How to hedge Asian options in fractional Black-Scholes model

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Outline of the talk

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1. Introduction

- Asian options
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- Fractional Black-Scholes
- Pathwise stochastic integration
- Hedging problem
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Introduction

- Change of variables formulas [Itô formulas] for both arithmetic and geometric averages of geometric fractional Brownian motion.
- Valid for all convex functions, not only for smooth ones.
- Can be used for obtaining hedges (but not prices) for Asian options in fractional Black-Scholes model.
- Explicit hedges in some cases where hedges are not known explicitly even in the ordinary Black-Scholes model.
Asian options

Let \( S(t) \) be the price of the underlying asset. Asian options depend on the time average of the underlying.

The payoff of the arithmetic Asian option is

\[
f \left( \frac{1}{T} \int_0^T S(s) \, ds \right)
\]

and the payoff of the geometric Asian option

\[
f \left( \exp \left( \frac{1}{T} \int_0^T \log S(s) \, ds \right) \right).
\]
Arithmetic Asian options are important in practise.
  - Used for example in commodity markets.

Geometric Asian options are easier to consider analytically e.g. in ordinary Black-Scholes model.

The problem with arithmetic Asian options is that sum of lognormals is not lognormal.

Here we overcome this problem by using pathwise methods.
Fractional Brownian motion

- Fractional Brownian motion (fBM) $B^H$ with Hurst index $H \in (0, 1)$ is a Gaussian process satisfying

$$\mathbb{E}B^H(t) = B^H(0)$$

and having the following covariance structure

$$\text{Cov}(B^H(t), B^H(s)) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}) .$$

- If $H = \frac{1}{2}$, we are in the case of ordinary BM.
For $H > \frac{1}{2}$ the process has long range dependence property and for $H < \frac{1}{2}$ the increments are negatively correlated.

FBM is self-similar with parameter $H$.

FBM is not semi-martingale nor Markov process (unless $H = \frac{1}{2}$).

$B^H$ has Hölder continuous sample paths of any order $\delta \in (0, H)$.

For $H > \frac{1}{2}$, fBM $B^H$ has zero quadratic variation over a sequence of subdivisions where the mesh goes to zero.
The price of the underlying is modeled as $S(t) = \exp B^H(t)$, where $\frac{1}{2} < H < 1$.

$S$ has Hölder continuous sample paths of any order $\delta \in (0, H)$.

$S$ has zero quadratic variation property.

The hedging results remain true if we add any deterministic drift to $B^H$ as long as the path properties (Hölder continuity and quadratic variation) are not changed.
The stochastic integrals considered here are pathwise:

- Riemann-Stieltjes integrals (RS)
- generalized Lebesgue-Stieltjes integrals (gLS)

If not mentioned otherwise the integrals are gLS.
Hedging problem

- Given claim \( F(S) = f((S(s) | s \in [0, T])) \).
- Find adapted \((H(s))_{s \in [0, T]}\) such that
  \[ F(S) = A + \int_0^T H(s) dS(s). \]
  - Integral should be economically justified.
- Separate problem of mathematical finance compared to pricing.
Theorem

Let $f$ be convex function. Then

$$f(S(T)) = f(S(0)) + \int_0^T f'(S(t)) S(t) dB^H(t)$$

in generalized Lebesgue-Stieltjes sense.

- Note that $dS(t) = S(t) dB^H(t)$.
- (Azmoodeh-Mishura-Valkeila 2009).
2. Main results

- Replication of the averages
- Options depending on geometric average
- Options depending on arithmetic average
- Fractional Bachelier model
Replication of geometric average

\[ G(t) = \exp \left( \frac{1}{T} \int_0^t \log S(s) \, ds \right) S(t) \frac{T-t}{T}. \]

Proposition

For all \( t \in [0, T] \) it holds almost surely that

\[ G(t) = S(0) + \int_0^t \frac{T-s}{T} G(s) dB^H(s), \]

in Riemann-Stieltjes sense.
Replication of geometric average (continued)

**Corollary**

*In particular*

\[
\exp \left( \frac{1}{T} \int_0^T B^H(s) \, ds \right) = S(0) + \int_0^T \frac{T - s}{T} \exp \left( \frac{1}{T} \int_0^s B^H(u) \, du + \frac{T - s}{T} B^H(s) \right) dB^H(s).
\]
Replication of arithmetic average

Proposition

For all \( t \in [0, T] \) it holds almost surely that

\[
\frac{T - t}{T} S(t) + \frac{1}{T} \int_{0}^{t} S(s) ds = S(0) + \int_{0}^{t} \frac{T - s}{T} S(s) dB^{H}(s),
\]

in Riemann-Stieltjes sense.

