

# A note on Olsson's Conjecture

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Dedicated to Geoffrey Robinson on the occasion of his 60th birthday

## Abstract

For a  $p$ -block  $B$  of a finite group  $G$  with defect group  $D$  Olsson conjectured that  $k_0(B) \leq |D : D'|$ , where  $k_0(B)$  is the number of characters in  $B$  of height 0 and  $D'$  denotes the commutator subgroup of  $D$ . Brauer deduced Olsson's Conjecture in the case where  $D$  is a dihedral 2-group using the fact that certain algebraically conjugate subsections are also conjugate in  $G$ . We generalize Brauer's argument for arbitrary primes  $p$  and arbitrary defect groups. This extends two results by Robinson. For  $p > 3$  we show that Olsson's Conjecture is satisfied for defect groups of  $p$ -rank 2 and for minimal non-abelian defect groups.

## 1 Introduction

In order to state Olsson's Conjecture we need some notations. Let  $\mathbf{R}$  be a complete discrete valuation ring with quotient field  $\mathbf{K}$  of characteristic 0. Moreover, let  $(\pi)$  be the maximal ideal of  $\mathbf{R}$  and  $\mathbf{F} := \mathbf{R}/(\pi)$ . We assume that  $\mathbf{F}$  is algebraically closed of characteristic  $p > 0$ . We fix a finite group  $G$ , and assume that  $\mathbf{K}$  contains all  $|G|$ -th roots of unity. Let  $B$  be a  $p$ -block of  $\mathbf{R}G$  (or simply of  $G$ ) with defect group  $D$ . We denote the set of irreducible ordinary characters by  $\text{Irr}(B)$  and its cardinality by  $k(B)$ . These characters split in  $k_i(B)$  characters of height  $i \in \mathbb{N}_0$ . Here the *height* of a character  $\chi$  in  $B$  is the largest integer  $h(\chi) \geq 0$  such that  $p^{h(\chi)}|G : D|_p \mid \chi(1)$ , where  $|G : D|_p$  denotes the highest  $p$ -power dividing  $|G : D|$ . We set  $\text{Irr}_0(B) := \{\chi \in \text{Irr}(B) : h(\chi) = 0\}$ . Finally, let  $\text{IBr}(B)$  be the set of irreducible Brauer characters and  $l(B) := |\text{IBr}(B)|$ .

In the situation above, Olsson conjectured in 1975 that we always have  $k_0(B) \leq |D : D'|$ , where  $D'$  denotes the derived subgroup of  $D$  (see [40]). This conjecture has been verified in some cases, but remains open in general. For example it was shown in [28] that Olsson's Conjecture for  $B$  would follow from the Alperin-McKay Conjecture for  $B$  (see also [54, 20]). Recall that the Alperin-McKay Conjecture predicts that  $k_0(B) = k_0(b)$ , where  $b$  is the Brauer correspondent of  $B$  in  $\mathbf{R}N_G(D)$ . In particular Olsson's Conjecture holds for  $p$ -solvable, symmetric or alternating groups by [39, 42, 34]. If  $D$  is abelian, Olsson's Conjecture follows from Brauer's  $k(B)$ -Conjecture  $k(B) \leq |D|$ . Moreover, Olsson's Conjecture is satisfied if  $D$  is metacyclic (see [50, 59]) or if  $p = 2$  and  $D$  is minimal non-abelian (see [51]). Hendren verified Olsson's Conjecture for some, but not all  $p$ -blocks with a non-abelian defect group of order  $p^3$  (see [23, 22]).

This paper is organized as follows. In the second section we introduce two results by Robinson and extend them in some sense using ideas of [52, 53]. In the third and fourth sections we generalize an argument of Brauer regarding a Galois action on subsections. In Section 5 we show that Olsson's Conjecture is fulfilled for controlled blocks with certain defect groups. In the last section we use the classification of finite simple groups to prove Olsson's Conjecture for defect groups of  $p$ -rank 2 and for minimal non-abelian defect groups if  $p > 3$  (in both cases). In particular, our results here settle most of the cases of Olsson's Conjecture left open in Hendren's papers [23, 22].

## 2 Subsection

The notion of  $B$ -subsections provides one tool in order to attack Olsson's Conjecture. Here a  $B$ -subsection is a pair  $(u, b_u)$ , where  $u \in D$  and  $b_u$  is a Brauer correspondent of  $B$  in  $\mathbf{R}C_G(u)$ . Robinson showed the following proposition (see [45]).

**Proposition 2.1** (Robinson). *If  $b_u$  has defect  $d$ , then  $k_0(B) \leq p^d \sqrt{l(b_u)}$ .*

We mention another result by Robinson which will be improved later (see Theorem 3.4 in [44]). Recall that a  $B$ -subsection  $(u, b_u)$  is called *major* if  $b_u$  and  $B$  have the same defect.

**Proposition 2.2** (Robinson). *If  $(u, b_u)$  is a major  $B$ -subsection such that  $l(b_u) = 1$ , then*

$$\sum_{i=0}^{\infty} p^{2i} k_i(B) \leq |D|.$$

In order to make these propositions clearer, we introduce the fusion system  $\mathcal{F}$  of  $B$ . For this we use the notation of [41, 32], and we assume that the reader is familiar with these articles. Let  $b_D$  be a Brauer correspondent of  $B$  in  $\mathbf{R}DC_G(D)$ . Then for every subgroup  $Q \leq D$  there is a unique block  $b_Q$  of  $\mathbf{R}QC_G(Q)$  such that  $(Q, b_Q) \leq (D, b_D)$ . We denote the inertial group of  $b_Q$  in  $N_G(Q)$  by  $N_G(Q, b_Q)$ . Then  $\text{Aut}_{\mathcal{F}}(Q) \cong N_G(Q, b_Q)/C_G(Q)$ .

