ON THE NUMBER OF CONJUGACY CLASSES OF A FINITE SOLVABLE GROUP II

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Abstract

We prove that a finite solvable group G has at least $(49p+1)/60$ conjugacy classes whenever p is a prime such that p^2 divides the order of G. We also construct an infinite family of finite solvable groups where this bound is attained.

For a finite group G, we denote by $k(G)$ the number of conjugacy classes of G. Thus $k(G)$ is also the number of irreducible complex characters of G. In the first part [3] of this paper we showed that $k(G) \geqslant 2\sqrt{p-1}$ whenever G is solvable and p is a prime dividing the order $|G|$ of G. Here we prove:

Theorem. *Let p be a prime number, and let G be a finite solvable group such that p^2 divides $|G|$. Then $k(G) \geqslant (49p+1)/60$.*

Our motivation for such a result comes from a question of L. Pyber, and also from the problem of finding explicit lower bounds for the number of irreducible characters in a block. (This aim is not achieved.) We stress that, as was shown in [3], the same bound does not hold for non–solvable groups, in general.

Proof of the Theorem. Let G be a minimal counterexample, let P be a Sylow p–subgroup of G, and write $|P| = p^n$. Then

$$(49p+1)/60 > k(G) \geqslant k_0(G)$$

where $k_0(G)$ denotes the number of irreducible characters of G whose degree is prime to p. As the Alperin–McKay conjecture is known to hold for solvable groups [7], we have $k_0(G) = k_0(N_G(P)) = k(N_G(P)/P')$. Since p^2 divides $|N_G(P)/P'|$, $N_G(P)/P'$ is also a counterexample to the theorem. By the choice of G, this means that $G = N_G(P)$ and $P' = 1$. Then P is abelian and normal in G.
Since $G/O_{p'}(G)$ is also a counterexample to the theorem, we must have $O_{p'}(G) = 1$.

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Then $G = PH$ where H is a solvable p'–group acting faithfully on P.

Suppose that P is cyclic. Then $x := |H|$ divides $p-1$, and

$$k(G) = x + (p^n - 1)/x \geqslant x + (p^2 - 1)/x \geqslant 2\sqrt{p^2 - 1} \geqslant (49p + 1)/60\,,$$

a contradiction.

Thus P is noncyclic. Then $G/\Phi(P)$ is also a counterexample to the theorem, so $\Phi(P) = 1$ by the choice of G. Hence P is elementary abelian.

We write $P = P_1 \times \ldots \times P_r$ with minimal normal subgroups $P_1, \ldots, P_r$ of G. If $r > 2$ then G/P_1 is still a counterexample to the theorem contradicting the choice of G. This shows that $r \leqslant 2$.

Suppose that $r = 2$. If $|P_2| \geqslant p^2$ then G/P_1 is again a counterexample contradicting the choice of G. Thus we must have $|P_2| = p$ and, similarly, $|P_1| = p$. For $i = 1, 2$, H acts on P_i. Thus $H/C_H(P_i)$ is cyclic, and $H' \leqslant C_H(P_1) \cap C_H(P_2) = 1$. Let $x := |H|$. Then

$$k(G) \geqslant x + (p^2 - 1)/x \geqslant (49p + 1)/60$$

as above, a contradiction.

We have thus shown that P is the only minimal normal subgroup of G. Hence we can view H as an irreducible p'–subgroup of $GL(n, p)$.

Let us discuss the case $n = 2$. In this case the structure of H is described by Theorem 2.11 in [6]. There are various possibilities: In cases (a) and (b) H contains an abelian normal subgroup X such that $x := |X| \leqslant p^2 - 1$ and $|H : X| \leqslant 2$. Thus

$$k(G) \geqslant (p^2 - 1)/(2x) + \frac{x}{2} \geqslant \sqrt{p^2 - 1} \geqslant (49p + 1)/60\,,$$

a contradiction.

Thus suppose now that case (c) of Theorem 2.11 in [6] holds. Then $F(H) = Q * X$ and $Q \cap X = Z(Q)$ where Q is a quaternion group of order 8 which is normal in H, and $X = Z(H)$ is cyclic and $|X|$ divides $p - 1$. Moreover, $H/F(H)$ acts irreducibly on $Q/Z(Q)$, so $H/F(H) \cong Z_3$ or $H/F(H) \cong S_3$. Let $x := |X : Z(Q)|$ so that x divides $\frac{p-1}{2}$. We will count the irreducible characters of H using Clifford theory with respect to $F(H)$.

