j, k \to \infty.$$ 

Hence, by the Fatou lemma,

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} nn! \|f^{(k)}_n - f_n\|_{L^2(\lambda^n)}^2 \leq \lim_{k \to \infty} \lim_{j \to \infty} \sum_{n=1}^{\infty} nn! \|f^{(k)}_n - f^{(j)}_n\|_{L^2(\lambda^n)}^2 = 0.$$ 

This yields $F \in D_{1,2}$ and $D_tF_k \longrightarrow D_tF$, $k \to \infty$, in $L^2(P \times \lambda)$. \qed
Fundamental rules of calculus

Let $F = \int_0^T f(s) dW(s)$, where $f \in L^2([0, T])$. Then

- $D_tF = f(t)$;
- $D_t(F^n) = nF^{n-1}D_tF = nF^{n-1}f(t)$.

**Theorem: Chain rule.**

Let $F \in \mathbb{D}_{1,2}$ and $\varphi$ be a continuously differentiable function with bounded derivative. Then $\varphi(F) \in \mathbb{D}_{1,2}$ and

$$D_t\varphi(F) = \varphi'(F)D_tF.$$

(The chain rule can be extended to the case $\varphi$ Lipschitz).
Let $G$ be Borel in $[0, T]$. Define $\mathcal{F}_G \subseteq \mathcal{F}_T$ as the completed $\sigma$-algebra generated by $\int_0^T 1_A(t)dW(t)$, for all Borel sets $A \subseteq G$. If $F \in \mathbb{D}_{1,2}$, then $E[F|\mathcal{F}_G] \in \mathbb{D}_{1,2}$ and

$$D_tE[F|\mathcal{F}_G] = E[D_tF|\mathcal{F}_G] \cdot 1_G(t).$$

In particular, if $u$ is an $\mathbb{F}$-adapted stochastic process with $u(s) \in \mathbb{D}_{1,2}$ for all $s$. Then $D_t u(\cdot)$ is $\mathbb{F}$-adapted for all $t$ and

$$D_t u(s) = 0, \text{ for } t > s.$$ 

In fact, $D_t u(s) = D_t E[u(s)|\mathcal{F}_s] = E[D_t u(s)|\mathcal{F}_s] \cdot 1_{[0,s]}(t)$. 


The Skorohod integral is the adjoint operator to the Malliavin derivative:

**Theorem: Duality formula**

Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable and let $u(t), t \in [0, T],$ be a Skorohod integrable process. Then

$$E \left[ F \int_0^T u(t) \delta W(t) \right] = E \left[ \int_0^T u(t) D_t F \, dt \right].$$

By the duality formula it is easy to see that the Skorohod integral is a closed operator.
The duality formula is at the core of the proof of the integration by parts formula for the Skorohod integral and Malliavin derivative.

**Theorem: Integration by parts**

Let \( u(t), t \in [0, T], \) be a Skorohod integrable stochastic process and \( F \in \mathbb{D}_{1,2} \) such that the product \( Fu(t), t \in [0, T], \) is Skorohod integrable. Then

\[
F \int_0^T u(t) \delta W(t) = \int_0^T Fu(t) \delta W(t) + \int_0^T u(t) D_t F dt.
\]
The fundamental theorem of calculus.

Let $u$ be a stochastic process such that $E\left[\int_0^T u^2(s)ds\right] < \infty$ and assume:

1. $u(s) \in D_{1,2}$ for all $s \in [0, T]$,
2. $D_t u \in \text{Dom}(\delta)$ for all $t \in [0, T]$,
3. $E\left[\int_0^T (\delta(D_t u))^2 dt\right] < \infty$.

Then $\delta(u) := \int_0^T u(s)\delta W(s)$ is well-defined, belonging to $D_{1,2}$, and

$$D_t \delta(u) := D_t \left( \int_0^T u(s)\delta W(s) \right) = \int_0^T D_t u(s)\delta W(s) + u(t).$$

In particular, if $u$ is $\mathbb{F}$-adapted, then

$$D_t \left( \int_0^T u(s)dW(s) \right) = \int_T^t D_t u(s)dW(s) + u(t).$$
2. Clark-Ocone formula and hedging in complete markets

The Clark-Ocone formula is a representation theorem for square integrable random variables in terms of Itô stochastic integrals in which the integrand is explicitly characterized in terms of the Malliavin derivative.