The fusion of subsections is given by the following proposition (see [49]).

**Proposition 2.3.** *Let  $\mathcal{R}$  be a set of representatives for the  $\mathcal{F}$ -conjugacy classes of elements of  $D$  such that  $\langle u \rangle$  is fully  $\mathcal{F}$ -normalized for  $u \in \mathcal{R}$  ( $\mathcal{R}$  always exists). Then*

$$\{(u, b_u) : u \in \mathcal{R}\}$$

*is a set of representatives for the  $G$ -conjugacy classes of  $B$ -subsections, where  $b_u := b_{\langle u \rangle}$  has defect group  $C_D(u)$ .*

Brauer proved Olsson's Conjecture for 2-blocks with dihedral defect groups using a Galois action on the generalized decomposition numbers (see [10]). We provide the necessary definitions for that purpose. Let  $p^k$  be the order of  $u$ , and let  $\zeta := \zeta_{p^k}$  be a primitive  $p^k$ -th root of unity. Then the generalized decomposition numbers  $d_{\chi\varphi}^u$  for  $\chi \in \text{Irr}(B)$  and  $\varphi \in \text{IBr}(b_u)$  lie in the ring of integers  $\mathbb{Z}[\zeta]$ . Hence, there exist integers  $a_i^\varphi := (a_i^\varphi(\chi))_{\chi \in \text{Irr}(B)} \in \mathbb{Z}^{k(B)}$  such that

$$d_{\chi\varphi}^u = \sum_{i=0}^{\varphi(p^k)-1} a_i^\varphi(\chi) \zeta^i \quad (2.1)$$

(see Satz I.10.2 in [37]). Here  $\varphi(p^k)$  denotes Euler's totient function.

Let  $\mathcal{G}$  be the Galois group of the cyclotomic field  $\mathbb{Q}(\zeta)$  over  $\mathbb{Q}$ . Then  $\mathcal{G} \cong \text{Aut}(\langle u \rangle) \cong (\mathbb{Z}/p^k\mathbb{Z})^\times$  and we will often identify these groups. We will also interpret the elements of  $\mathcal{G}$  as integers in  $\{1, \dots, p^k\}$  by a slight abuse of notation. Then  $(u^\gamma, b_u)$  for  $\gamma \in \mathcal{G}$  is also a (*algebraically conjugate*) subsection and

$$\gamma(d_{\chi\varphi}^u) = d_{\chi\varphi}^{u^\gamma} = \sum_{i=0}^{\varphi(p^k)-1} a_i^\varphi(\chi) \zeta^{i\gamma}.$$

We use the opportunity to present a slight generalization of Lemma 1 in [53]. Here we call two matrices  $A, B \in \mathbb{Z}^{l \times l}$  *equivalent* if there exists a matrix  $S \in \text{GL}(l, \mathbb{Z})$  with  $A = S^T B S$ , where  $S^T$  denotes the transpose of  $S$ . This is just Brauer's notion of basic sets.

**Theorem 2.4.** *Let  $B$  be a  $p$ -block of  $G$ , and let  $(u, b_u)$  be a  $B$ -subsection. Let  $C_u = (c_{ij})$  be the Cartan matrix of  $b_u$  up to equivalence. Then for every positive definite, integral quadratic form  $q(x_1, \dots, x_{l(b_u)}) = \sum_{1 \leq i \leq j \leq l(b_u)} q_{ij} x_i x_j$  we have*

$$k_0(B) \leq \sum_{1 \leq i \leq j \leq l(b_u)} q_{ij} c_{ij}.$$

In particular

$$k_0(B) \leq \sum_{i=1}^{l(b_u)} c_{ii} - \sum_{i=1}^{l(b_u)-1} c_{i,i+1}.$$

If  $(u, b_u)$  is major, we can replace  $k_0(B)$  by  $k(B)$  in these formulas.

*Proof.* We imitate the proof of Lemma 1 in [53]. For this let  $\varphi_1, \dots, \varphi_l$  ( $l := l(b_u)$ ) be the irreducible Brauer characters of  $b_u$ . Then we have rows  $d_\chi := (d_{\chi\varphi_1}^u, \dots, d_{\chi\varphi_l}^u)$  for  $\chi \in \text{Irr}(B)$ , and it follows that

$$\sum_{1 \leq i \leq j \leq l} q_{ij} c_{ij} = \sum_{1 \leq i \leq j \leq l} \sum_{\chi \in \text{Irr}(B)} q_{ij} d_{\chi\varphi_i}^u \overline{d_{\chi\varphi_j}^u} = \sum_{\chi \in \text{Irr}(B)} q(d_\chi) \geq \sum_{\chi \in \text{Irr}_0(B)} q(d_\chi),$$

since  $q$  is positive definite. Hence, it suffices to show

$$\sum_{\chi \in \text{Irr}_0(B)} q(d_\chi) \geq k_0(B). \quad (2.2)$$

Let  $\chi \in \text{Irr}_0(B)$ . Then Corollary 2 in [11] implies that the contribution  $m_{\chi\chi}^{(u, b_u)}$  does not vanish. On the other hand we have  $m_{\chi\chi}^{(u, b_u)} = d_\chi C_u^{-1} \overline{d_\chi}^T$  by Eq. (5.2) in [9]. Hence, the rows  $d_\chi$  do not vanish for  $\chi \in \text{Irr}_0(B)$ .