Let us first look at an irreducible character of $F(H) = Q * X$ of the form $1_Q * \xi$ where 1_Q denotes the trivial character of Q and ξ denotes an irreducible character of X which is trivial on $Z(Q)$. Such a character is H–stable and thus yields 3 irreducible characters of H. Since there are x irreducible characters of $F(H)$ of the form $1_Q * \xi$, we obtain $3x$ irreducible characters of H in this way.

Next we look at an irreducible character of $F(H) = Q * X$ of the form $\lambda * \xi$ where λ is a nontrivial linear character of Q and where ξ is an irreducible character of X which is trivial on $Z(Q)$. Such a character has a stabilizer of index 3 in H. In case $H/F(H) \cong Z_3$ it yields one irreducible character of H, in case $H/F(H) \cong S_3$ it yields two irreducible characters of H. Since there are $3x$ irreducible characters of $F(H)$ of the form $\lambda * \xi$ (in H–orbits of length 3 each) we obtain x ($2x$, respectively) irreducible characters of H in this way.

It remains to consider an irreducible character of $F(H) = Q * X$ of the form $\varphi * \xi$ where φ is the irreducible character of Q of degree 2 and where ξ is an irreducible character of X which is nontrivial on $Z(Q)$. Such a character is again H–stable and therefore yields 3 irreducible characters of H. Since there are x irreducible characters of H of the form $\varphi * \xi$ we obtain $3x$ irreducible characters of H in this way.

Hence we have $k(H) = 7x$ in case $H/F(H) \cong Z_3$, and $k(H) = 8x$ in case $H/F(H) \cong S_3$. In the first case, $|H| = 24x$ and

$$k(G) \geqslant 7x + (p^2-1)/(24x) \geqslant \sqrt{7/6}\sqrt{p^2-1} \geqslant (49p+1)/60\,.$$

In the second case, we have $|H| = 48x$ and

$$k(G) \geqslant 8x + (p^2-1)/(48x).$$

Now, the function $f : x \mapsto 8x + (p^2-1)/(48x)$ decreases for $x \leq \frac{p-1}{20}$ and increases for $x \geq \frac{p-1}{18}$. Moreover, we have

$$f(\frac{p-1}{20}) = (49p+1)/60 < (59p-5)/72 = f(\frac{p-1}{18})$$

if $p \geq 37$. (Recall also that x divides $\frac{p-1}{2}$, so there is no need to consider $f(\frac{p-1}{19})$.)

Finally, if $p \leqslant 31$ then $\sqrt{(p^2-1)/384} \leqslant 2$. However, we have $f(2) = 16 + (p^2 - 1)/96 > (49p+1)/60$ for all $p \leqslant 31$, and $f(1) = 8 + (p^2-1)/48 > (49p+1)/60$ for all $p \leqslant 31$ except for $p = 19$. For $p = 19$, we have $f(1) = 31/2$, so $k(G) \geqslant 16 > 233/15 = (49p+1)/60$. This means that there is no counterexample with $n = 2$, so in the following we may assume $n \geq 3$.

The numbers in [9] show that $k(G) \geqslant 13$. This implies that $p \geqslant 17$.

Let us consider the case $n = 3$. In this case H can be viewed as a subgroup of $GL(3,p)$. The structure of H is described by Theorem 2.12 in [6]. There are various possibilities.

In cases (a) and (b) H has an abelian normal subgroup X such that $x := |X| \leqslant p^3-1$

and $|H:X| \leqslant 6$. Thus

$$k(G) \geqslant (p^3-1)/(6x) + \frac{x}{6} \geqslant \sqrt{p^3-1}/3 \geqslant (49p+1)/60\,.$$

In case (c) we have $F(H) = D * X$ and $D \cap X = Z(D)$ where D is extraspecial of order 27 and normal in H, and X is cyclic and central in H. Moreover, $D/Z(D)$ is a faithful irreducible module for $H/F(H)$, and $H/F(H)$ is isomorphic to a subgroup of $SL(2,3)$; in particular, $|H/F(H)| \in \{4,8,24\}$. Then $|H:X| \in \{36,72,216\}$ and $k(H/X) \geqslant 6$. Let $x := |X|$. Then

$$\begin{aligned} k(G) &\geqslant (p^3-1)/(216x) + x - 1 + 6 \geqslant \sqrt{(p^3-1)/54} + 5 \\ &\geqslant (49p+1)/60\,. \end{aligned}$$

(The real function $f(x) = \sqrt{(x^3-1)/54} + 5 - (49x+1)/60$ is positive for $x = 17$ and increasing for $x \geqslant 17$.) This shows that we must have $n \geqslant 4$.

