Numerical bifurcation analysis of the asymmetric spring-mass model for running

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Abstract

In this study, we transform the spring-mass model for running into a parametrized boundary value problem. We show that the new approach can be extended for investigations of the asymmetric spring-mass model. The new approach allows the computation of bifurcations and points on the event hyperplanes. Hence, the study of the region of the stable solutions can be reduced to the calculation of its boundaries giving a considerable benefit for computation time.

Keywords: spring-mass model, leg asymmetry, bifurcations, running, boundary-value problem

1. Introduction

In a previous paper, we showed, how the planar bipedal spring-mass model can be transformed into a two-point boundary value problem (BVP) [1]. In this work, we complete our investigations and discuss the application of this method to the single-legged spring-mass model for running [2]. Additionally, we extend our approach on the asymmetric spring-mass model [3].

Beside the three parameters of the symmetric model (angle of attack, leg stiffness and leg length), the asymmetric model contains three asymmetric parameters. The additional complexity results in a considerably increased computational effort. To reduce the computation costs, it appears more appropriate and efficient to calculate a couple of bifurcations instead of computing the manifolds of periodic solutions. Hence, we extend the transformed model for study of bifurcations, which serve as limits of the stability regions. For instance, the turning point is one of the limits of the region of stable periodic running [4]. Hence, we reduce the study of regions of stable periodic solutions to the computation of their boundaries.

The most obvious example of the asymmetric locomotion can be found in humans with prosthetic limbs [5]. Asymmetric gait patterns are also observed in healthy humans [6]. One reason could be the existence of a preferred limb, which executes a manipulative or mobilizing action while the other one provides stabilizing support [7]. Asymmetry is also important for investigation of gait transitions, since the steps between walking and running have an asymmetric behavior [8].
In the present study, we investigate, to what extent the leg asymmetry may challenge running stability. We transform the model into a BVP, apply the single shooting method to compute the stability region of the symmetric model and use it as reference for further investigations. Afterwards, we introduce the asymmetry parameters into the model and study their influence on the stability of running.

2. Methods

2.1. The spring-mass model

The detailed description of the symmetric planar spring-mass model can be found e.g. in [2, 9]. Here, we give only a short introduction.

The spring-mass model consists of a massless leg spring supporting the point mass \( m \), which represents the center of mass (CoM) of the human body [2]. The location and velocity of the CoM in the real plane \( \mathbb{R}^2 \) are given by \((x, y)^T\) and \((\dot{x}, \dot{y})^T\), respectively. Any running gait of the symmetric model is completely characterized by four fundamental system parameters (the leg stiffness \( k_0 \), the angle of attack \( \alpha_0 \), the leg length \( L_0 \), the system energy \( E_0 \)) and the four-dimensional vector of initial conditions \((x_0, \dot{x}_0, y_0, \dot{y}_0)^T\).

The system is energy-conservative, i.e. the system energy \( E_0 \) remains constant during the whole step. In the apex, where the vertical velocity \( \dot{y}_0 \) is zero, the system energy is given by

\[
E_0 = mg y_0 + \frac{m \dot{x}_0^2}{2}.
\] (1)

For this, the vector of initial conditions in apex depends only on the initial height \( y_0 \).

One single running step comprises the first flight phase, the stance phase and the second flight phase [2, 9]. The trajectory of the CoM in each phase is the solution of an initial value problem (IVP). Events of touch-down and take-off are transitions between the phases. The step begins in the first apex, i.e. in the highest point of the flight curve, and ends in the next one.

The study of asymmetric periodic patterns requires investigation of the full stride comprising two subsequent steps. Like in the case of the bipedal model [3], we define three asymmetry...
parameters $\varepsilon_\alpha$, $\varepsilon_k$ and $\varepsilon_L$ of the angle of attack $\alpha_0$ (Fig. 1(a)), the leg stiffness $k_0$ (Fig. 1(b)) and the rest length $L_0$ (Fig. 1(c)), respectively. During the first step of a stride, these parameters are subtracted:

\[ \alpha_1 = \alpha_0 - \varepsilon_\alpha, \quad k_1 = k_0 - \varepsilon_k, \quad L_1 = L_0 - \varepsilon_L. \]  

(2)