Now let  $p^k$  be the order of  $u$ . Then  $d_{\chi\varphi_i}^u$  lies in the ring of integers  $\mathbb{Z}[\zeta]$  of the  $p^k$ -th cyclotomic field  $\mathbb{Q}(\zeta)$  for  $\zeta := e^{2\pi i/p^k}$ . Since  $q$  is positive definite,  $q(d_\chi)$  is a positive algebraic integer for  $\chi \in \text{Irr}_0(B)$ . Let  $\mathcal{G}$  be the Galois group of  $\mathbb{Q}(\zeta)$  over  $\mathbb{Q}$ . Then it is known that  $\mathcal{G}$  permutes the set  $\{q(d_\chi) : \chi \in \text{Irr}_0(B)\}$ . Hence,  $\prod_{\chi \in \text{Irr}_0(B)} q(d_\chi) \in \mathbb{Z}[\zeta]$  is rational and thus integral. Since all  $q(d_\chi)$  are positive, we get  $\prod_{\chi \in \text{Irr}_0(B)} q(d_\chi) \geq 1$ . Now (2.2) follows from the inequality of the arithmetic and geometric means. For the second claim we take the quadratic form corresponding to the Dynkin diagram of type  $A_l$  for  $q$ . The last claim is just Lemma 1 in [53].  $\square$

We present an application.

**Proposition 2.5.** *Let  $(u, b_u)$  be a  $B$ -subsection such that  $b_u$  has defect group  $Q$ . Then the following hold:*

(i) *If  $Q/\langle u \rangle$  is cyclic, we have*

$$k_0(B) \leq \left( \frac{|Q/\langle u \rangle| - 1}{l(b_u)} + l(b_u) \right) |\langle u \rangle| \leq |Q|.$$

(ii) *If  $|Q/\langle u \rangle| \leq 9$ , we have  $k_0(B) \leq |Q|$ .*

(iii) *Suppose  $p = 2$ . If  $Q/\langle u \rangle$  is metacyclic or minimal non-abelian or isomorphic to  $C_4 \wr C_2$ , we have  $k_0(B) \leq |Q|$ .*

*Proof.*

(i) It is well-known that  $b_u$  dominates a block  $\overline{b_u}$  of  $C_G(u)/\langle u \rangle$  with cyclic defect group  $Q/\langle u \rangle$  and  $l(\overline{b_u}) = l(b_u)$ . By [13, 46] the Cartan matrix  $b_u$  has the form  $|\langle u \rangle|(m + \delta_{ij})_{1 \leq i, j \leq l(b_u)}$  up to equivalence, where  $m := (|Q/\langle u \rangle| - 1)/l(b_u)$  is the multiplicity of  $\overline{b_u}$ . Now the claim follows from Theorem 2.4.

(ii) See Theorem 1 in [53].

(iii) If  $Q/\langle u \rangle$  is metacyclic, the claim follows as in Theorem 2 of [52]. If  $Q/\langle u \rangle$  is minimal non-abelian, the claim can easily be deduced from the results in [51, 15]. Finally, for  $D/\langle u \rangle \cong C_4 \wr C_2$  the result follows from [27].  $\square$

Since  $u \in Z(Q)$  in Proposition 2.5(i), the condition implies that  $Q$  is abelian of rank at most 2. It is known that the number  $l(b_u)$  in Proposition 2.5(i) equals the inertial index of  $\overline{b_u}$  (see [13]).

### 3 The case $p = 2$

Let  $p = 2$ , and let  $(u, b_u)$  be a  $B$ -subsection for a block  $B$  of  $G$ . Then by Lemma 2.3 we may assume that  $\langle u \rangle$  is fully  $\mathcal{F}$ -normalized, where  $\mathcal{F}$  is the fusion system of  $B$ . By Proposition 2.5 in [32]  $\langle u \rangle$  is also fully  $\mathcal{F}$ -centralized and

$$\text{Aut}_{\mathcal{F}}(\langle u \rangle) = \text{Aut}_D(\langle u \rangle) = \text{N}_D(\langle u \rangle) C_G(u) / C_G(u) \cong \text{N}_D(\langle u \rangle) / C_D(u).$$

Hence, Theorem 2.4(ii) in [31] implies that  $C_D(u)$  is a defect group of  $b_u$ .

**Theorem 3.1.** *Let  $B$  be a 2-block of a finite group  $G$  with defect group  $D$  and fusion system  $\mathcal{F}$ , and let  $(u, b_u)$  be a  $B$ -subsection such that  $\langle u \rangle$  is fully  $\mathcal{F}$ -normalized and  $b_u$  has Cartan matrix  $C_u = (c_{ij})$ . Let  $\text{IBr}(b_u) = \{\varphi_1, \dots, \varphi_{l(b_u)}\}$  such that  $\varphi_1, \dots, \varphi_m$  are stable under  $\text{N}_D(\langle u \rangle)$  and  $\varphi_{m+1}, \dots, \varphi_{l(b_u)}$  are not. Then  $m \geq 1$ . Suppose further that  $u$  is conjugate to  $u^{-5^n}$  for some  $n \in \mathbb{Z}$  in  $D$ . Then*

$$k_0(B) \leq \frac{|\text{N}_D(\langle u \rangle) / C_D(u)|}{\varphi(|\langle u \rangle|)} \sum_{1 \leq i \leq j \leq m} q_{ij} c_{ij} \quad (3.1)$$

for every positive definite, integral quadratic form  $q(x_1, \dots, x_m) = \sum_{1 \leq i \leq j \leq m} q_{ij} x_i x_j$ . In particular if  $l(b_u) = 1$ , we get

$$k_0(B) \leq \frac{|\text{N}_D(\langle u \rangle)|}{\varphi(|\langle u \rangle|)}. \quad (3.2)$$

If  $l(b_u) = 2$ , we may replace  $C_u$  by an equivalent matrix such that  $|C_D(u)| c_{11} / \det C_u$  is even and as small as possible. In this case (with the hypothesis above) we have

$$k_0(B) \leq \frac{|\text{N}_D(\langle u \rangle) / C_D(u)| c_{11}}{\varphi(|\langle u \rangle|)} \leq \frac{|\text{N}_D(\langle u \rangle)|}{\varphi(|\langle u \rangle|)}. \quad (3.3)$$