Suppose next that H is isomorphic to a subgroup of

$$\Gamma(p^n) := \{\mathbb{F}_{p^n} \to \mathbb{F}_{p^n},\ x \mapsto a\sigma(x) : 0 \neq a \in \mathbb{F}_{p^n},\ \sigma \in \mathrm{Aut}(\mathbb{F}_{p^n})\}\,.$$

Then, as in [3], we have

$$k(G) \geqslant (2/n)\sqrt{p^n-1} \geqslant (2/4)\sqrt{p^4-1} \geqslant (49p+1)/60\,.$$

Thus, in the following, we may assume that H is not isomorphic to a subgroup of $\Gamma(p^n)$.

We observe that H has at most $(49p+1)/60 < 5p/6$ orbits on P. Each orbit has length at most $|H|$. Then $p^n = |P| \leqslant (5p/6) \cdot |H|$ and

$$|H| \geqslant (6/5) \cdot p^{n-1} \geqslant (6/5) \cdot 17^3 > 5895\,.$$

The tables in [9] imply that $k(H) \geqslant 13$. Hence $14 \leqslant k(G) < (49p+1)/60$, so $p \geqslant 19$.

Suppose next that $p = 19$. Then

$$k(G) < (49 \cdot 19 + 1)/60 < 16\,,$$

so

$$15 \geqslant k(G) = \sum_{u \in R} k(C_H(u))$$

where R denotes a set of representatives for the conjugacy classes of p–elements in G. (The last equality comes from decomposing every element in G according to its p-factor and its p'-factor.) Thus $\sum_{1 \neq u \in R} k(C_H(u)) \leqslant 2$. Suppose that $|R| = 2$. Then H acts transitively on $P^\sharp$. Hence Theorem XII.7.3 in [5] gives a contradiction. Thus we must have $|R| = 3$ and $C_H(u) = 1$ for all $u \in P^\sharp$. Then G is a Frobenius group with kernel P and complement H. Also, we have $k(H) = 13$ in this case.

If $|H|$ is even then $|Z(H)| = 2$ since H is a Frobenius complement. Moreover, we have $k(H/Z(H)) \leqslant 12$ and $|H/Z(H)| \leqslant 3240$ by the tables in [9], so $|H| \leqslant 6480$. On the other hand, we have

$$|H| \geqslant (19^4 - 1)/2 = 65160\,,$$

a contradiction.

It remains to consider the case where $|H|$ is odd. In this case H is metacyclic. Let X be a cyclic normal subgroup of H such that H/X is cyclic, and set $x := |X|$, $y := |H/X|$. Then $13 = k(H) \geqslant (x-1)/y + y$. It is easy to check that this implies that $|H| = xy \leqslant 333$. Since $|H| \geqslant (19^4 - 1)/2 = 65160$ we have a contradiction again.

We conclude that $p \geqslant 23$ (and $n \geqslant 4$).

Suppose that H acts primitively on P. Then, by the main result of [8], H has at least $p^{n/2}/(12n)$ orbits on $P^\sharp$. Thus we have

$$k(G) \geqslant p^{n/2}/(12n) + 13 \geqslant p^2/48 + 13 \geqslant (49p+1)/60\,.$$

It remains to consider the case where H acts imprimitively on P. We write $P = P_1 \times \ldots \times P_r$ where $P_1, \ldots, P_r$ are transitively permuted under the action of H and r is as large as possible (so $r \geqslant 2$). For $i = 1, \ldots, r$ we set $H_i := N_H(P_i)$, $K_i := C_H(P_i)$ and $T := H_1 \cap \ldots \cap H_r$. By Dixon's Theorem (Theorem 36.2. on page 215 in [1]), we have $|H : T| \leqslant (\sqrt[3]{24})^{r-1} < 3^{r-1}$. Furthermore, we have $K_1 \cap \ldots \cap K_r = C_H(P) = 1$. We write $|P_1| = p^m$ (so that $mr = n$). We have to discuss several cases again.

