Jena, Oberseminar „Geometrie“

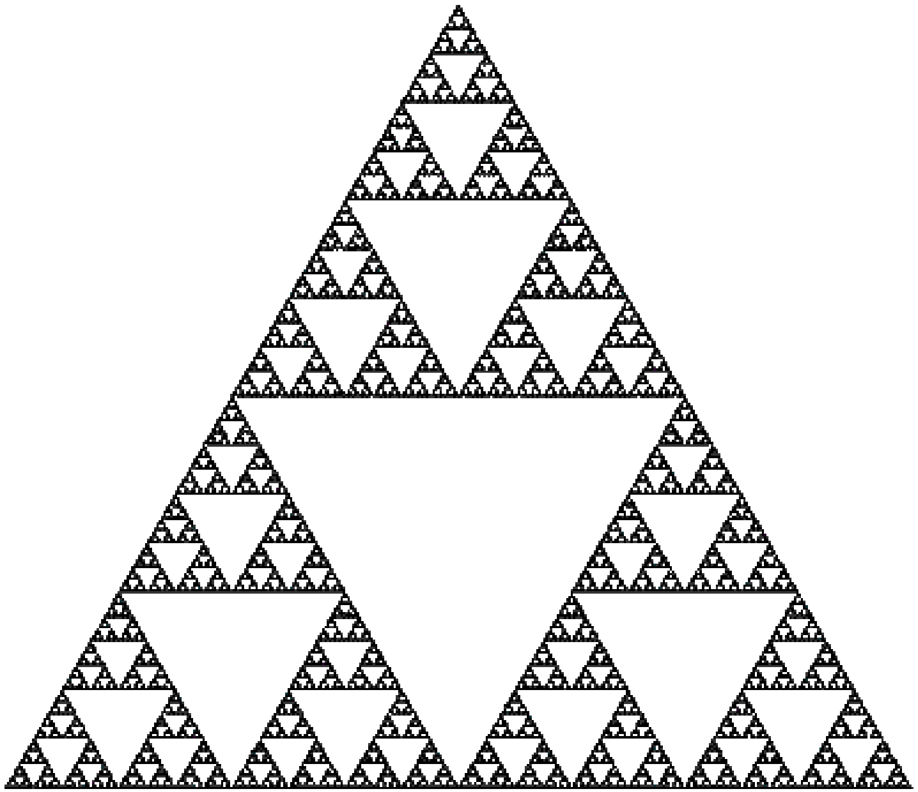
A Markov chain Algorithm for determining crossing times through nested graphs

Uta Freiberg
Mathematisches Institut, Friedrich-Schiller-Universität Jena, D
joint work with Christoph Thäle, Département de Mathématiques, Université de Fribourg, CH

18. November 2008

1. The Problem

SG(2)–Sierpinski gasket:



Aim: Definition of a Brownian motion (or, Laplacian Δ) as a limit of suitable normalized random walks on pre–gaskets (or, via discrete Dirichlet forms, resp.)

Crucial step: Find out the right time–space–scaling of the „natural" stochastic process

Important assumptions: Self–similarity and finite ramification

Pose $A := (0,0)$, $B := (1,0)$ and $C := (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

$SG(2)$ is the unique nonempty compact set, which is self–similar with respect to the family of similarity contractions $\Psi := \{\psi_1, \psi_2, \psi_3\}$, i.e.

$$SG(2) = \bigcup_{i=1}^{3} \psi_i SG(2),$$

where the mappings $\psi_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are just given by the unique contractive similitude with contraction ratio $\frac{1}{2}$ and fixed points A, B and C respectively.