*Proof.* Let  $\chi \in \text{Irr}_0(B)$  and  $|\langle u \rangle| = 2^k$  for some  $k \geq 0$ . We write  $d_{\chi}^u := (d_{\chi\varphi_1}^u, \dots, d_{\chi\varphi_l}^u)$ , where  $l := l(b_u)$ . Then we have  $|C_D(u)| m_{\chi\chi}^{(u, b_u)} = d_{\chi}^u |C_D(u)| C_u^{-1} \overline{d_{\chi}^u}$  for the contribution  $m_{\chi\chi}^{(u, b_u)}$  (see Eq. (5.2) in [9]). By Corollary 2 in [11] it follows that

$$|C_D(u)| m_{\chi\chi}^{(u, b_u)} = |C_D(u)| (\chi^{(u, b_u)}, \chi)_G \not\equiv 0 \pmod{(\pi)}.$$

Since  $\zeta \equiv 1 \pmod{(\pi)}$ , we see that

$$d_{\chi\varphi_i}^u \equiv \gamma(d_{\chi\varphi_i}^u) \equiv \sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi) \pmod{(\pi)}$$

for  $\gamma \in \mathcal{G}$ . In particular  $d_{\chi\varphi_i}^u \equiv \overline{d_{\chi\varphi_i}^u} \pmod{(\pi)}$ . We write  $|C_D(u)| C_u^{-1} = (\tilde{c}_{ij})$ . Then it follows that

$$\begin{aligned} 0 \not\equiv |C_D(u)| m_{\chi\chi}^{(u, b_u)} &\equiv \sum_{1 \leq i, j \leq l} \tilde{c}_{ij} d_{\chi\varphi_i}^u \overline{d_{\chi\varphi_j}^u} \equiv \sum_{1 \leq i \leq l} \tilde{c}_{ii} (d_{\chi\varphi_i}^u)^2 \\ &\equiv \sum_{1 \leq i \leq l} \tilde{c}_{ii} \sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi)^2 \equiv \sum_{1 \leq i \leq l} \tilde{c}_{ii} \sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi) \pmod{(\pi)} \end{aligned}$$

Now every  $g \in \text{N}_D(\langle u \rangle)$  induces a permutation on  $\text{IBr}(b_u)$ . Let  $P_g$  be the corresponding permutation matrix. Then  $g$  also acts on the rows  $d_i^u := (d_{\chi\varphi_i}^u : \chi \in \text{Irr}(B))$  for  $i = 1, \dots, l$ , and it follows that  $C_u P_g = P_g C_u$ . Hence, we also have  $C_u^{-1} P_g = P_g C_u^{-1}$  for all  $g \in \text{N}_D(\langle u \rangle)$ . If  $\{\varphi_{m_1}, \dots, \varphi_{m_2}\}$  ( $m < m_1 < m_2 \leq l$ ) is an orbit under  $\text{N}_D(\langle u \rangle)$ , it follows that  $d_{\chi\varphi_{m_1}}^u \equiv \dots \equiv d_{\chi\varphi_{m_2}}^u \pmod{(\pi)}$  and  $\tilde{c}_{m_1 m_1} = \dots = \tilde{c}_{m_2 m_2}$ . Since the length of this orbit is even, we get

$$\sum_{1 \leq i \leq m} \tilde{c}_{ii} \sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi) \not\equiv 0 \pmod{2}.$$

In particular  $m \geq 1$ . In case  $|\langle u \rangle| \leq 2$  this simplifies to

$$\sum_{1 \leq i \leq m} \tilde{c}_{ii} a_0^i(\chi) \not\equiv 0 \pmod{2}.$$

We show that this holds in general. Thus, let  $k \geq 2$  and  $i \in \{1, \dots, m\}$ . Since  $(u, b_u)$  is conjugate to  $(u^{-5^n}, b_u)$  and  $\varphi_i$  is stable, we have

$$\sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi) \zeta^j = d_{\chi\varphi_i}^u = d_{\chi\varphi_i}^{u^{-5^n}} = \sum_{j=0}^{2^{k-1}-1} a_j^i(\chi) \zeta^{-5^n j}.$$

Moreover, for every  $j \in \{0, \dots, \varphi(2^k) - 1\}$  there is some  $j_1 \in \{0, \dots, \varphi(2^k) - 1\}$  such that  $\zeta^{-5^n j} = \pm \zeta^{j_1}$ . In order to compare coefficients observe that

$$\zeta^j = \zeta^{-5^n j} \implies j \equiv -5^n j \pmod{2^k} \implies 1 \equiv -5^n \pmod{2^k / \gcd(2^k, j)} \implies j = 0.$$

Hence, the set  $\{\pm \zeta^j : j = 1, \dots, \varphi(2^k) - 1\}$  splits under the action of  $\langle -5^n + 2^k \mathbb{Z} \rangle$  into orbits of even length. This shows  $\sum_{j=0}^{\varphi(2^k)-1} a_j^i(\chi) \equiv a_0^i(\chi) \pmod{2}$ . Hence,

$$\sum_{1 \leq i \leq m} \tilde{c}_{ii} a_0^i(\chi) \not\equiv 0 \pmod{2} \quad (3.4)$$

for every  $\chi \in \text{Irr}_0(B)$ . In particular, there is an  $i \in \{1, \dots, m\}$  such that  $a_0^i(\chi) \neq 0$ . This gives

$$k_0(B) \leq \sum_{1 \leq i \leq j \leq m} q_{ij}(a_0^i, a_0^j)$$

(see proof of Theorem 2.4).