Corollary

In particular

\[
\frac{1}{T} \int_{0}^{T} S(s) ds = S(0) + \int_{0}^{T} \frac{T - s}{T} S(s) dB^{H}(s).
\]
Geometric Asian options

\[ G(t) = \exp \left( \frac{1}{T} \int_{0}^{t} B^H(s) ds \right) S(t)^{\frac{T-t}{T}}. \]

**Theorem**

Let \( f \) be a convex function. Then it holds almost surely that

\[ f(G(t)) = f(S(0)) + \int_{0}^{t} \frac{T-s}{T} f'(G(s)) G(s) dB^H(s), \]

where the stochastic integral in the right side is understood in generalized Lebesgue-Stieltjes sense.
Geometric Asian options (continued)

Corollary

*In particular,*

\[
 f \left( \exp \left( \frac{1}{T} \int_0^T B^H(s) \, ds \right) \right) \\
 = f(S(0)) + \int_0^T \frac{T - s}{T} f'(G(s)) \, G(s) \, dB^H(s).
\]
**Theorem**

Let $f$ be a convex function. Then it holds almost surely that

$$f \left( \frac{T - t}{T} S(t) + \frac{1}{T} \int_0^t S(s) ds \right)$$

$$= f(S(0)) + \int_0^t f' \left( \frac{T - s}{T} S(s) + \frac{1}{T} \int_0^s S(u) du \right) \frac{T - s}{T} S(s) dB^H(s),$$

where the stochastic integral in the right side is understood in the sense of generalized Lebesgue-Stieltjes integral.
Corollary

In particular,

\[
f \left( \frac{1}{T} \int_0^T S(s) ds \right) = f(S(0)) + \int_0^T f'(s) \left( \frac{T - s}{T} S(s) + \frac{1}{T} \int_0^s S(u) du \right) \frac{T - s}{T} S(s) dB^H(s).
\]
Fractional Bachelier model

The case of arithmetic average can be written also when the geometric price process $S$ is replaced by a fractional Brownian motion $B^H$ with $H \in \left(\frac{1}{2}, 1\right)$. In that case we obtain for a convex function $f$ that

$$f \left( \frac{T - t}{T} B^H(t) + \frac{1}{T} \int_0^t B^H(s) ds \right)$$

$$= f(B^H(0)) + \int_0^t \frac{T - s}{T} f' \left( \frac{T - s}{T} B^H(s) + \frac{1}{T} \int_0^s B^H(u) du \right) dB^H(s)$$

almost surely as a generalized Lebesgue-Stieltjes integral.
3. Methodology

- Pathwise philosophy
- Functional change of variables formula
- Fractional Besov space techniques and generalized Lebesgue-Stieltjes integral
Pathwise philosophy

- Stochastic integrals are pathwise.
  - In practice one observes the path, not distribution.
- Drift can be added, if path properties do not change.
  - In practice one does not observe the drift.
- Arithmetic Asian options can be considered.
  - The problem in classical Black-Scholes is that sum of lognormals is not lognormal anymore.
  - Here it does not matter.
Let $B^H_t = (B^H(s) | s \in (0, t))$ be the whole path of $B^H$ up to $t$. Then by (Cont-Fournié 2010)

$$F_T(B^H_T) = F_0(B^H(0)) + \int_0^T \mathcal{D}_t F_t(B^H_u) du + \int_0^T \partial_x F_u(B^H_u) dB^H(u)$$

$$\left( + \int_0^T \frac{1}{2} \partial^2_x F_t(B^H_u) d[B^H](u) \right)$$

in Riemann-Stieltjes sense.

- $F_t$ is called non-anticipative functional and $(F_t)$ is so called non-anticipative flow.
- $\mathcal{D}_t$ is horizontal and $\partial_x$ vertical derivative.
- $[\cdot]$ denotes the pathwise quadratic variation.
In the fBM setup the quadratic variation term vanishes.