During the second one, they are added:

\[ \alpha_2 = \alpha_0 + \varepsilon_\alpha, \quad k_2 = k_0 + \varepsilon_k, \quad L_2 = L_0 + \varepsilon_L. \]  

(3)

We call $\varepsilon_\alpha$ $\alpha$-asymmetry, $\varepsilon_k$ $k$-asymmetry and $\varepsilon_L$ consequently $L$-asymmetry. We consider positive perturbations of leg asymmetry only ($\varepsilon_\alpha > 0$, $\varepsilon_k > 0$, $\varepsilon_L > 0$). Switching the order of the steps, i.e. beginning with step 2 followed by step 1, we achieve the case of negative perturbation with exact the same stability behavior. All calculations in this investigation are done with constant dimensional energy $E_0 = 1800$ J corresponding to the average horizontal velocity $v_x \approx 5$ m/s. The leg stiffness $k_0 = 20$ kN/m and the leg length $L_0 = 1$ m were derived from experimental data in [9]. Moreover, we set $m = 80$ kg and $g = 9.81$ m/s$^2$.

Periodic solutions are found using the Poincaré return map with the apex of the flight curve as Poincaré section [10]. Here, the vector of initial conditions depends only on the initial height $y_0$. Thus, the return map has dimension one. For the detailed description of the system analysis, we further refer to [1, 4, 9].

2.2. Boundary value problem

Like in [1], the three phases of the step are scaled to the unit interval $[0, 1]$ and solved at once. The BVP computing a single symmetric step is then

\[ \dot{z}(t) = f(z(t)), \quad t \in [0, 1] \]

\[ r(y(0), y(1); \alpha_0) = 0, \]  

(4)

with functions $f$ and $r$ as listed in Tab. 1. The dimension of the boundary value problem is $n = 16$. To find a periodic solution to the given set of parameters, the functions $r_2$ and $r_3$ have to be replaced by the relations

\[ \dot{r}_2 = E_0 - mgz_3(0) - \frac{mz_3(0)^2}{2} \]

\[ \dot{r}_3 = z_3(0) - z_{14}(1). \]  

(5)

The new boundary condition $\dot{r}_2 = 0$ defines the relationship between the height and the velocity of the CoM in the apex of the flight curve with respect to the system energy $E_0$. The function $\dot{r}_3$ corresponds to the Poincaré return map of the original model, i.e. it describes the periodic condition to the apex height $y_0$ of the CoM.

To compute period-2 solutions and solutions of the asymmetric model, we have to extend the BVP for the computation of a double step. The new functions $\tilde{f}$ and $\tilde{r}$ have the dimension 32, i.e. they consist of the functions $f$ and $r$ from Tab. 1, which are used twice. The first 16 equations compute the first step and the last 16 equations compute the second one. The BVP for the asymmetric model uses modified functions $\tilde{f}$ and $\tilde{r}$ with the parameters $\alpha_1 \neq \alpha_2$, $k_1 \neq k_2$ and $L_1 \neq L_2$. 


In some cases, the computation of the points on the touch-down line $y_{TD} = L_i \sin(\alpha_i)$ (Fig. 2(b)) or on the take-off line $L_i = L_0$ (Fig. 4) is required. To find these points, we replace all $\alpha_i$’s in the functions $\tilde{f}$ and $\tilde{r}$ by the new variable $z_{33}$. Then, we add one equation and, consequently, one boundary condition to the BVP (4):

$$f_{33} = 0 \quad r_{33} = z_5(0)z_{16}(0)z_{21}(0)z_{32}(0). \tag{6}$$

In case of $z_5 = 0$ or $z_{16} = 0$, the first or the second step of the stride begins right with stance phase (touch-down point). In case of $z_{21} = 0$ or $z_{32} = 0$, the corresponding step has no second flight phase (take-off point).

2.3. Implementation

All one-parameter bifurcations are computed using the techniques of extended systems as it described in [1]. The two-parameter bifurcations (i.e. double turning points) are found using the secant-predictor continuation [11].