Now let  $k$  again be arbitrary. Observe that  $a_0^i = \varphi(2^k)^{-1} \sum_{\gamma \in \mathcal{G}} \gamma(d_i^u)$  for  $i \in \{1, \dots, m\}$ . By the orthogonality relations for generalized decomposition numbers we have  $(d_i^{u^\gamma}, d_i^{u^\delta}) = c_{ii}$  for  $\gamma, \delta \in \mathcal{G}$  if  $u^\gamma$  and  $u^\delta$  are conjugate under  $N_D(\langle u \rangle)$  (see Theorem 5.4.11 in [35] for example). Otherwise we have  $(d_i^{u^\gamma}, d_i^{u^\delta}) = 0$ . This implies

$$(a_0^i, a_0^j) = \frac{1}{\varphi(2^k)^2} \sum_{\gamma, \delta \in \mathcal{G}} (d_i^{u^\gamma}, d_i^{u^\delta}) = \frac{|N_D(\langle u \rangle) / C_D(u)|}{\varphi(2^k)} c_{ij},$$

and (3.1) follows. In case  $l = 1$  we have  $C = (|C_D(u)|)$ , and (3.2) is also clear.

Now assume  $l = 2$ . Here we can use (3.4) in a stronger sense. We have  $m = 2$ . Since  $|C_D(u)|$  occurs as elementary divisor of  $C_u$  exactly once, we see that the rank of  $\frac{|C_D(u)|}{\det C_u} C_u \pmod{2}$  is 1. Hence,  $\frac{|C_D(u)|}{\det C_u} C_u \pmod{2}$  has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pmod{2}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}, \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \pmod{2}.$$

Now it is easy to see that we may replace  $C_u$  by an equivalent matrix (still denoted by  $C_u = (c_{ij})$ ) such that  $|C_D(u)|c_{11}/\det C_u$  is even and as small as possible. Then we also have to replace the rows  $d_1^u$  and  $d_2^u$  by linear combinations of each other. This gives rows  $\widehat{d}_i^u$  and  $\widehat{a}_j^i$  for  $i = 1, 2$  and  $j = 0, \dots, \varphi(2^k) - 1$ . Observe that the contributions do not depend on the representative of the equivalence class of  $C_u$ . Moreover,  $\tilde{c}_{11}$  is odd and  $\tilde{c}_{22}$  is even. Hence, (3.4) takes the form

$$\widehat{a}_0^1(\chi) \not\equiv 0 \pmod{2}$$

for all  $\chi \in \text{Irr}_0(B)$ . Since both  $\varphi_1$  and  $\varphi_2$  are stable under  $N_D(\langle u \rangle)$ , we have  $\gamma(\widehat{d}_1^u) = \widehat{d}_1^u$  for all  $\gamma \in \text{Aut}_{\mathcal{F}}(\langle u \rangle)$ . Hence,

$$k_0(B) \leq (\widehat{a}_0^1, \widehat{a}_0^1) = \frac{|N_D(\langle u \rangle) / C_D(u)| c_{11}}{\varphi(2^k)}$$

as above. It remains to show that  $c_{11} \leq |C_D(u)|$ . The reduction theory of quadratic forms gives an equivalent matrix  $C'_u = (c'_{ij})$  such that  $0 \leq 2c'_{12} \leq \min\{c'_{11}, c'_{22}\}$ . In case  $c'_{12} = 0$  we may assume  $c_{11} \leq c'_{11} = |C_D(u)|$ , since  $|C_D(u)|$  is the largest elementary divisor of  $C'_u$ . Hence, let  $c'_{12} \geq \det C_u / |C_D(u)| =: \alpha$ . It follows that

$$3\alpha^2 \leq 3(c'_{12})^2 \leq c'_{11}c'_{22} - (c'_{12})^2 = \det C'_u \leq \frac{|C_D(u)|^2}{2}$$

and  $\alpha \leq |C_D(u)|/4$ . It was shown in the proof of Theorem 1 of [52] that

$$\max\{c'_{11}, c'_{22}\} \leq c'_{11} + c'_{22} - c'_{12} \leq c'_{11} + c'_{22} - \alpha \leq \alpha \frac{|C_D(u)|/\alpha + 3}{2} = \frac{|C_D(u)| + 3\alpha}{2} \leq |C_D(u)|.$$

If  $\alpha^{-1}c'_{11}$  or  $\alpha^{-1}c'_{22}$  is even, the result follows from the minimality of  $c_{11}$ . Otherwise we replace  $C'_u$  by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} C'_u \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} c'_{11} + c'_{22} - 2c'_{12} & c'_{12} - c'_{22} \\ c'_{12} - c'_{22} & c'_{22} \end{pmatrix}.$$

Then  $c_{11} \leq c'_{11} + c'_{22} - 2c'_{12} \leq |C_D(u)|$ . This finishes the proof.  $\square$

In the situation of Theorem 3.1 we have  $u \in Z(C_G(u))$ . Hence, all Cartan invariants  $c_{ij}$  are divisible by  $|\langle u \rangle|$ . This shows that the right hand side of (3.1) is always an integer. It is also known that  $k_0(B)$  is divisible by 4 unless  $|D| \leq 2$ .

Observe that the subsection  $(u, b_u)$  in Theorem 3.1 cannot be major unless  $|\langle u \rangle| \leq 2$ , since then  $u$  would be contained in  $Z(D)$ .

If  $m = l(b_u)$  in Theorem 3.1, it suffices to know the Cartan matrix  $C_u$  only up to equivalence. For, replacing  $C_u$  by an equivalent matrix is essentially the same as taking another quadratic form  $q$ . However, for  $m < l(b_u)$  we really have to use the ‘‘exact’’ Cartan matrix  $C_u$  which is unknown in most cases. For  $p > 2$  there are not always stable characters in  $\text{IBr}(b_u)$  (see Proposition (2E)(ii) and the example following it in [26]).

