Credit Risk Premia and quadratic BSDEs with a single jump

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Credit Risk Premium $c$ defined via maximal expected utility

$$V^\xi(0) = \sup \left\{ EU(0 + G_T + \xi) \right\} = V^\xi_{1\{\tau > T\}}(c).$$

$\xi$ defaultable contingent claim, $\tau$ default time
$U(x) = -\exp(-\eta x)$ with risk aversion coef $\eta > 0$
New class of BSDE

Solve the BSDE with quadratic growth generators and jumps at random times

\[ Y_t = F - \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s) \, ds, \]

\( W \) multidimensional Brownian Motion, \( M \) the compensated default process \( 1\{\tau>t\} \) with a single jump and \( f \) has a quadratic growth in \( z \).

The aims are

- Existence and uniqueness established by a simple constructive algorithm
- \( V^F(v) = U(v - Y_0) \).
Outline

1. The model
2. Quadratic BSDEs with one possible jump
3. Credit Risk premia
Defaultfree market

k risky assets, 1 riskless asset (numeraire).
Probability space $(\Omega, \mathcal{F}, P)$.
Prices of risky asset

$$dS_t^i = S_t^i(\alpha_i(t)dt + \sigma_i(t)dW_t), \quad i = 1, \ldots, k,$$

- $W$ a d-dimensional Brownian motion and $\mathcal{F}_t$ the filtration generated by $W$,
- $\alpha_i(t)$ is the $i$th component of a predictable and vector-valued map $\alpha : \Omega \times [0, T] \rightarrow \mathbb{R}^k$.
- $\sigma_i(t)$ is the $i$th row of a predictable and matrix-valued map $\sigma : \Omega \times [0, T] \rightarrow \mathbb{R}^{k \times d}$

AOA and completeness : standard hypotheses.
Defaultable Contingent Claim

- Default time $\tau$: a random time
- Available information: $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(1_{\tau \leq t})$
- A defaultable contingent Claim $F$:

$$F = X_1 1_{\tau > T} + X_2(\tau) 1_{\tau \leq T}$$

$X_1$ is a bounded $\mathcal{F}_T$-measurable random variable, $X_2$ is a $\mathcal{F}$-predictable process.

**Hypothesis (H) (Immersion property)**

Any square integrable $(\mathcal{F}, P)$-martingale is a square integrable $(\mathcal{G}, P)$-martingale
Notations:
- $D_t = 1_{\{\tau \leq t\}}$.
- $K$, the compensator of $D$, such that $K_0 = 0$.
- $M_t = D_t - \int_0^t (1 - D_s^-)dK_s$ ($\mathcal{G}_t$)-martingale.

$k_s$ a ($\mathcal{F}_t$)-predictable non-negative bounded process and $A$ a ($\mathcal{F}_t$)-predictable increasing process, $\{0, 1\}$-valued, such that

$$dK_s = k_s ds + dA_s.$$

Remark: $A$ and $M$ do not have any common jump.
Examples

1. \( \tau_1 = \inf\{t, \int_0^t k_s ds > \Theta\} \), where \( k \) a bounded non-negative \( (\mathcal{F}_t) \)-predictable process and \( \Theta \) an exponentially distributed random variable independent of the Brownian Motion \( W \).

\[
dK_t = k_t dt
\]

2. \( \tau = \tau_1 \wedge \tau_2 \) with \( \tau_2 = \inf\{t \geq 0 : S_t^i \leq a \text{ for one } 1 \leq i \leq k\} \)
and \( \tau_1 \) defined as previously.

\[
dK_t = k_t dt + d1_{\tau_2 \leq t}, \text{ in other words } A_t = 1_{\tau_2 \leq t}
\]
Defaultable Zero-Coupon Bond