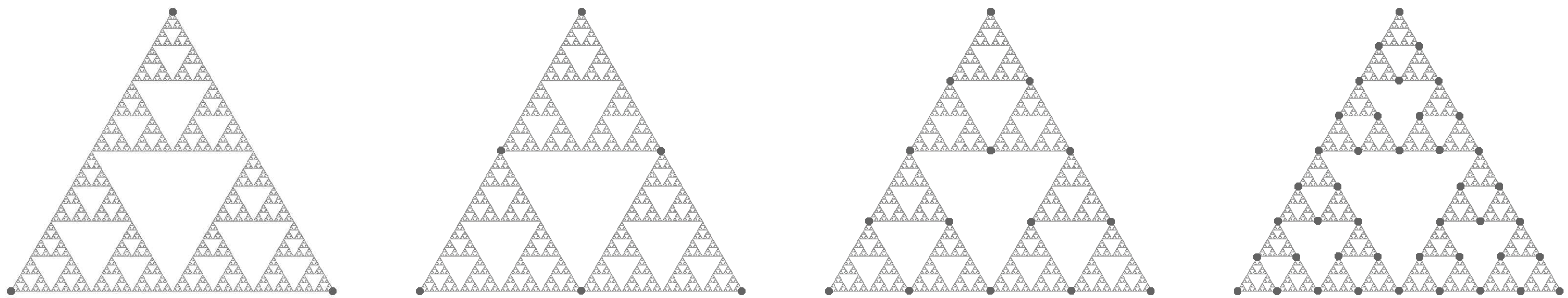
Hausdorff dimension: $d_H = \frac{\ln 3}{\ln 2}$

Approximation by an increasing sequence of finite sets $(V_n)_{n\geq 0}$:

Set $V_0 := \{A, B, C\}$ and define inductively

$$V_n := \bigcup_{i=1}^{3} \psi_i(V_{n-1}), n \geq 1.$$

Setting $V_* := \bigcup_{n\geq 0} V_n = \lim_{n\to\infty} V_n$, it holds that $SG(2) = \overline{V_*}$ (the bar denotes the closure with respect to Euclidean norm).



Important notion: Walk dimension d_W

Recall: Hausdorff dimension d_H relates the volume of small balls with their radii, i.e. $\mu(B(x,R)) \sim r^{d_H}$.

While: Walk dimension relates mean exit times (of the 'canonical Brownian motion') from balls with the radii of these balls, i.e. i.e. $\tau(B(x,R)) \sim r^{d_W}$, where $\tau(B(x,R))$ denotes the exit time of a stochastic process starting at time 0 in x from a ball with radius R. So define

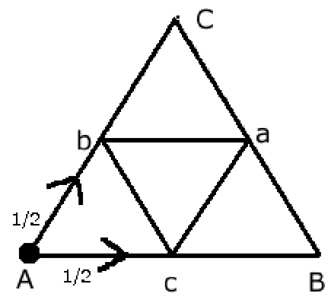
$$d_W := \frac{\ln \mathbb{E}\tau^x(B(x,R))}{\ln R}$$

Remark: graph theory: limit as $R \to \infty$ (metric spaces: limit as $R \to 0$) of the latter term is taken.

Here: unimportant by self–similarity.

local symmetry, self–similarity, strong reflection principle $\rightsquigarrow$ exit times from balls = crossing times through subgraphs of V_n.

Reformulation of the problem: Supposing that the random walk starts in a vertex of V_0, we ask for the expected time of the first hitting of *another* vertex of V_0 provided that we move along the edges of V_1. To be more concrete, we assume that we start in A and want to pass through the graph of V_1 until we reach B or C. In the following considerations, $\mathbb{E}\tau^P$ denotes the expected time moving from a point P to the set $\{B, C\}$.



$$\mathbb{E}\tau^A = \frac{1}{2}\left(\mathbb{E}\tau^b + \mathbb{E}\tau^c\right) + 1 = \mathbb{E}\tau^c + 1.$$

$$\mathbb{E}\tau^c = \frac{1}{4}\left(\mathbb{E}\tau^A + \mathbb{E}\tau^b + \mathbb{E}\tau^a + \mathbb{E}\tau^B\right) + 1 = \frac{1}{4}\left(\mathbb{E}\tau^A + \mathbb{E}\tau^c + \mathbb{E}\tau^a\right) + 1$$

$$\mathbb{E}\tau^a = \frac{1}{4}\left(\mathbb{E}\tau^C + \mathbb{E}\tau^b + \mathbb{E}\tau^c + \mathbb{E}\tau^B\right) + 1 = \frac{1}{2}\mathbb{E}\tau^c + 1.$$

Solution: $T = \mathbb{E}\tau^A = 5$.