We give an example. Let  $D$  be a (non-abelian) 2-group of maximal class. Then there is an element  $x \in D$  such that  $|D : \langle x \rangle| = 2$  and  $x$  is conjugate to  $x^{-5^n}$  for some  $n \in \{0, |\langle x \rangle|/8\}$  under  $D$ . Since  $\langle x \rangle \trianglelefteq D$ , the subgroup  $\langle x \rangle$  is fully  $\mathcal{F}$ -normalized, and  $b_x$  has cyclic defect group  $C_D(x) = \langle x \rangle$ . Thus, Dade’s Theorem on blocks with cyclic defect groups implies  $l(b_x) = 1$ . Hence, Theorem 3.1 shows Olsson’s Conjecture  $k_0(B) \leq 4 = |D : D'|$ . This was already proved in [10, 40].

On the other hand, we cannot improve Theorem 3.1 or Theorem 2.4 if  $u$  is not conjugate to  $u^{-5^n}$  in  $D$ . Indeed, if  $D$  a modular 2-group and  $x \in D$  such that  $|D : \langle x \rangle| = 2$ , then  $B$  is nilpotent (see [16]) and  $k_0(B) = |D : D'| = |D|/2 = |C_D(x)|$ .

We give a more general example.

**Proposition 3.2.** *Let  $D$  be a 2-group and  $x \in D$  such that  $|D : \langle x \rangle| \leq 4$ , and suppose that one of the following holds:*

- (i)  $x$  is conjugate to  $x^{-5^n}$  in  $D$  for some  $n \in \mathbb{Z}$ ,
- (ii)  $\langle x \rangle \trianglelefteq D$ .

*Then Olsson’s Conjecture holds for all blocks with defect group  $D$ .*

*Proof.* Let  $B$  be a block with defect group  $D$  and fusion system  $\mathcal{F}$ . By [50] we may assume that  $D$  is non-metacyclic.

- (i) By hypothesis,  $x$  is conjugate to  $x^{-5^n}$  in  $\mathcal{F}$ . This condition is preserved if we replace  $x$  by an  $\mathcal{F}$ -conjugate. Hence, we may assume that  $\langle x \rangle$  is fully  $\mathcal{F}$ -normalized. Then  $x$  is conjugate to  $x^{-5^n}$  in  $D$ . In particular  $|C_D(x)/\langle x \rangle| \leq 2$ . Hence,  $b_x$  dominates a block of  $C_G(x)/\langle x \rangle$  with cyclic defect group  $C_D(x)/\langle x \rangle$ . This shows  $l(b_x) = 1$ . Now we can apply Theorem 3.1 which gives  $k_0(B) \leq 8$ . In case  $|D : D'| = 4$  a theorem of Taussky (see for example Proposition 1.6 in [8]) implies that  $D$  has maximal class which was excluded.
- (ii) We consider the order of  $C_D(x)$ .

**Case 1:**  $C_D(x) = \langle x \rangle$ .

Since  $D$  is non-metacyclic,  $D/\langle x \rangle$  is non-cyclic. Hence, we are in case (i).

**Case 2:**  $x \in Z(D)$ .

If  $D$  is abelian, the result follows from Theorem 2 in [52]. Thus, we may assume that  $D$  is non-abelian. Then every conjugacy class of  $D$  has length at most 2. By a result of Knoche (see for example Aufgabe III.24b in [24]) this is equivalent to  $|D'| = 2$ . Let  $y \in D \setminus Z(D)$ . Then  $C_D(y)$  is non-cyclic. After replacing  $y$  by

$xy$  if necessary, we have  $|\langle x \rangle| = |\langle y \rangle|$ . By Proposition 2.5 it suffices to show that  $\langle y \rangle$  is fully  $\mathcal{F}$ -normalized. By Alperin's Fusion Theorem (see [32]) every  $\mathcal{F}$ -isomorphism on  $\langle y \rangle$  is a composition of automorphisms of  $\mathcal{F}$ -essential subgroups containing  $y$  or of  $D$  itself. Assume that  $E < D$  is  $\mathcal{F}$ -essential such that  $\langle y \rangle \leq E$ . Since  $E$  is metacyclic and  $\text{Aut}(E)$  is not a 2-group, Lemma 1 in [33] implies  $E \cong Q_8$  or  $E \cong C_2 \times C_2$ ; in particular  $|D| \leq 16$ . Moreover, Proposition 1.8 and Proposition 10.17 in [8] imply that  $D$  has maximal class, because every  $\mathcal{F}$ -essential subgroup is self-centralizing. This contradiction shows that there are no  $\mathcal{F}$ -essential subgroups containing  $y$ . Then of course  $\langle y \rangle$  is fully  $\mathcal{F}$ -normalized.

**Case 3:**  $|C_D(x)/\langle x \rangle| = 2$ .

Let  $y \in C_D(x) \setminus \langle x \rangle$  be of order 2. If  $z \in D \setminus C_D(x)$ , we may assume that  $\langle x, z \rangle$  is a modular 2-group by (i). In particular we have  $|\langle z \rangle| = 2$  after replacing  $z$  by  $zx^m$  for some  $m \in \mathbb{Z}$  if necessary. Let  $|\langle x \rangle| = 2^r$  for some  $r \in \mathbb{N}$ . Since  $\langle x \rangle \trianglelefteq D$ , we have  $zyz^{-1} \in \{y, yx^{2^{r-1}}\}$ . In case  $zyz^{-1} = yx^{2^{r-1}}$  it is easy to see that  $|D : \langle xy \rangle| = 4$  and  $xy \in Z(D)$ . Then we are done by Case 2. Thus, we may assume that  $zyz^{-1} = y$  and  $y \in Z(D)$ . Then  $D$  is given as follows:

$$D = \langle x, z \rangle \times \langle y \rangle \cong M_{2^{r+1}} \times C_2,$$

where  $M_{2^{r+1}}$  denotes the modular 2-group of order  $2^{r+1}$  and  $C_2$  denotes a cyclic group of order 2. Now we have  $|D'| = 2$  and the claim follows from Proposition 2.5 applied to the subsection  $(x, b_x)$ . Here observe that  $\langle x \rangle$  is fully  $\mathcal{F}$ -normalized, since  $\langle x \rangle \trianglelefteq D$ .  $\square$

We like to point out that every subgroup of  $D$  is fully  $\mathcal{F}$ -normalized whenever  $\mathcal{F}$  is controlled by  $\text{Aut}_{\mathcal{F}}(D)$ . The groups in Proposition 3.2 were given explicitly by generators and relations in [38].