Both, the original and transformed spring-mass model are implemented in MatLab (version R2011a, The MathWorks Inc., Natick, MA, USA). The boundary value problems in this study are solved using the latest MatLab version of the BVP solver RWPM [12]. All periodic solutions and extended systems are calculated using the simple shooting method. The associated initial value problems are solved using the MatLab solver ode45 with an absolute and relative tolerance of $10^{-8}$. The Matlab function fsolve from the Optimization Toolbox is then applied to find the zeros of the system of non-linear equations.

3. Results

3.1. Symmetric model

The region of symmetric stable solutions is located between the turning point and the period-doubling bifurcation (Fig. 2(a)). The branch of period-2 solutions is short, because the lower part is limited by the touch-down line (Fig. 2(b)). Here, the apex heights are located between 0.93 and 0.95 m. All symmetric period-2 solutions are becoming fast very unstable. For instance, the Floquet multiplier of the periodic solution at $\alpha_0 = 68.7^\circ$ is 3.6. The Floquet multiplier of the periodic solution at $\alpha_0 = 68.69^\circ$ is already 6.7. Thus, the computation of these solutions by the original model requires very precise initial conditions. On the other side, to compute them using the boundary value problem (4), it is sufficient to use any other point from this branch as an initial guess.

The complete manifold of stable periodic solutions of the spring-mass model for running is shown in Fig. 3. It is confined between the curve of the turning points and the curve of the period-doubling bifurcations. Periodic stable solutions exist for the system energy between $E_0 = 1140$ J and $9120$ J. Both curves are computed using the secant-predictor continuation [11] with the system energy $E_0$ as the second bifurcation parameter.

In the following, we investigate the size of the region where locomotion is stable. Like in [3], we define the continuous range of stable solutions $\Delta \alpha = \alpha_{\text{max}} - \alpha_{\text{min}}$ and call it the $\alpha$-range. The symmetric model exhibits an $\alpha$-range of $\Delta \alpha = 2.7^\circ$, which provides a reference for investigations on the asymmetric model.
3.2. Asymmetric model

Stable periodic solutions exist for $\varepsilon_\alpha < 18.2^\circ$, i.e. as long as the turning points exist. For $\varepsilon_\alpha < 1.1^\circ$, they are located between two turning points (Fig. 4). For $1.2^\circ < \varepsilon_\alpha < 4.2^\circ$, the right boundary of the stability area is a TO-point. At $\varepsilon_\alpha < 4.2^\circ$, the transcritical bifurcation vanishes. From here until $\varepsilon_\alpha = 18.2^\circ$, the right limit of the stability area is the point on the touch-down line $y_{TD} = L_0 \sin(\alpha_0)$.

Compared to the symmetric case, we observed an increase of the $\alpha$-range for values of $\varepsilon_\alpha$ less than $1.7^\circ$ (Fig. 6). The maximum value of $\Delta\alpha = 3^\circ$ was found at $\varepsilon_\alpha = 2.8^\circ$. Here, the left boundary of the region of stable solutions is the turning point at $\alpha_{min} = 66.2^\circ$ (Fig. 5). The right one is the touch-down point at $\alpha_{max} = 69.2^\circ$. For $\varepsilon_\alpha > 1.7^\circ$, the $\alpha$-range monotonically decreases.

Like the bipedal model, the single-legged asymmetric model is very sensitive to the asymmetry of the leg length $\varepsilon_L$. With increasing $\varepsilon_L$, the $\alpha$-range decreases rapidly (Fig. 6(a)). Stable solutions are located either between the turning point and a touch-down point ($\varepsilon_L < 0.0063$) or between two turning points ($0.0064 < \varepsilon_L < 0.0074$). For $\varepsilon_L = 0.008$ m, which is less than 1% of the symmetric leg length $L_0 = 1$ m, no stable solutions were found anymore.

The asymmetry of stiffness $\varepsilon_k$ affects the system in a similar way as the leg length asymmetry $\varepsilon_L$ (Fig. 6(b)). Again, we observe no increase of the $\alpha$-range. However, the model is less sensitive to the $k$-asymmetry than to the $L$-asymmetry. Here, stable solutions exist for values of $\varepsilon_k < 6.3$ kN/m, which is $31.5\%$ of the reference symmetric stiffness $k_0 = 20$ kN/m. Again, they are located either between the turning point and a touch-down point ($\varepsilon_k < 3.6$) or between two turning points ($3.7 < \varepsilon_k < 6.3$).