Hence, it takes in expectation 5 steps (of length one) to leave a ball of radius 2.

Self similarity + Markov property $\rightsquigarrow$ this time–space–scaling will occur on all magnification scales, i.e. it takes 5^n steps to leave a ball of radius 2^n.

The same holds in the limit approaching continuous time (Lindstrøm, 1990).

Thus, the walk dimension of the Sierpinski gasket equals $d_W = \frac{\ln 5}{\ln 2}$.

Summarize: Use of the walk dimension:

- construction of the BM: $2^{-n}X_{5^n t} \Rightarrow X_t$ in the case of Sierpinski gasket (compare with the scaling in the Euclidean case)

- heat kernel estimates

- Einstein's relation $\frac{d_H}{d_S} = \frac{d_W}{2}$

2. The Idea –Example: Sierpinski gasket

Step 1: Graph with vertex set V_0

Denote the set of vertices $V_0 = \{v_1, v_2, v_3\}$. The set V_0 can be considered as a graph with adjacency matrix

$$A_0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Step 2: Build isolated copies of this graph

This corresponds to the (9×9)-block matrix

$$A := \begin{pmatrix} A_0 & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_0 \end{pmatrix}.$$

Step 3: Connect the copies – The connection matrix C

The so-called *connection matrix* of the graph V_1 is the following symmetric (9×9)–matrix

$$C := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Each row i and each column j represent vertices $v, w \in V_1$. The numbers i and j have a unique representation $i = 3(i_1 - 1) + i_2$ and $j = 3(j_1 - 1) + j_2$, where $i_1, i_2, j_1, j_2 \in \{1, 2, 3\}$.

Matrix C is obtained in the following way:

If $\psi_{i_2}(v_{i_1}) = \psi_{j_2}(v_{j_1})$ the $c_{ij} = c_{ji} = 1$.

Note: mathematically precise: notion of *code spaces* (Barnsley): Every point in the Sierpinski gasket $SG(2)$ has an address in the space $\{1,2,3\}^{\mathbb{N}}$. The *address function* $\pi : \{1,2,3\}^{\mathbb{N}} \longrightarrow SG(2)$ given by

$$\pi(\sigma) := \lim_{n\to\infty} \psi_{\sigma_1} \circ \ldots \circ \psi_{\sigma_n}(x_0), \quad \sigma = \sigma_1\sigma_2\sigma_3 \ldots \in \{1,2,3\}^{\mathbb{N}}, \tag{1}$$

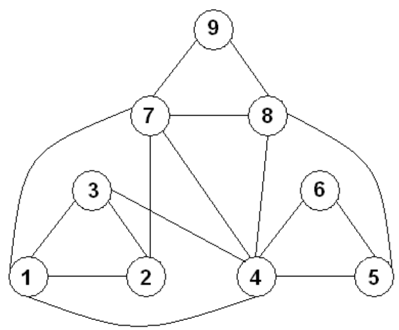
is well defined as the limit in (1) is a singleton which does not depend on $x_0 \in \mathbb{R}^2$. Note that π is onto, but in general not one–to–one (unless the fractal under consideration is totally disconnected as for example famous Cantor set). In terms of the address function, the connectivity property above reads

$$\pi(1\overline{2}) = \pi(2\overline{1}),\ \pi(1\overline{3}) = \pi(3\overline{1}) \text{ and } \pi(2\overline{3}) = \pi(3\overline{2}).$$

Step 4: Coupling of the subgraphs with the help of C

If $c_{ij} = 1$ then add row i to row j and column i to column j.