By the propositions in [53] it is easy to see that Olsson's Conjecture holds for 2-blocks with defect at most 4. For defect groups  $D$  of order 32 one can show with GAP [18] that there is always an element  $x \in D$  such that  $|C_D(x)| = |D : D'|$ . If in addition  $D$  is abelian, Olsson's Conjecture follows from Corollary 2 in [53] for every block with defect group  $D$ . If  $D$  is non-abelian, then  $|C_D(x)/\langle x \rangle| \leq 8$ . Thus, by Proposition 2.5(ii) Olsson's Conjecture also holds for controlled 2-blocks of defect 5.

## 4 The case $p > 2$

Now we turn to the case where  $B$  is a  $p$ -block of  $G$  for an odd prime  $p$ . We fix some notations for this section. As before  $(u, b_u)$  is a  $B$ -subsection such that  $|\langle u \rangle| = p^k$ . Moreover,  $\zeta \in \mathbb{C}$  is a primitive  $p^k$ -th root of unity. Since the situation is more complicated for odd primes, we assume further that  $l(b_u) = 1$ . We write  $\text{IBr}(b_u) = \{\varphi_u\}$ . Then the generalized decomposition numbers  $d_{\chi\varphi_u}^u$  for  $\chi \in \text{Irr}(B)$  form a column  $d(u)$ . As in the case  $p = 2$  we can write

$$d(u) = \sum_{i=0}^{\varphi(p^k)-1} a_i^u \zeta^i$$

with  $a_i^u \in \mathbb{Z}^{k(B)}$  (change of notation!). We define the following matrix

$$A := (a_i^u(\chi) : i = 0, \dots, \varphi(p^k) - 1, \chi \in \text{Irr}(B)) \in \mathbb{Z}^{\varphi(p^k) \times k(B)}.$$

The next lemma uses the same idea as in case  $p = 2$ .

**Lemma 4.1.** *Let  $(u, b_u)$  be a  $B$ -subsection with  $|\langle u \rangle| = p^k$  and  $l(b_u) = 1$ .*

(i) *For  $\chi \in \text{Irr}_0(B)$  we have*

$$\sum_{i=0}^{\varphi(p^k)-1} a_i^u(\chi) \not\equiv 0 \pmod{p}.$$

(ii) *If  $(u, b_u)$  is major and  $\chi \in \text{Irr}(B)$ , then  $p^{h(\chi)} \mid a_i^u(\chi)$  for  $i = 0, \dots, \varphi(p^k) - 1$  and*

$$\sum_{i=0}^{\varphi(p^k)-1} a_i^u(\chi) \not\equiv 0 \pmod{p^{h(\chi)+1}}.$$

*Proof.*

- (i) Let  $d$  be the defect of  $b_u$ . Since  $l(b_u) = 1$ , we have  $p^d m_{\chi\chi}^{(u,b_u)} = d_{\chi\varphi_u}^u \overline{d_{\chi\varphi_u}^u}$  for the contribution  $m_{\chi\chi}^{(u,b_u)}$  (see Eq. (5.2) in [9]). By Corollary 2 in [11] it follows that

$$p^d m_{\chi\chi}^{(u,b_u)} = p^d (\chi^{(u,b_u)}, \chi)_G \not\equiv 0 \pmod{(\pi)}$$

and  $d_{\chi\varphi_u}^u \not\equiv 0 \pmod{(\pi)}$ . Since  $\zeta \equiv 1 \pmod{(\pi)}$ , the claim follows from (2.1).

- (ii) Let  $\psi \in \text{Irr}_0(B)$ . Then (5G) in [9] implies

$$h(\chi) = \nu(|D| m_{\chi\psi}^{(u,b_u)}) = \nu(d_{\chi\varphi_u}^u) + \nu(\overline{d_{\psi\varphi_u}^u}),$$

where  $\nu$  is the  $p$ -adic valuation. Thus,  $h(\chi) = \nu(d_{\chi\varphi_u}^u)$  follows from (i). Now the claim is easy to see.  $\square$

The proof of the main theorem of this section is an application of the next proposition.

**Proposition 4.2.** *For every positive definite, integral quadratic form  $q(x_1, \dots, x_{\varphi(p^k)}) = \sum_{1 \leq i \leq j \leq \varphi(p^k)} q_{ij} x_i x_j$  we have*

$$k_0(B) \leq \sum_{1 \leq i \leq j \leq \varphi(p^k)} q_{ij} (a_{i-1}^u, a_{j-1}^u). \quad (4.1)$$

If (in addition)  $(u, b_u)$  is major, we can replace  $k_0(B)$  by  $\sum_{i=0}^{\infty} p^{2i} k_i(B)$  in (4.1).