**Theorem: Clark-Ocone formula.**
Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable. Then

$$F = E[F] + \int_0^T E[D_tF|\mathcal{F}_t] \, dW(t).$$

**Remark.** The formula can only be applied to random variables in $\mathbb{D}_{1,2}$. Extensions beyond this domain to the whole $L^2(P)$ are possible in the white noise framework.

Other Itô integral representations exist where the integrand is given in terms of the *non-anticipating derivative*. This operator is defined on the whole $L^2(P)$. See e.g. Di Nunno (2002, 2007). Some rules of calculus are given for this operator, however much has still to be discovered.
Proof. Write $F = \sum_{n=0}^{\infty} l_n(f_n)$ with $f_n \in \tilde{L}^2([0, T]^n)$, $n = 1, 2, \ldots$. Hence,

$$
\int_0^T E[D_t F | \mathcal{F}_t] dW(t) = \int_0^T E\left[ \sum_{n=1}^{\infty} n l_{n-1}(f_n(\cdot, t)) | \mathcal{F}_t \right] dW(t)
$$

$$
= \int_0^T \sum_{n=1}^{\infty} n E[l_{n-1}(f_n(\cdot, t)) | \mathcal{F}_t] dW(t)
$$

$$
= \int_0^T \sum_{n=1}^{\infty} n l_{n-1}[f_n(\cdot, t) \cdot \chi_{[0,t]}(\cdot)] dW(t)
$$

$$
= \int_0^T \sum_{n=1}^{\infty} n(n-1)! J_{n-1}[f_n(\cdot, t) \chi_{[0,t]}(\cdot)] dW(t)
$$

$$
= \sum_{n=1}^{\infty} n! J_n[f_n(\cdot)] = \sum_{n=1}^{\infty} l_n[f_n]
$$

$$
= \sum_{n=0}^{\infty} l_n[f_n] - l_0[f_0] = F - E[F].
$$
A Black-Scholes type market model

Market model.

risk free asset \[
\begin{cases}
  dS_0(t) = \rho(t)S_0(t)dt \\
  S_0(0) = 1
\end{cases}
\]

risky asset \[
\begin{cases}
  dS_1(t) = S_1(t)[\mu(t)dt + \sigma(t)dW(t)] \\
  S_1(0) = x > 0.
\end{cases}
\]

Here \(\rho(t) > 0, \mu(t)\) and \(\sigma(t), t \in [0, T]\), are \(\mathbb{F}\)-adapted processes such that
\[
E\left[\int_0^T \{|\rho(t)| + |\mu(t)| + \sigma^2(t)\} dt\right] < \infty.
\]

Moreover, let \(\sigma(t) \neq 0, t \in [0, T]\).
Let \( \theta_0(t), \theta_1(t), t \in [0, T], \) denote the number of units invested at time \( t \) in the two investment possibilities, respectively. Then the corresponding value at time \( t, V^\theta(t), t \in [0, T], \) of this portfolio \( \theta(t) := (\theta_0(t), \theta_1(t)), t \in [0, T], \) is given by

\[ V^\theta(t) = \theta_0(t)S_0(t) + \theta_1(t)S_1(t). \]

The portfolio \( \theta(t) \) is called self-financing if

\[ dV^\theta(t) = \theta_0(t)dS_0(t) + \theta_1(t)dS_1(t). \]

Since

\[ \theta_0(t) = \frac{V^\theta(t) - \theta_1(t)S_1(t)}{S_0(t)}, \]

the value process dynamics can be equivalently written as

\[ dV^\theta(t) = \rho(t)(V^\theta(t) - \theta_1(t)S_1(t))dt + \theta_1(t)dS_1(t) \]
\[ = [\rho(t)V^\theta(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t)]dt + \sigma(t)\theta_1(t)S_1(t)dW(t). \]
Let $Q \sim P$ be a risk-neutral probability measure for the market model.

