L^2 Betti numbers were defined by Atiyah in 1974 as analytic invariants of a compact, Riemannian manifold M (von Neumann dimensions of spaces of L^2 harmonic forms on the universal covering $\tilde{M}$ of M). The original definition quickly morphed into a combinatorial one where the differential forms are replaced by L^2 cochains. I will attempt to motivate the definition, describe properties of the L^2 Betti numbers, and present some applications.

These invariants are difficult to compute and one of the main questions, already raised by Atiyah, was to determine their possible values. Atiyah asked specifically whether the L^2 Betti numbers were necessarily rational, and this question was dubbed "the Atiyah conjecture." The problem itself reduces to a question about matrices with coefficients in the integral group ring $\mathbb{Z}[\Gamma]$ of the covering group $\Gamma = \pi_1(M)$. Recent results by Austin, Grabowski, Pichot, Schick, and Żuk show that the Atiyah conjecture fails completely. Their examples come from groups with a great deal of torsion and it is possible that the conjecture is true for all torsion-free groups. As a matter of fact the conjecture has been verified for a very large class of torsion-free groups. I will describe how this leads to a solution of Kaplansky's zero divisor conjecture for groups of this class (absence of zero divisors in the complex group ring $\mathbb{C}[\Gamma]$ for torsion-free Γ).