- Payoff of a defaultable zero-coupon bond is 1 if $\tau > T$, and 0 otherwise.
- Arbitrage free dynamics of a defaultable zero-coupon bond:
  \[
  d\rho_t = \rho_t (a_t dt + c_t dW_t - dM_t)
  \]
  where $(a_t, c_t)$ are $\mathbb{R} \times \mathbb{R}^d$-valued measurable processes.
An investment strategy \((p, q)\) leads to the following gain at time \(t\):

\[
G_{t}^{p,q} = \int_{0}^{t} (p_s \vartheta_s + q_s a_s) \, ds + \int_{0}^{t} (p_s + q_s c_s) \, dW_s - \int_{0}^{t} q_s \, dM_s.
\]

Let \(\mathcal{A}\) denote the set of \textit{admissible strategies} \((p, q)\), satisfying

\[
E \int_{0}^{T} |p_s|^2 \, ds + E \int_{0}^{T} |q_s|^2 \, ds < \infty.
\]
Constraints are imposed to the investor.
A strategy \((p_t, q_t) \in C_t = C^1_t \times C^2_t \subset \mathbb{R}^{k\sigma_t} \times \mathbb{R}^{\rho_t}\) satisfying
\((0, 0) \in C^1_t \times C^2_t \text{ for all } t, \quad C^1_t \text{ is closed, } \quad C^2_t \text{ is bounded.}\)

If the investor has a defaultable position \(F\) in his portfolio, then his maximal expected utility is given by

\[
V^F(v) = \sup_{(p, q) \in A} \left\{ \mathbb{E}U(v + G^{p,q}_T + F) : (p_s, q_s) \in C_s \text{ for all } s \in [0, T] \right\}.
\]
Outline

1. The model

2. Quadratic BSDEs with one possible jump

3. Credit Risk premia
We consider the following BSDE

\[ Y_t = \xi - \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s) \, ds \]

where \( \xi \) is a bounded \( G_T \)-measurable random variable, and the generator \( f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) satisfies the following property:

**P1** There exist 3 predictable functions \( l, j \) and \( m \) such that

\[ f(s, z, u) = [l(s, z) + j(s, u)](1 - D_{s-}) + m(s, z)D_{s-}, \]

- There exists a constant \( L \in \mathbb{R}_+ \) such that \( \forall z, z' \in \mathbb{R}^d \)

\[ |l(s, z) - l(s, z')| + |m(s, z) - m(s, z')| \leq L(1 + |z| + |z'|)|z - z'|, \]
- \( j \geq 0 \), and \( j \) is Lipschitz on \( (-K, \infty) \) for every \( K \in \mathbb{R}_+ \),
- \( l(., 0), m(., 0) \) and \( j(., 0) \) are bounded, say by \( \Lambda \in \mathbb{R}_+ \).
In addition, we will assume sometimes that the generator \( f \) satisfies also

\[ \textbf{(P2)} \quad \text{There exists a continuous increasing function } \gamma \text{ such that for all } s \in [0, T] \text{ and } u, u' \in [-n, n], \ n \in \mathbb{N}, \]

\[ |j(s, u) - j(s, u')| \leq \gamma(n) \sqrt{k_s} |u - u'|, \]
Notations: $(\mathcal{J}_t)$ be an arbitrary filtration.

- $\mathcal{H}^2(\mathcal{J}_t)$ the set of all $(\mathcal{J}_t)$-predictable processes $X_t$ satisfying $E \int_0^T |X_t|^2 ds < \infty$
- $\mathcal{H}^\infty(\mathcal{J}_t)$ the set of essentially bounded $(\mathcal{J}_t)$-predictable processes.
- $\mathcal{R}^\infty(\mathcal{J}_t)$ the set of all bounded $(\mathcal{J}_t)$-optional processes.
A priori estimates

Let \( \xi_1 \) and \( \xi_2 \) two bounded \( \mathcal{G}_T \)-measurable r.v., \( f^1 \) and \( f^2 \) two generators satisfying properties (P1) and (P2), and let
\[
(Y^i, Z^i, U^i) \in \mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)
\]
be solutions of the BSDEs associated. Let
\[
\delta \xi = \xi_1 - \xi_2, \quad \delta f_s = f^1(s, Z^1_s, U^1_s) - f^2(s, Z^1_s, U^1_s),
\]
\[
\delta Y = Y^1 - Y^2, \quad \delta Z = Z^1 - Z^2 \quad \text{and} \quad \delta U = U^1 - U^2.
\]