Suppose first that $m = 1$ (so $r = n$). For $i = 1, \ldots, r$, H_i/K_i is cyclic, so $H_i' \subseteq K_i$ and $T' \subseteq H_1' \cap \ldots \cap H_r' \subseteq K_1 \cap \ldots \cap K_r = 1$. Hence T is an abelian normal subgroup of H. Since $|H : T| \leqslant 3^{n-1}$ and $|H| \geqslant 6p^{n-1}/5$ we have

$$|T| = |H|/|H : T| \geqslant (6/5) \cdot (p^{n-1}/3^{n-1})\,.$$

Thus

$$\begin{aligned} 5p/6 \geqslant (49p+1)/60 > k(H) &\geqslant |T|/|H : T| \\ &\geqslant (6/5) \cdot (p^{n-1}/9^{n-1})\,. \end{aligned}$$

Since $n \geqslant 4$ and $p \geqslant 23$ this is a contradiction.

This means that we must have $m \geqslant 2$. Suppose that H_i/K_i is isomorphic to a subgroup of $\Gamma(p^m)$ for $i = 1, \ldots, r$. Then H_i/K_i contains a cyclic normal subgroup X_i/K_i such that $|H_i : X_i| \leqslant m$. We set $L := X_1 \cap \ldots \cap X_r$. Then

$$L' \subseteq X_1' \cap \ldots \cap X_r' \subseteq K_1 \cap \ldots \cap K_r = 1\,,$$

so L is an abelian normal subgroup of H such that $|T:L| \leqslant m^r$ and $|H:L| \leqslant 3^{r-1}\,m^r$. By assumption, H has at most $k(G) < 5p/6$ orbits on $P^\sharp$, and the length of each orbit is bounded by $|H|$. Thus $p^{mr} - 1 \leqslant (5p/6)\cdot|H|$ and

$$|H| \geqslant (p^{mr}-1)/(5p/6) \geqslant p^{mr-1} - 1/(5p/6),$$

i.e. $|H| \geqslant p^{mr-1}$. Hence

$$|L| = |H|/|H:L| \geqslant p^{mr-1}/(3^{r-1}m^r)$$

and

$$5p/6 \geqslant k(G) \geqslant k(H) \geqslant |L|/|H:L| \geqslant p^{mr-1}/(9^{r-1}m^{2r})\,.$$

This implies that

$$(5/6)\cdot 9^{r-1}\cdot m^{2r} \geqslant p^{mr-2} \geqslant 23^{mr-2}\,.$$

Hence

$$1 \leqslant (5/6)\cdot 9^{r-1}\,m^{2r}/23^{mr-2} \leqslant (5/6)\cdot 9^{r-1}\,(m-1)^{2r}/23^{(m-1)r-2}$$
$$\leqslant \ldots \leqslant (5/6)\cdot 9^{r-1}\cdot 2^{2r}/23^{2r-2} \leqslant 4\cdot 6^{2r-2}/23^{2r-2} < 1\,,$$

a contradiction.

Thus, in the following, we may assume that H_i has at least $[p^{m/2}/(12m)]$ orbits on $P_i^\sharp$, for $i=1,\ldots,r$, by the main result of [8].

Suppose next that $m=2$. In this case the structure of H_i/K_i is described by Theorem 2.11 (c) in [6]. Thus H_i/K_i contains a central subgroup Z_i/K_i such that $|H_i:Z_i| \leqslant 24$. Let $Z := Z_1\cap\ldots\cap Z_r$. Then $Z' \subseteq Z_1'\cap\ldots\cap Z_r' \subseteq K_1\cap\ldots\cap K_r = 1$, so Z is an abelian normal subgroup of H which is central in T. Therefore

$$|H:Z| \leqslant 24^r\left(\sqrt[3]{24}\right)^{r-1} \leqslant 24^r\cdot 3^{r-1}$$

and

$$|Z| = |H|/|H:Z| \geqslant (6/5)\cdot p^{2r-1}/\left(24^r\cdot\left(\sqrt[3]{24}\right)^{r-1}\right)\,.$$

Hence

$$5p/6 > k(H) \geqslant |Z|/|H:T| \geqslant (6/5)\,p^{2r-1}/\left(24^r\cdot\left(\sqrt[3]{24}\right)^{2r-2}\right)\,.$$

This is a contradiction for $r \geqslant 4$.

Suppose that $r=3$. Then the analysis above can be improved in the following way: We certainly have $|H:T| \leqslant 6$, so $|H:Z| \leqslant 6\cdot 24^3$ and

$$|Z| = |H|/|H:Z| \geqslant (6p^5)/(30\cdot 24^3)\,.$$

Hence

$$5p/6 > k(H) \geqslant |Z|/|H:T| \geqslant (6p^5)/(180\cdot 24^3)$$

which is a contradiction unless $p = 23$.