**Definition**

A claim $F$ is an $\mathcal{F}_T$-measurable random variable such that $E_Q[F^2] < \infty$. The claim $F$ is *replicable* if there exists a self-financing portfolio $\theta(t)$, $t \in [0, T]$, such that

- $V^\theta(T) = F$
- $\bar{V}^\theta(t) := \frac{V^\theta(t)}{S_0(t)}$, $t \in [0, T]$, is an $\mathbb{F}$-martingale under $Q$.

**Definition**

A market model is *complete* if all claims are replicable, otherwise it is called *incomplete*.

Recall that completeness of the market corresponds to the uniqueness of the risk-neutral measure.

**Remark.** The market model under consideration is complete.
Hedging

Hedging problem
Given the (replicable) claim $F$, the problem is to find explicitly the stochastic process $\theta_1(t)$, $t \in [0, T]$, such that $(V^\theta(t), \theta_1(t))$, $t \in [0, T]$ solves the (BSDE):

$$dV^\theta(t) = \left[ \rho(t)V^\theta(t) + (\mu(t) - \rho(t))\theta_1(t)S_1(t) \right] dt + \sigma(t)\theta_1(t)S_1(t)dW(t)$$

$$V^\theta(T) = F.$$ 

Then the portfolio $\theta(t) = (\theta_0(t), \theta_1(t))$, $t \in [0, T]$, where

$$\theta_0(t) = \frac{V^\theta(t) - \theta_1(t)S_1(t)}{S_0(t)},$$

is then the replicating or hedging portfolio.

Remark. The general theory of BSDEs (conditions on $F$, $\rho$, $\mu$, and $\sigma$ provided) guarantees the existence and uniqueness of the solution to this problem. However computational effectiveness is reduced to the case of Markovian set-up (see later). This is when integral representation theorems as the Clark-Ocone formula come into good use.
Let us consider
\[ u(t) := \frac{\mu(t) - \rho(t)}{\sigma(t)}, \quad t \in [0, T], \]
and assume that
\[ Z(t) = \exp \left\{ - \int_0^t u(s) dW(s) - \frac{1}{2} \int_0^t u^2(s) ds \right\}, \quad t \in [0, T], \]
is an \( \mathbb{F} \)-martingale. Then, by the Girsanov theorem,
\[ \tilde{W}(t) = \int_0^t u(s) ds + W(t), \quad t \in [0, T], \]
is a Brownian motion with respect to \( \mathbb{F} \) under the measure \( Q \sim P \) given by
\[ dQ := Z(T) dP \quad \text{on} \ (\Omega, \mathcal{F}_T). \]
In this setting we can write the dynamics of \( V^\theta(t), \ t \in [0, T] \), under \( Q \) as:
\[ dV^\theta(t) = \rho(t)V^\theta(t)dt + \sigma(t)\theta_1(t)S_1(t)d\tilde{W}(t). \]
Thus the original BSDE under $Q$ becomes

$$
d\tilde{V}^\theta(t) := d\left(S_0^{-1}(t) V^\theta(t)\right)
= e^{-\int_0^t \rho(s)ds} \sigma(t) \theta_1(t) S_1(t) d\tilde{W}(t)
$$

$$
\tilde{V}^\theta(T) = S_0^{-1}(T) F = e^{-\int_0^T \rho(s)ds} F =: G.
$$

Hence,

$$
G = \tilde{V}^\theta(0) + \int_0^T \varphi(t) d\tilde{W}(t)
$$

where

$$
\varphi(t) := e^{-\int_0^t \rho(s)ds} \sigma(t) \theta_1(t) S_1(t).
$$

The solution can be given by applying the Clark-Ocone formula with respect to the Brownian motion $\tilde{W}$ under $Q$.