**Theorem**

There exist constants \( q \geq 1 \) and \( C \in \mathbb{R}_+ \), depending only on \( T, L, \| k \|_\infty, \| \sup_{s \in [0, T]} Y^1_s \|_\infty, \| \sup_{s \in [0, T]} Y^2_s \|_\infty \), the function \( \gamma \) and the \( L^2[0, T] \) norm of \( l(s, 0) \), such that
\[
E \left[ \sup_{t \in [0, T]} \left| \delta Y_t \right|^2 + \left( \int_0^T \left( \left| \delta Z_s \right|^2 + \left| \delta U_s \right|^2 (1 - D_{s-}) \frac{k_s}{2} \right) ds \right) \right] \leq C \left( E \left[ \left| \delta \xi \right|^{2q} + \left( \int_0^T \left| \delta f_s \right| ds \right)^{2q} \right] \right)^{\frac{1}{q}}.
\]
Sketch of the proof

1. Under (P1), show that if $(Y, Z, U) \in \mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)$ is a solution of (1), then $\int_0^\cdot Z_s dW_s$ is a BMO-martingale.

2. Derive a priori estimates with respect to the auxiliary measure $Q$, defined by $\frac{dQ}{dP} = \mathcal{E}(H \cdot W)_T$, where

$$H_s = \frac{f^2(s, Z^1_s, U^1_s) - f^2(s, Z^2_s, U^1_s)}{\delta Z_s}.$$ 

3. Use BMO-martingale properties and Hölder’s inequality.
**Theorem**

*(Existence)* Let $F = \xi 1_{\{\tau > T\}} + \zeta 1_{\{\tau \leq T\}}$ where $\xi$ and $\zeta$ be two bounded $\mathcal{F}_T$-measurable random variables, and let $f$ be a generator satisfying (P1). Then there exists a solution $(Y, Z, U) \in \mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)$ of

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s) ds.$$
Proof

Main Idea: Construct a solution of BSDE starting from two continuous quadratic BSDEs with terminal conditions $\xi$ and $\zeta$.

1. Take a solution $(\hat{Y}, \hat{Z}) \in \mathcal{H}^\infty(\mathcal{F}_t) \times \mathcal{H}^2(\mathcal{F}_t)$ of the BSDE

$$\hat{Y}_t = \zeta - \int_t^T \hat{Z}_s dW_s + \int_t^T m(s, \hat{Z}_s) ds.$$  \hspace{1cm} (1)

2. Define the stopping time

$$\tau_A = \inf\{ t \geq 0 : A_t = 1 \}, \quad ((\mathcal{F}_t)\text{-predictable})$$

with convention $\inf \emptyset = \infty$. 
Consider a BSDE with generator

\[ h(s, y, z) = l(s, z) + j(s, (\hat{Y}_s - y)1_{\{\tau_A \geq s\}}) + (\hat{Y}_s - y)1_{\{\tau_A \geq s\}} k_s. \]

and terminal condition

\[ \psi = \xi 1_{\{\tau_A > T\}} + \zeta 1_{\{\tau_A \leq T\}}. \]

However \( h \) is not Lipschitz in \( y \).

Let \((Y^g, Z^g) \in \mathcal{H}^\infty(\mathcal{F}_t) \times \mathcal{H}^2(\mathcal{F}_t)\) be a solution of the BSDE with generator \( g \)

\[ g(s, y, z) = l(s, z) + (\hat{Y}_s - y)1_{\{\tau_A \geq s\}} k_s. \]

and terminal condition \( \psi \).