However, in case $p = 23$ (and $r = 3$) we have $k(G) < (49 \cdot 23 + 1)/60 < 19$. Since $k(H) \geqslant 13$, H has at most 5 orbits on $P^\sharp$. Hence $23^6 - 1 = |P^\sharp| \leqslant 5 \cdot |H|$, so $|H| \geqslant 29607178$. Since $|H/Z| \leqslant 6 \cdot 24^3$ this implies that $|Z| \geqslant 357$. Thus $k(H) \geqslant |Z|/6 > 59$, a contradiction again.

Suppose that $r = 2$. Then the analysis above can be improved in the following way: We certainly have $|H : T| \leqslant 2$ so $|H : Z| \leqslant 2 \cdot 24^2 = 1152$ and

$$|Z| = |H|/|H : Z| \geqslant 6p^3/(5 \cdot 1152)\,.$$

Hence

$$5p/6 > k(H) \geqslant |Z|/|H : T| \geqslant 6p^3/11520\,.$$

and $p^2 < 1600$. This is a contradiction unless $p \leqslant 37$.

Suppose that $p = 23$ (and $r = 2$). Then $k(G) \leqslant 18$ and $k(H) \geqslant 13$. Thus H has at most 5 orbits on $P^\sharp$. Hence $23^4 - 1 \leqslant 5 \cdot |H|$ and $|H| \geqslant 55968$. Thus

$$|Z| = |H|/|H : Z| \geqslant 55968/1152 > 48\,.$$

Hence $k(H) \geqslant |Z|/|H : T| \geqslant 49/2 > 24$, a contradiction.

In a similar way, the cases $p = 29$, 31, 37 (and $r = 2$) lead to a contradiction.

Thus, in the following, we may assume that $m \geqslant 3$. Suppose first that $m \geqslant 4$. In this case we are going to apply Foulser's Lemma (cf. Lemma 2.6 in [2]): If H_1 has h orbits on P_1 then H has at least $\binom{h+r-1}{h}$ orbits on P. Since the binomial coefficent $\binom{h+r-1}{h}$ is monotonically increasing with r, this means that H has at least $\binom{h+1}{h} = h + 1 \geqslant p^2/48$ orbits on P. Thus

$$p^2/48 + 12 \leqslant k(G) < 5p/6$$

which is a contradiction (since $p \geqslant 23$).

It remains to consider the case $m = 3$. In this case, H_1 has at least $p^{3/2}/36$ orbits on $P_1^\sharp$. Suppose first that $r \geqslant 3$. Then, by Foulser's Lemma again, H has at least

$$\begin{aligned}\binom{h+r-1}{h} \geqslant \binom{h+2}{h} &= \frac{(h+2)(h+1)}{2}\\ &\geqslant (p^{3/2}/36 + 1)(p^{3/2}/36 + 2)/2\end{aligned}$$

orbits on P. Thus

$$12 + (p^{3/2}/36 + 1)(p^{3/2}/36 + 2)/2 \leqslant k(G) < (49p + 1)/60$$

which is a contradiction (since $p \geqslant 23$).

Thus we may assume that $r = 2$ (and $m = 3$), i.e. $n = 6$. Then we have $|H : H_1| =$

$r = 2$, so H_1 is normal in H; in particular, $H_1 = H_2$. Let $Z_1/K_1 = Z(H_1/K_1)$. By Theorem 2.12 (c) in [6], we have $|H_1 : Z_1| \leqslant 216$. Similarly, $|H_1 : Z_2| = |H_2 : Z_2| \leqslant 216$ where $Z_2/K_2 = Z(H_2/K_2)$. Let $Z := Z_1 \cap Z_2$. Then

$$Z' \subseteq Z_1' \cap Z_2' \subseteq K_1 \cap K_2 = 1\,,$$

so Z is an abelian normal subgroup of H with $Z \subseteq Z(H_1)$. Hence $|H_1 : Z| \leqslant 216^2$ and $|H : Z| \leqslant 2 \cdot 216^2$. It follows that

$$|Z| = |H|/|H : Z| \geqslant 6p^5/(10 \cdot 216^2)$$

and

$$5p/6 > k(H) \geqslant |Z|/2 \geqslant 6p^5/(20 \cdot 216^2)$$

which gives the final contradiction. Next we show that the bound $(49p+1)/60$ is attained for infinitely many primes p.