However, there is need for carefulness!
Remark. The given Clark-Ocone formula is applicable to random variables $G$ that are measurable with respect to the filtration generated by the noise. If $\tilde{F}$ is the $(P \sim Q)$- augmented filtration generated by $\tilde{W}$, then we have that, in general,

$$\tilde{F}_t \subset F_t \quad \text{with} \quad \tilde{F}_t \neq F_t.$$ 

Need for a Clark-Ocone formula fitting this setting!
Theorem: Clark-Ocone formula under change of measure
Suppose $G \in \mathcal{D}_{1,2}$ is $\mathcal{F}_T$-measurable and

$$E_Q[|G|] < \infty$$

$$E_Q \left[ \int_0^T |D_t G|^2 dt \right] < \infty$$

$$E_Q \left[ |G| \int_0^T \left( \int_0^T D_t u(s) d\tilde{W}(s) + \int_0^T u(s) D_t u(s) ds \right)^2 dt \right] < \infty$$

where the measure $dQ = Z(T) dP$ is the one given by the Girsanov theorem with $u(t) := \frac{\mu(t) - \rho(t)}{\sigma(t)}$, $t \in [0, T]$.

Then

$$G = E_Q[G] + \int_0^T E_Q \left[ (D_t G - G \int_t^T D_t u(s) d\tilde{W}(s)) |\mathcal{F}_t \right] d\tilde{W}(t).$$
Hence the hedging problem can be solved by application of the Clark-Ocone formula under change of measure to the random variable:

\[ G := e^{-\int_0^T \rho(s) ds} F. \]

Then the uniqueness of the integrands

\[ \varphi(t) = e^{-\int_0^t \rho(s) ds} \sigma(t) \theta_1(t) S_1(t) = E_Q \left[ (D_t G - G \int_t^T D_t u(s) d\tilde{W}(s)) | F_t \right] \]

yields the final result:

\[ V^\theta(t) = e^{\int_0^t \rho(s) ds} \left[ V^\theta(0) + \int_0^t e^{-\int_0^r \rho(r) dr} \sigma(s) \theta_1(s) S_1(s) d\tilde{W}(s) \right] \]

with

\[ \theta_1(t) = e^{\int_0^t \rho(s) ds} \sigma^{-1}(t) S_1^{-1}(t) E_Q \left[ (D_t G - G \int_t^T D_t u(s) d\tilde{W}(s)) | F_t \right] \]

and

\[ \bar{V}^\theta(0) = V^\theta(0) = E_Q[G]. \]
We try out the method above to give an alternative proof to the classical Black-Scholes formula.

**Example** We consider a classical Black-Scholes market where \( \rho(t) = \rho > 0, \mu(t) = \mu \) and \( \sigma(t) = \sigma \neq 0, t \in [0, T] \), are constants. Then

\[
    u = \frac{\mu - \rho}{\sigma} \quad \text{and} \quad D_t u = 0.
\]

Let us consider the European call option

\[
    F = (S_1(T) - K)^+ \in \mathbb{D}_{1,2}
\]

In this case the formula for \( \theta_1(t), t \in [0, T] \), reduces to

\[
    \theta_1(t) = e^{\rho(t-T)}\sigma^{-1}S_1^{-1}(t) E_Q[D_tF \mid \mathcal{F}_t].
\]

So we only need to compute \( D_tF \).
The function \( f(x) = (x - K)^+ \) is not differentiable at \( x = K \), so we cannot use the chain rule directly to evaluate \( D_t F \). However, we can approximate \( f \) by \( C^1(\mathbb{R}) \) functions \( f_n \) with the property that

\[
f_n(x) = f(x) \quad \text{for} \quad |x - K| \geq \frac{1}{n}
\]

and

\[
0 \leq f'_n(x) \leq 1 \quad \text{for all} \ x.
\]

We can apply the chain rule to the random variables \( F_n := f_n(S_1(T)) \). Recalling that the Malliavin derivative is a closed operator we have:

\[
D_t F = \lim_{n \to \infty} D_t F_n \\
= \lim_{n \to \infty} f'_n(S_1(T))D_t S_1(T) \\
= \chi_{[K, \infty]}(S_1(T))D_t S_1(T) \\
= \chi_{[K, \infty]}(S_1(T))S_1(T)\sigma.
\]