In particular, there exists a \( K \in \mathbb{R}_+ \) such that

\[ \sup_{t \in [0, T]} Y^g_t \geq -K, \text{ a.s.} \]
In addition consider an auxiliary generator $h^a$ satisfying, for all $z \in \mathbb{R}^d$ and $s \in [0, T]$,

$$h^a(s, y, z) = \begin{cases} h(s, y, z), & \text{if } y \in (-K, \infty) \\ g(s, y, z) + j(s, \hat{Y}_s + K) [1 - (y + K)], & \text{else.} \end{cases}$$

and there exists a solution $(Y^a, Z^a) \in \mathcal{H}^\infty(\mathcal{F}_t) \times \mathcal{H}^2(\mathcal{F}_t)$ of the BSDE with generator $h^a$ and terminal condition $\psi$.

We next show that $Y^a_t \geq Y^g_t$ so there exist solutions of BSDEs with bounded terminal condition $\psi$ and generator $h$ denoted by $(Y^a, Z^a)$. 
Finally, find a solution by setting

\[ Y_t = \begin{cases} 
Y_t^a, & (\tau > t) \vee (\tau \leq t, \tau = \tau_A), \\
\hat{Y}_t, & (\tau \leq t, \tau \not\equiv \tau_A), 
\end{cases} \]

\[ Z_t = \begin{cases} 
Z_t^a, & (\tau > t) \vee (\tau \leq t, \tau = \tau_A), \\
\hat{Z}_t, & (\tau \leq t, \tau \not\equiv \tau_A), 
\end{cases} \]

and

\[ U_t = \begin{cases} 
\hat{Y}_t - Y_t^a, & t \leq \tau, \\
0, & t > \tau. 
\end{cases} \]
Theorem

(Existence) Let \((\bar{\zeta}(t))_{0 \leq t \leq T}\) be a \((\mathcal{F}_t)\)-predictable bounded process, such that \(t \mapsto \bar{\zeta}(t)\) is almost surely right-continuous on \([0, T]\). Let \(f\) satisfy (P1) and (P2). Then there exists a solution \((Y, Z, U) \in \mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)\) of the BSDE

\[
Y_t = \xi 1\{\tau > T\} + \bar{\zeta}(\tau) 1\{\tau \leq T\}
- \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s) ds
\]
Proof

1. Assume that the process $A$ is equal to zero, so $dK_s = k_s ds$.
2. Let $\tau_n$, $n \in \mathbb{N}$, be the discrete approximation of the default time $\tau$ defined by

$$
\tau_n(\omega) = \frac{k}{n} \quad \text{if} \quad \tau(\omega) \in \left[ \frac{k-1}{n}, \frac{k}{n} \right], \ k \in \mathbb{Z}_+.
$$
3. Observe that $\tau_n$ is a $(\mathcal{G}_t)$-stopping time.
4. Construct a Cauchy sequence $(Y^n, Z^n, U^n)$ which converges to $(Y, Z, U)$, solution of the BSDE.
(Uniqueness) Let $\xi$ be a bounded $\mathcal{G}_T$-measurable random variable and a generator satisfying $P1$ and $P2$, then the BSDE has a unique solution in $\mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)$. 
Outline

1. The model
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Credit Risk premia

- No tradable defaultable asset and \( C_t = C_t^1 \times \{0\} \).
- \( \xi \) a bounded \( \mathcal{F}_T \)-measurable random variable (value of a position if no default occurs).
- \( V^{\xi}(v) \) and \( V^{\xi^1\{\tau>T\}}(v) \) the maximal expected utility of an investor with initial wealth \( v \), and endowment \( \xi \) and \( \xi^1\{\tau>T\} \) respectively.