PROPOSITION. *There are infinitely many primes p with the following property: There exists a finite solvable group G such that p^2 divides $|G|$ and $k(G) = (49p + 1)/60$.*

Proof. By Dirichlet's Theorem on arithmetic progressions there are infinitely many primes p such that $p \equiv 1 \pmod{40}$. Let p be such a prime. We are going to construct a finite solvable group G of order $48p^2(p-1)/20$ with $k(G) = (49p+1)/60$. By Theorem II.8.18 in [4], $PSL(2, p)$ contains a subgroup S isomorphic to the symmetric group S_4 of degree 4. Let B be the inverse image of S in $SL(2, p)$. Then B has order 48, and $B/Z(B)$ is isomorphic to S_4. Moreover, B has generalized quaternion Sylow 2–subgroups, and $F(B)$ is a quaternion group of order 8.
Next let X be the group of scalar matrices in $SL(2, p)$ of order $(p-1)/10$, and let $H = B * X$ be the central product of B and X, so that $|H| = 48(p-1)/20$. The number of conjugacy classes of H is computed in the proof of the theorem above. Thus

$$k(H) = 8 \cdot |X|/2 = 4|X| = 2(p-1)/5\,.$$

We claim that every element $h \neq 1$ in H acts fixed–point–freely in its natural action on the 2–dimensional vector space V over $\mathbb{F}_p$. Indeed, otherwise h is similar to a matrix of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right)$, so $\alpha = 1$ since $h \in SL(2, p)$.
Then the semidirect product $G = HV$ is a solvable Frobenius group of order

$48p^2(p-1)/20$. Hence

$$\begin{aligned} k(G) &= (p^2-1)/|H| + k(H) = 5(p+1)/12 + 2(p-1)/5 \\ &= (49p+1)/60. \end{aligned}$$

□

We note that for every odd prime p there exists a finite solvable group G such that p^3 divides $|G|$ and $k(G) = 2p+4$, i.e. $k(G)$ is "close" to $(49p+1)/60$.

Example. Let p be an odd prime, let P be an extraspecial p–group of order p^3 and exponent p, and write $P = \langle a, b\rangle$. Let k be a primitive root $\mod p$, let α be the automorphism of P such that $\alpha(a) = a^k$ and $\alpha(b) = b^k$, and let $H := \langle\alpha\rangle$. Then H acts freely on $(P/Z(P))^\sharp$, and the restriction of α to $Z(P)$ has order $(p-1)/2$.
Let $G = PH$ be the semidirect product of P and H, and let R denote a set of representatives for the conjugacy classes of p–elements in G. Then $|R \smallsetminus Z(P)| = p+1$, $|R \cap Z(P)^\sharp| = 2$, and

$$\begin{aligned} k(G) &= \sum_{u\in R} k(C_H(u)) = k(H) + |R \smallsetminus Z(P)| + 2|R \cap Z(P)^\sharp| \\ &= p-1+p+1+4 = 2p+4\,. \end{aligned}$$

We close with some remarks.

Remarks.

(i) Let p be a prime, and let G be a finite solvable group such that p^2 divides $|G|$. Then there is always a finite solvable group G_1 such that p divides $|G_1|$ and $k(G_1) < k(G)$. Indeed, if p is odd then let G_1 be the semidirect product of a group P of order p with a group of order 2 acting faithfully on P. Then

$$k(G_1) = \frac{p-1}{2} + 2 < (49p+1)/60 \leqslant k(G)$$

unless $p = 3$. However, if 9 divides $|G|$ then $k(G) \geqslant 6 > 3 = |Z_3|$, and if 4 divides $|G|$ then $k(G) \geqslant 4 > 2 = |Z_2|$.

(ii) Let p be a prime, and set

$$k_{\min}(p) := \min\{k(G) : G \text{ finite solvable group, } p \mid |G|\}\,.$$

Moreover, let G be a finite solvable group such that p divides $|G|$ and $k(G) = k_{\min}(p)$. Then, by (i), $|G|$ is not divisible by p^2. Furthermore, we have $O_{p'}(G) = 1$. It follows that $G = PX$ is the semidirect product of a group P

of order p with a subgroup X of $\mathrm{Aut}(P)$. Thus $x := |X|$ has to be a divisor of $p-1$ "close" to $\sqrt{p-1}$, and

$$k_{\min}(p) = x + (p-1)/x\,.$$

There are either one or two isomorphism classes of finite solvable groups G such that $k(G) = k_{\min}(p)$, depending on whether $p-1$ is a square or not.

(iii) We note that the function $p \mapsto k_{\min}(p)$ is not monotonically increasing. For example, we have

$$k_{\min}(23) = 13 > 11 = k_{\min}(29)\,.$$

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