Hence

\[
\theta_1(t) = e^{\rho(t-T)}S_1^{-1}(t)E_Q \left[ S_1(T)\chi_{[K, \infty]}(S_1(T))|\mathcal{F}_t \right].
\]
To complete the computation, we can continue the evaluation of the conditional expectation thanks to the Markov property of $S_1(t)$, $t \in [0, T]$:

\[
\theta_1(t) = e^{\rho(t-T)} S_1^{-1}(t) E_Q \left[ S_1(T) \chi_{[K, \infty]}(S_1(T)) | S_1(t) \right] \\
= e^{\rho(t-T)} S_1^{-1}(t) E_Q^{y} [S_1(T-t) \chi_{[K, \infty]}(S_1(T-t))] | y = S_1(t)
\]

where $E_Q^{y}$ is the expectation when $S_1(0) = y$.

Under $Q$ the dynamics of $S_1(t)$ are written:

\[
dS_1(t) = \rho S_1(t) dt + \sigma S_1(t) d\tilde{W}(t),
\]

thus

\[
S_1(t) = S_1(0) \exp \left\{ \left( \rho - \frac{1}{2} \sigma^2 \right) t + \sigma \tilde{W}(t) \right\}
\]

where $\tilde{W}$ is a Brownian motion under $Q$.

Moreover, we can see that

\[
V^{\theta}(0) = E_Q[e^{-\rho T} F] \\
= e^{-\rho T} E_Q[(S_1(T) - K)^+].
\]
Hedging of Markovian setting: $\Delta$-hedging

Market model.

risk free asset
\[
\begin{cases}
    dS_0(t) = \rho S_0(t)dt \\
    S_0(0) = 1
\end{cases} 
\quad (\rho > 0)
\]

risky asset
\[
\begin{cases}
    dS_1(t) = S_1(t)[\mu(S_1(t))dt + \sigma(S_1(t))dW(t)] \\
    S_1(0) = x > 0.
\end{cases}
\]

Suppose we want to replicate an $\mathcal{F}_T$-measurable Markovian payoff
\[
F = \varphi(S_1(T))
\]

where, for convenience, $\varphi$ is considered bounded.
To this end, we have to find a portfolio $\theta(t) = (\theta_0(t), \theta_1(t))$, $t \in [0, T]$, such that

$$V^\theta(T) := \theta_0(T)S_0(T) + \theta_1(T)S_1(T) = \varphi(S_1(T))$$

and it is self-financing

$$dV^\theta(t) = \theta_0(t)S_0(t)\rho dt + \theta_1(t)dS_1(t).$$

Seen the structure of the market model, it is reasonable to suppose that there is a function $f(t, x)$, $t \in [0, T]$, $x > 0$, such that

$$V^\theta(t) = f(t, S_1(t)), \quad \forall t \in [0, T], \quad \text{and} \quad f(T, S_1(T)) = \varphi(S_1(T)).$$

We assume that $f$ is $C^{1,2}$. Then by the Itô formula we have

$$dV^\theta(t) = \frac{\partial f}{\partial t}(t, S_1(t))dt + \frac{\partial f}{\partial x}(t, S_1(t))dS_1(t)$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_1(t))\sigma^2(S_1(t))S_1^2(t)dt.$$

Comparing with the self-financing condition we obtain two equations.
One corresponding to the “dt” terms:

$$\theta_0(t)S_0(t)\rho + \theta_1(t)S_1(t)\mu(S_1(t))$$

$$= \frac{\partial f}{\partial t}(t, S_1(t)) + \frac{\partial f}{\partial x}(t, S_1(t))S_1(t)\mu(S_1(t)) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, S_1(t))\sigma^2(S_1(t))S_1^2(t)$$

and one corresponding to the “dW(t)” terms:

$$\theta_1(t)\sigma(S_1(t))S_1(t) = \frac{\partial f}{\partial x}(t, S_1(t))\sigma(S_1(t))S_1(t).$$