4. Summary

In this work, we have presented the new implementation of the spring-mass model for running as a two-point boundary value problem. We have also shown that the BVP approach can be extended for the study of the asymmetric model. Using the shooting method, we have been able to calculate even highly unstable solutions of the model. We have applied the technique of extended systems to find bifurcations. Moreover, a simple extension of the BVP allows the computation of the points on the event hyperplanes. Hence, the study of the region of stable solutions can be reduced to the calculation of its boundaries. In particular, this gives a considerable benefit for the next step of our investigations, namely to analyze how the combinations of the asymmetry parameters $\varepsilon_\alpha$, $\varepsilon_k$, $\varepsilon_L$ affect the stability of running.

The discussion on the biomechanical side of this study is similar to the case of bipedal model [3]: we observe the increased $\alpha$-range for the $\alpha$-asymmetry and high sensitivity to the $L$-asymmetry. The decrease of the $\alpha$-range in case of walking is less critical than for running. In the running model, the $k$-asymmetry of 2 kN/m, which is $10\%$ of the reference stiffness $k_0 = 20$ kN/m, causes the loss of the $\alpha$-range by more than $50\%$ (Fig. 6(a)). The reason for this is that during walking both leg springs are present in the both steps of a stride (for example in double-support phases). That makes the overall stiffness of the system larger than the stiffness $k_1 = k_0 - \varepsilon_k$ of the single first leg.

Figure 2: Initial apex height $y_0$ of symmetric periodic running patterns dependent on the angle of attack $\alpha_0$. Thick lines indicate stable periodic solutions. Large green points are bifurcations: TP = turning point, PD = period-doubling bifurcation point. Blue points are intersections of branches of periodic solutions with the touch-down line $y_{TD} = L_0 \sin(\alpha_0)$. The subfigure (a) shows the branch of period-1 solutions with an example of the CoM-trajectory of a period-1 solution. The subfigure (b) is the enlarged inset from (a) with the branch of period-2 solutions.

Figure 3: Initial apex height $y_0$ of periodic running solutions dependent on the angle of attack $\alpha_0$ for different values of the system energy $E_0$. The grey region represents stable periodic solutions. The blue line consists of turning points. The green one is the curve of period-doubling bifurcations.

Figure 4: Initial apex height $y_0$ of stable periodic solutions dependent on the reference angle of attack $\alpha_0$ for $\epsilon_0 = 1^\circ$, $3^\circ$, and $5^\circ$. The gray curves represent the initial conditions of the reference symmetric gait patterns (Fig. 2(a)). The blue dots are touch-down points; the red one is the take-off point. The grey areas are the insets to the right hand side.
Figure 5: The $\alpha$-range $\Delta\alpha$ of stable asymmetric solutions dependent on the asymmetry of the angle of attack $\varepsilon_\alpha$.

Figure 6: Initial apex height $y_0$ of stable asymmetric solutions dependent on the angle of attack $\alpha_0$ with (a) $k$-asymmetry and (b) $L$-asymmetry. The gray curves represent the initial conditions of the reference symmetric gait patterns (Fig. 2(a)). The green curves are touch-down points; the red ones consist of turning points. The blue dots are double turning points.
Table 1: The functions \( f \) and \( r \) for the boundary value problem (4) computing a single running step. The lengths of the compressed leg springs during the stance phase is given by 
\[
L_1 = \sqrt{(z_1 - x_{11})^2 + z_2^2}.
\]
The scaling parameters are \( z_5, z_{10} \) and \( z_{16} \). The functions \( f_1 \) to \( f_4 \) and \( f_{12} \) to \( f_{15} \) describe the motion of the CoM during the both flight phases. \( f_6 \) to \( f_9 \) describe the stance phase. \( f_{11} \) together with the boundary condition \( r_{11} = 0 \) determine the horizontal position \( x_f = L_0 \cos(\alpha_0) \) of the foot point. The boundary functions \( r_1 \ldots r_4 \) define the initial location and velocity of the CoM. The switches between the phases are \( r_6 \ldots r_9 \) and \( r_{12} \ldots r_{15} \). Finally, \( r_5, r_{10}, r_{16} \) determine the events of touch-down, take-off and second apex.