$$A' = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



Step 5: Remove double points from the graph

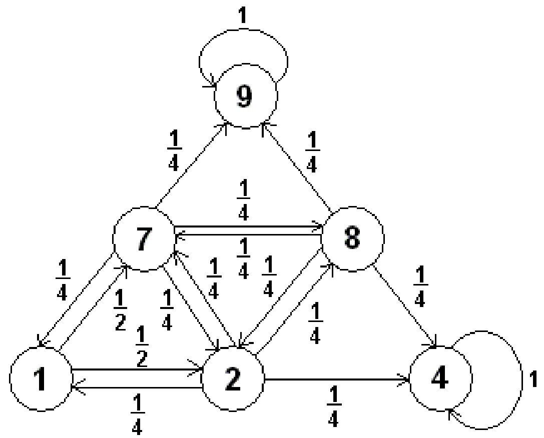
If in the connection matrix $c_{ij} = 1$ for some $1 \leq j < i \leq 9$ then delete row and column number i.

Step 6: Make the resulting matrix stochastic

Divide each entry by the number of non-zero elements in its row:

$$A'' = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Step 7: Remove cemeteries from the matrix



The resulting (substochastic) matrix is given by

$$A''' = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

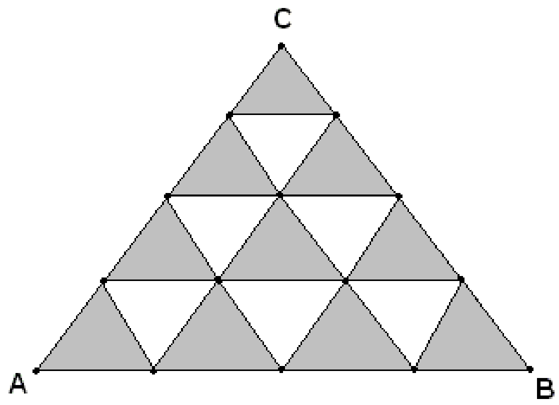
To calculate the crossing time of the graph V_1, we have to solve the following linear system

$$\vec{\mathbb{E}\tau} = A'''\vec{\mathbb{E}\tau} + \mathbf{1},$$

where $\mathbf{1} = (1,1,1,1)^T$ and $\vec{\mathbb{E}\tau}$ is the vector $\vec{\mathbb{E}\tau} = (\mathbb{E}\tau^A, \mathbb{E}\tau^b, \mathbb{E}\tau^c, \mathbb{E}\tau^a)^T$ (the upper T denotes the transpose) and $\mathbb{E}\tau^v$ is the mean number of steps a walker needs from vertex v to one of the vertices B or C. The crossing time T we are looking for is given by $T = \mathbb{E}\tau^A = 5$, which is easily calculated by hand or using a computer–algebra–system as Maple®. By doing so, we get that the whole vector $\vec{\mathbb{E}\tau}$ equals $\vec{\mathbb{E}\tau} = (5,4,4,3)$.