This last one holds if and only if

$$\theta_1(t) = \frac{\partial f}{\partial x}(t, S_1(t)).$$

which actually gives the form of the $\Delta$-hedge.
Substituting the result above into the first equation we see that the function $f$ must satisfy the Black-Scholes equation

$$
\begin{aligned}
\frac{\partial f}{\partial t} (t, x) - \rho f(t, x) + \rho x \frac{\partial f}{\partial x} (t, x) + \frac{1}{2} \sigma^2(x) x^2 \frac{\partial^2 f}{\partial x^2} (t, x) &= 0, \quad t < T, \\
\end{aligned}
$$

The solution of this equation can be achieved using the Feynman-Kac formula. Then we have

$$
f(t, S_1(t)) = E^x \left[ e^{-\rho(T-t)} \varphi(X(T - t)) \right]_{x=S_1(t)}
$$

where $X(t) = X^x(t), \ 0 \leq t \leq T$, is the solution of the stochastic differential equation

$$
dX(t) = X(t) \left[ \rho dt + \sigma(X(t)) dW(t) \right]; \quad X(0) = x.
$$

Note that law of the process $X$ under $P$ is the same as the low of $S_1$ under the risk-neutral probability measure $Q \sim P$. Thus we could have written:

$$
f(t, S_1(t)) = E^x_Q \left[ e^{-\rho(T-t)} \varphi(S_1(T - t)) \right]_{x=S_1(t)}
$$
The ∆-hedge is then given by:

\[ \theta_1(t) = \frac{\partial f}{\partial x}(t, S_1(t)) = e^{-\rho(T-t)} \frac{\partial}{\partial x} E^x_Q[\varphi(S_1(T - t))] \big|_{x=S_1(t)} \]

\[ = e^{-\rho(T-t)} \frac{\partial}{\partial x} E^x[\varphi(X(T - t))] \big|_{x=S_1(t)}. \]

This indicates the importance of the computation of quantities given in the form

\[ \Delta = \frac{\partial}{\partial x} E^x[\varphi(X(T))]. \]

This topic merges with the problem of computation of the so-called Greek ∆.
3. Sensitivity analysis: the “Greeks”

The “Greeks” are quantities representing the market sensitivities of financial derivatives to the variation of the model parameters. In this sense they are important tools in risk management and hedging. The name “Greeks” was given because these quantities are often denoted by Greek letters.

If \( V_t, t \in [0, T] \), is the payoff process of some derivative, we have, e.g.,

- “delta” measures the sensitivity to changes in the initial price \( x \) of the underlying asset: \( \Delta = \frac{\partial V}{\partial x} \) (important for hedging purposes)
- “gamma” measures the rate of change in the delta: \( \Gamma = \frac{\partial^2 V}{\partial x^2} \)
- “rho” measures the sensitivity to the applicable interest rate \( r \): \( \rho = \frac{\partial V}{\partial r} \)
- “theta” measures the sensitivity to the amount of time to expiration date: \( \Theta = -\frac{\partial V}{\partial T} \)
- “vega” (indicated by \( \nu \)) measures the sensitivity to volatility \( \sigma \): \( \nu = \frac{\partial V}{\partial \sigma} \)
Computation of the Greeks: the ∆

Though very interesting quantities to be considered, in many cases, the Greeks cannot be expressed in closed form and require numerical methods for the computation. Qualitatively speaking, being $V$ computed as an expectation, the Greeks are basically derivatives of expectations.

From Glynn (1989) we read that one of the most flexible methods is the application of Monte Carlo simulation on top of a finite difference approximation of the derivatives. However, this contains intrinsically two kinds of errors: one on the approximation of the derivatives and the other on the numerical computation of the expectations. In particular, the most of the inefficiency is revealed when dealing with discontinuous payoffs e.g. digital and exchange options.
Other methods are in use to overcome a generally poor convergence rate.