**Definition**

Then the indifference credit risk premium is the amount \( c \) defined as the unique real number satisfying

\[
V^{\xi}(0) = V^{\xi^1\{\tau>T\}}(c).
\]

Remark: As \( U \) is exponential function, \( c \) doesn’t depend on the initial wealth of the investor.
Expected Utility and optimal investment

\[ V^\xi(0) = U(-\tilde{Y}_0), \]

\[ g(t, z) = \frac{1}{2} \eta \text{dist}^2(z + \frac{1}{\eta} \vartheta_s, C_s^1) - \vartheta z - \frac{1}{2\eta} |\vartheta|^2 \] and \((\tilde{Y}, \tilde{Z})\) solution of the BSDE

\[ \tilde{Y}_t = \xi - \int_t^T \tilde{Z}_s \, dW_s + \int_t^T g(s, \tilde{Z}_s) \, ds. \]

\[ V^{\xi_{\{\tau > \tau\}}}(c) = U(c - Y_0), \]

\[ f(t, z, u) = \frac{1}{2} \eta \text{dist}^2(z + \frac{1}{\eta} \vartheta_s, C_s^1) - \vartheta z - \frac{1}{2\eta} |\vartheta|^2 + \frac{1}{\eta} (1 - D_s) k_s [e^{\eta u} - 1 - \eta u] \]

and \((Y, Z, U)\) solution of our BSDE with generator \(f\).
Proposition

The indifference credit risk premium satisfies

\[ c = \bar{Y}_0, \]

where \((\bar{Y}, \bar{Z}, \bar{U})\) is the solution of the BSDE

\[ \bar{Y}_t = \xi 1_{\{\tau \leq T\}} - \int_t^T \bar{Z}_s d\hat{W}_s - \int_t^T \bar{U}_s dM_s + \int_t^T h(s, \bar{Z}_s, \bar{U}) ds, \tag{2} \]

with generator \(h(t, z, u) = -\vartheta z + \frac{1}{\eta} (1 - D_s) k_s [e^{\eta u} - 1 - \eta u],\)

\[ \frac{d\hat{P}}{dP} = \mathcal{E}(\int_0^\tau \gamma_s dW_s)_T \] and \(\hat{W}_t = W_t + \int_0^t \gamma_s ds\) is a Brownian motion with respect to \(\hat{P} \).
We suppose that our financial market consists in one tradable asset with dynamics

\[ dS_t = S_t \alpha dt + S_t \sigma dW_t. \]

Assume \( k = d = 1 \) and \( C_t = \mathbb{R} \times \{0\} \) (no contraints and no defaulatable asset).

Credit risk is the only source for market incompleteness.

Suppose the compensator \( K \) satisfies \( dK_t = k(S_t)dt + dA_t \) (for ex. \( k \) is a positive continuous function).
Proposition

Conditionally to $S_t = x$ and $\tau > t$, the credit risk premium at time $t$ of a defaultable put option with strike $C$ and maturity $T > t$ is given by $\nu(t, x)$, where $\nu$ is the solution of the following PDE

$$
\nu_t + \frac{1}{2} \sigma^2 x^2 \nu_{xx} + \frac{k(x)}{\eta} (e^{\eta(u-\nu)} - 1) = 0, \quad \nu(T, x) = 0.
$$

Remark : Notice that the PDE does not depend on the drift parameter $\mu$, which is almost impossible to estimate in practice.
Conclusion

- Studied a new class of BSDE with 1 possible jump.
- Obtained Existence and Uniqueness for a BSDE with a jump at a random time and satisfying a quadratic growth condition.
- Derived BSDE representations of the maximal expected exponential utility of investors endowed with defaultable contingent claims.
- Same arguments can be used to determine indifference value of the defaultable contingent claim, allow to write hedging formulas in terms of derivatives of the indifference value w.r.t. the market price process.

The paper is available on Arxiv http://fr.arxiv.org/abs/0907.1221 and will be published in IJTAF.
Thank you for your attention.
Hypothesis

In order to exclude arbitrage opportunities in the financial market we assume $d \geq k$. For technical reasons we suppose that

(M1) $\alpha$ is bounded,

(M2) there exist constants $0 < \varepsilon < K$ such that $\varepsilon l_k \leq \sigma(t)\sigma^*(t) \leq Kl_k$ for all $t \in [0, T]$,

(M1) and (M2) imply that the market price of risk, given by

$$\vartheta = \sigma^*(\sigma\sigma^*)^{-1} \alpha,$$

is bounded.