3.Generalized planar Sierpinski gaskets $SG(m)$

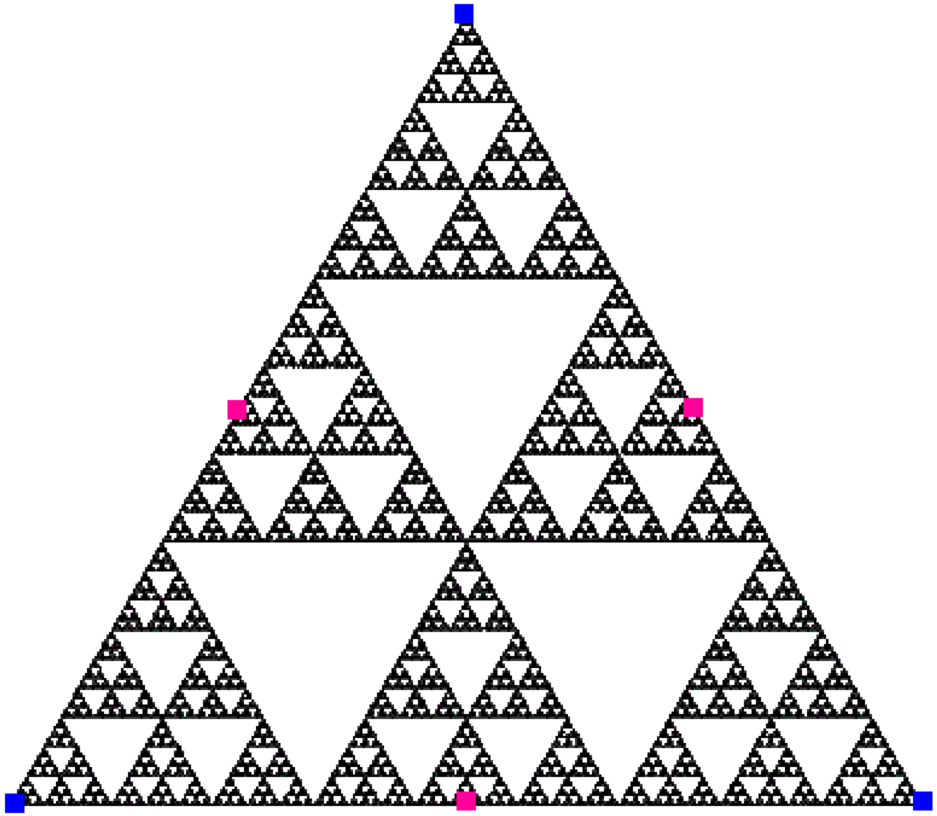
Divide a (regular) triangle Δ into m^2, $m \geq 2$, smaller triangles (for example, $m = 4$)



We have $M := m(m+1)/2$ triangles pointing up (keep) and $m(m-1)/2$ pointing down (remove).

$$d_H SG(m) = \frac{\ln m + \ln(m+1) - \ln 2}{\ln m}.$$

SG(3)–modified Sierpinski gasket:



self–similar wrt IFS $\mathcal{G} = \{g_1, g_2, g_3, g_4, g_5, g_6\}$

fixed points: $P_{1,2,3,4,5,6}$

essential f.p.'s: $P_{1,4,6}$

SO: the starting set V_0 remains the same!

4. Results

For generalized Sierpinski gaskets $SG(2), ..., SG(7)$

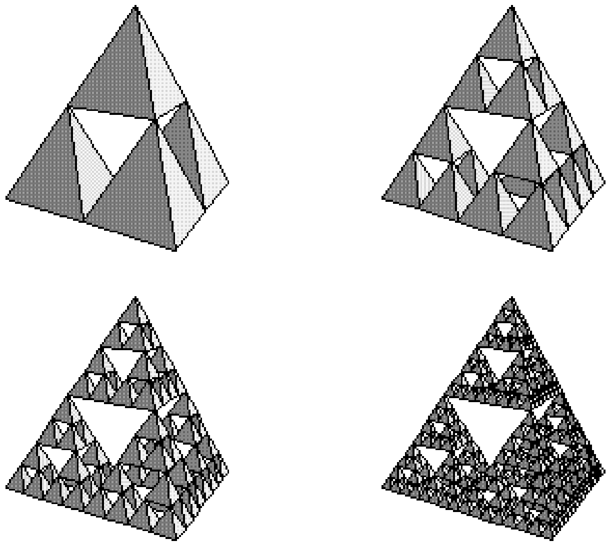
m	$T(m) = \mathbb{E}\tau^A$	$\ln T(m)/\ln m$
2	5	2,3219
3	90/7	2,3247
4	1030/41	2,3254
5	8315/197	2,3255
6	452739/7025	2,3249
7	904836/9823	2,3244

Note: Explicit formula for the crossing time $T(m)$ is unknown for arbitrary m.

Results of Hambly and Kumagai imply

$$\lim_{m\to\infty} \frac{\ln T(m)}{\ln m} = 2.$$

For Sierpinski spaces $SS(D)$ in $\mathbb{R}^D$



The mean crossing time through the graph belonging to $SS(D)$ equals $D + 3$, $D \in \mathbb{N}$.

5. Outlook

Extension to self similar graphs obtained from regular n–gons