First Curran (1994) suggested taking the differential of the payoff process inside the expectation. Then Broadie and Glasserman (1996) introduced the method of differentiation of the density function, moving in this way the differentiation from the payoff function to the density function and they introduced the so-called likelihood ratio, e.g.

\[
\Delta = \frac{\partial}{\partial x} E^x [\varphi(X(T))] = E \left[ \varphi(X^x(T)) \frac{\partial}{\partial x} \ln p(X^x(T)) \right].
\]

This method is very efficient, but has however the disadvantage of requiring an explicit expression of the density function.

Here we will present the method introduced by Fournié et al. (1999) based on Malliavin calculus.
Consider a general Itô diffusion $X^x(t), \ t \geq 0,$ given by

$$dX^x(t) = b(X^x(t))dt + \sigma(X^x(t))dW(t), \quad X^x(0) = x \in \mathbb{R},$$

where $b$ and $\sigma$ are given functions in $C^1(\mathbb{R})$ and $\sigma(x) \neq 0,$ for all $x \in \mathbb{R}.$

The first variation process $Y(t) := \frac{\partial}{\partial x} X^x(t), \ t \geq 0,$ is given by

$$dY(t) = b'(X^x(t))Y(t)dt + \sigma'(X^x(t))Y(t)dW(t), \quad Y(0) = 1.$$

Let us define

$$g(x) := E^x[\varphi(X(T))] = E[\varphi(X^x(T))],$$

thus

$$\Delta = g'(x) = \frac{d}{dx} E[\varphi(X^x(T))].$$

Then we have the following result:
Theorem
Let \( a(t), \ t \in [0, T], \) be a continuous deterministic function such that
\[
\int_0^T a(t)dt = 1.
\]

Then
\[
g'(x) = E^x \left[ \varphi(X(T)) \int_0^T a(t)\sigma^{-1}(X(t))Y(t)dW(t) \right].
\]

We call weight: \( w = \int_0^T a(t)\sigma^{-1}(X(t))Y(t)dW(t). \)

Note that:

- There is no differentiation of the payoff function!
- There is no need to know the density function, but we need to know the diffusion
- Weighting function is independent of the payoff
- The formula depends on a function \( a. \) There is an infinity of weights. How to choose \( a? \)
Proof. The proof is organized in three steps.

**Step A.** First of all we prove that

\[
D_s X(t) = \frac{Y(t)}{Y(s)} \sigma(X(s)) \chi_{[0,t]}(s).
\]

**Proof of Step A.** Recall that

\[
X(t) = x + \int_0^t b(X(u)) du + \int_0^t \sigma(X(u)) dW(u).
\]

Then the fundamental theorem of calculus gives

\[
D_s X(t) = \int_s^t b'(X(u)) D_s X(u) du + \int_s^t \sigma'(X(u)) D_s X(u) dW(u) + \sigma(X(s)), \quad t \geq s,
\]

or equivalently,

\[
dD_s X(t) = b'(X(t)) D_s X(t) dt + \sigma'(X(t)) D_s X(t) dW(t)
\]

\[
D_s X(s) = \sigma(X(s)).
\]
This is a SDE in $D_sX(t)$, $t \in [s, T]$, which has solution

$$D_sX(t) = \sigma(X(s))e^{\int_s^t \left[b'(X(u)) - \frac{1}{2}\left(\sigma'(X(u))\right)^2\right]du + \int_s^t \sigma'(X(u))dW(u)}, \quad t \geq s.$$ 

Then the proof of this step is complete:

$$D_sX(t) = \sigma(X(s))\frac{Y(t)}{Y(s)}, \quad t \geq s.$$
Step B. Now we show that

\[ Y(T) = \int_0^T D_sX(T)a(s)\sigma^{-1}(X(s))Y(s)ds, \]

Proof of Step B. Take Step A, with \( t = T \) and rearrange the terms, then we have

\[ Y(T) = D_sX(T)Y(s)\sigma(X(s))^{-1}, \quad s \in [0, T]. \]

Hence

\[ Y(T) = \int_0^T Y(T)a(s)ds = \int_0^T D_sX(T)a(s)\sigma^{-1}(X(s))Y(s)ds. \]