In the case of geometric Asian options, change of variables formula is applied to non-anticipative functional

\[ F_t(x_t) = \exp \left( \frac{1}{T} \int_0^t x(s) ds \right) e^{\frac{T-t}{T} x(t)}. \]

And in the case of arithmetic Asian options

\[ F_t(x_t) = \frac{T - t}{T} e^{x(t)} + \frac{1}{T} \int_0^t e^{x(s)} ds. \]
Horizontal derivative

The horizontal extension of $x_t$ for $h > 0$ is defined as

$$x_{t,h}(u) = x(u), \quad u \in [0, t]$$

and

$$x_{t,h}(u) = x(t), \quad u \in (t, t + h].$$

Now the horizontal derivative of $F$ at $x \in C([0, T])$ is defined as

$$\mathcal{D}_t F(x) = \lim_{h \downarrow 0} \frac{F_{t+h}(x_{t,h}) - F_t(x)}{h},$$

if the limit exists.
The vertical perturbation of path $x_t$ is defined for $h \in \mathbb{R}$ as

$$x_t^h(u) = x(u), \quad u \in [0, t) \quad \text{and} \quad x_t^h(t) = x(t) + h.$$ 

The vertical derivative is defined in the following way. A non-anticipative functional $F$ is vertically differentiable at $x \in C([0, t])$ if limit

$$\partial_x F_t(x) = \lim_{h \to 0} \frac{F_t(x_t^h) - F_t(x)}{h}$$

exists.
Fractional Besov spaces

Definition

Let \( f : [0, T] \mapsto \mathbb{R} \) be measurable. Then \( f \in W_1^\beta ([0, T]) \) if

\[
\| f \|_{1, \beta} = \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t - s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u - s)^{\beta+1}} \, du \right) < \infty.
\]

Definition

Let \( f : [0, T] \mapsto \mathbb{R} \) be measurable. Then \( f \in W_2^\beta ([0, T]) \) if

\[
\| f \|_{2, \beta} = \int_0^T \frac{|f(t)|}{t^\beta} \, dt + \int_0^T \int_0^t \frac{|f(t) - f(s)|}{(t - s)^{\beta+1}} \, ds \, dt < \infty.
\]
Let $\beta \in (0, 1)$. Generalized Lebesgue-Stieltjes integral is defined as

$$\int_0^t f dg := \int_0^t \left( D_0^\beta f \right)(x) \left( D_{t-}^{1-\beta} g_{t-} \right)(x) dx.$$ 

Operators $D_0^\beta$ and $D_{t-}^\beta$ are Riemann-Liouville fractional derivatives.

- $f \in W_2^\beta ([0, T]), \ g \in W_1^{1-\beta} ([0, T]).$
- $g_{t-}(x) = (g(t-) - g(x)) 1_{(0,t)}(x).$
Remark
\[ \int_0^s + \int_s^t = \int_0^t. \]

Remark
*The integral is the same for all \( \beta \) for which it can be defined.*

Remark
*For \( 0 < \beta < H \) the trajectories of fBm \( B^H \) belong to \( W^{\beta}_1([0, T]) \) almost surely by Hölder continuity.*
Convergence theorem

Theorem

Let \( f, (f_n)_{n=1}^\infty \in W_2^\beta ([0, T]) \) and \( g \in W_1^{1-\beta} ([0, T]) \). If

\[ \| f_n - f \|_{2,\beta} \rightarrow 0, \]

then

\[ \int_0^t f_n dg \rightarrow \int_0^t f dg, \]

in generalized Lebesgue-Stieltjes sense for all \( t \in (0, T] \).
4. Conclusions

- Extended the functional Itô formula of (Cont-Fournié 2010) for non-smooth convex functions in the special case of driving gfBm or fBm and functional depending on the average of the driving process.

- Obtained hedging strategies for Asian options in fractional Black-Scholes model. We were able to find hedges also for Asian options depending on the ordinary arithmetic average. Explicit hedges for such options are not known even in the case of Black-Scholes model.
Conclusions (continued)

- In the case of Asian options, fBm behaves as continuous function of bounded variation. However, this is not the case for all path-dependent options: see for example the case of lookback options (Azmoodeh-Tikanmäki-Valkeila 2010).

- Some related models behave differently. For example in exponential mixed Brownian motion and fractional Brownian motion market model the hedges of Asian options are the same as in ordinary Black-Scholes model, (Bender-Sottinen-Valkeila 2008).
References


Thanks for your attention!