Step C. First assume that \( \varphi \) is smooth. By the application of Step A, the chain rule and the duality formula, we obtain
\[ g'(x) = E\left[ \varphi'(X^x(T)) \frac{d}{dx} X^x(T) \right] = E\left[ \varphi'(X^x(T)) Y(T) \right] \]
\[ = E^x \left[ \int_0^T \varphi'(X(T)) D_s X(T) a(s) \sigma^{-1} X(s)) Y(s) ds \right] \]
\[ = E^x \left[ \int_0^T D_s (\varphi(X(T))) a(s) \sigma^{-1} X(s)) Y(s) ds \right] \]
\[ = E^x \left[ \varphi(X(T)) \int_0^T a(s) \sigma^{-1} (X(s)) Y(s) dW(s) \right]. \]

This completes the proof when \( \varphi \) is smooth.
The general case is treated by approximation:

\[ \varphi_m \rightarrow \varphi, \quad m \rightarrow \infty, \]

where \( \varphi_m \) are smooth functions bounded a.e. on \([0, T]\) and the convergence is pointwise. Define

\[ g_m(x) := E^x[\varphi_m(X(T))]. \]

Then by Step B we have

\[ g'_m(x) = E^x[\varphi_m(X(T))\Lambda], \quad m = 1, 2, \ldots, \]

where \( \Lambda = \int_0^T a(s)\sigma^{-1}(X(s)) Y(s) dW(s) \). Hence

\[ \lim_{m \rightarrow \infty} g'_m(x) = E^x[\varphi(X(T))\Lambda] =: h(x). \]

Thus

\[ g(x) = \lim_{m \rightarrow \infty} g_m(x) = \lim_{m \rightarrow \infty} g_m(0) + \int_0^x g'_m(t) dt = g(0) + \int_0^x h(t) dt. \]

Then \( g \) is differentiable and \( g'(x) = h(x) = E^x[\varphi(X(T))\Lambda]. \)
Example

Suppose

\[ dX(t) = \rho X(t)dt + \sigma_0 X(t)dW(t), \]

with \( \rho \) and \( \sigma_0 \neq 0 \) constants. Choose

\[ a(t) = \frac{1}{T}, \quad t \in [0, T]. \]

Then

\[ g'(x) = E^x \left[ \varphi(X(T)) \frac{W(T)}{x\sigma_0 T} \right]. \]

To see this, we can observe that in this case we have \( b(x) = \rho x \), \( \sigma(x) = \sigma_0 x \) and hence the first variation process is

\[ dY(t) = \rho Y(t)dt + \sigma_0 Y(t)dW(t), \quad Y(0) = 1. \]

Therefore \( g'(x) = E^x \left[ \varphi(X(T)) \int_0^T a(t)\sigma^{-1}(X(t))Y(t)dW(t) \right] \), where

\[ \sigma^{-1}(X(t))Y(t) = \sigma_0^{-1}x^{-1}Y^{-1}(t)Y(t) = \frac{1}{\sigma_0 x}. \]
Other comments

- Avellaneda et al. (2000) also worked in the direction of moving the differentiation from the payoff function to the inclusion of a weighting function. They propose another way of finding the weighting function inspired by the Kullback-Leibler relative entropy maximization.

- Benhamou (2003) and related works studies how to characterize and choose the weighs.
  - His study indicates that the weights can be expressed as Skorohod integrals and one can introduce the concept of a “weighting function generator”.
  - As for the choice of the weights, he focuses on the ones providing the minimum variance of the random variable: \( \varphi(X(T)) \cdot w. \)
  - It turns out that the minimum variance weight is the conditional expectation of any weight with respect to \( X(T) \).
  - This result provides also the link with the likelihood ration in the density method.
References

This presentation follows:
G. Di Nunno, B. Øksendal, and F. Proske, Malliavin Calculus for Lévy Processes with Applications to Finance, Springer 2008.

References on the topic at the base of this presentation include:
G. Di Nunno, On orthogonal polynomials and the Malliavin derivative for Lévy stochastic measures, SMF, Séminaires et Congrès. 15 (Nov-Dec 2007).