*Proof.* By Lemma 4.1(i) every column  $a^u(\chi)$  of  $A$  corresponding to a character  $\chi$  of height 0 does not vanish. Hence, we have

$$k_0(B) \leq \sum_{\chi \in \text{Irr}(B)} q(a^u(\chi)) = \sum_{\chi \in \text{Irr}(B)} \sum_{1 \leq i \leq j \leq \varphi(p^k)} q_{ij} a_{i-1}^u(\chi) a_{j-1}^u(\chi) = \sum_{1 \leq i \leq j \leq \varphi(p^k)} q_{ij} (a_{i-1}^u, a_{j-1}^u).$$

If  $(u, b_u)$  is major and  $\chi \in \text{Irr}(B)$ , then  $p^{-h(\chi)} a^u(\chi)$  is a non-vanishing integral column by Lemma 4.1(ii). In this case we have

$$\sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \sum_{\chi \in \text{Irr}(B)} p^{2h(\chi)} q(p^{-h(\chi)} a^u(\chi)) = \sum_{1 \leq i \leq j \leq \varphi(p^k)} q_{ij} (a_{i-1}^u, a_{j-1}^u).$$

The second claim follows.  $\square$

Notice that we have used only a weak version of Lemma 4.1 in the proof above.

In order to find a suitable quadratic form it is often very useful to replace  $A$  by  $UA$  for some integral matrix  $U \in \text{GL}(\varphi(p^k), \mathbb{Q})$  (observe that the argument in the proof of Theorem 4.2 remains correct).

However, we need a more explicit expression of the scalar products  $(a_i^u, a_j^u)$ . For this reason we introduce an auxiliary lemma about inverses of Vandermonde matrices. Let  $\mathcal{G} = \{\sigma_1, \dots, \sigma_{\varphi(p^k)}\}$ .

**Lemma 4.3.** *The inverse of the Vandermonde matrix  $V := (\sigma_i(\zeta)^{j-1})_{i,j=1}^{\varphi(p^k)}$  is given by*

$$V^{-1} = p^{-k} (\sigma_j(t_{i-1}))_{i,j=1}^{\varphi(p^k)},$$

where  $t_i = \zeta^{-i} - \zeta^{i'}$  and  $-i \equiv i' \pmod{p^{k-1}}$  for  $i' \in \{1, \dots, p^{k-1}\}$ .

*Proof.* For  $i, j \in \{0, \dots, \varphi(p^k) - 1\}$  we have

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(t_i) \sigma_l(\zeta)^j = \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i} - \zeta^{j+i'}).$$



Assume first that  $i = j$ . Then  $\zeta^{j-i} = 1$  and  $j + i' = i + i'$  is divisible by  $p^{k-1}$  but not by  $p^k$ . Hence,  $\zeta^{j+i'}$  is a primitive  $p$ -th root of unity. Since the second coefficient of the  $p$ -th cyclotomic polynomial  $\Phi_p(X) = X^{p-1} + X^{p-2} + \dots + X + 1$  is 1, we get  $\sum_{l=1}^{\varphi(p^k)} \zeta^{j+i'} = -p^{k-1}$ . This shows that

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(1 - \zeta^{i+i'}) = \varphi(p^k) + p^{k-1} = p^k.$$

Now let  $i \neq j$ . Then  $j-i \not\equiv 0 \pmod{p^k}$  and  $j+i' \not\equiv 0 \pmod{p^k}$ . Moreover,  $j-i \equiv j+i' \pmod{p^{k-1}}$ , since  $i+i' \equiv 0 \pmod{p^{k-1}}$ . Assume first that  $j-i \not\equiv 0 \pmod{p^{k-1}}$ . Then  $\zeta^{j-i}$  is a primitive  $p^s$ -th root of unity for some  $s \geq 2$ . Since the second coefficient of the  $p^s$ -th cyclotomic polynomial  $\Phi_{p^s}(X) = X^{(p-1)p^{s-1}} + X^{(p-2)p^{s-1}} + \dots + X^{p^{s-1}} + 1$  (see Lemma I.10.1 in [37]) is 0, we have  $\sum_{l=1}^{\varphi(p^k)} \zeta^{j-i} = 0$ . The same holds for  $j + i'$ . Finally let  $j - i \equiv 0 \pmod{p^{k-1}}$ . Then we have (as in the first part of the proof)

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i} - \zeta^{j+i'}) = -p^{k-1} + p^{k-1} = 0.$$

This proves the claim.  $\square$

Now let  $\mathcal{A} := \text{Aut}_{\mathcal{F}}(\langle u \rangle) \leq \mathcal{G}$ . The next proposition shows that the scalar products  $(a_i^u, a_j^u)$  only depend on  $p$ ,  $k$  and  $\mathcal{A}$ .

**Proposition 4.4.** *We have*

$$p^{k-d}(a_i^u, a_j^u) = |\{\tau \in \mathcal{A} : p^k \mid i - j\tau\}| - |\{\tau \in \mathcal{A} : p^k \mid i + j'\tau\}| + |\{\tau \in \mathcal{A} : p^k \mid i' - j'\tau\}| - |\{\tau \in \mathcal{A} : p^k \mid i' + j\tau\}|. \quad (4.2)$$

*Proof.* Let  $W := (d_{\chi}^{\sigma_i(u)} : i = 1, \dots, \varphi(p^k), \chi \in \text{Irr}(B))$  be a part of the generalized decomposition matrix. If  $V$  is the Vandermonde matrix in Lemma 4.3, we have  $VA = W$  and  $A = V^{-1}W$ . This shows

$$((a_{i-1}^u, a_{j-1}^u))_{i,j=1}^{\varphi(p^k)} = AA^T = V^{-1}WW^T V^{-T} = V^{-1}W\overline{W}^T \overline{V}^{-T}.$$

Now let  $S := (s_{ij})_{i,j=1}^{\varphi(p^k)}$ , where

$$s_{ij} := \begin{cases} 1 & \text{if } \sigma_i \sigma_j^{-1} \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the orthogonality relations (see proof of Theorem 3.1) imply  $W\overline{W}^T = p^d S$ . It follows that

$$\begin{aligned} p^{2k-d}(a_i^u, a_j^u) &= \sum_{l=1}^{\varphi(p^k)} \sigma_l(t_i) \sum_{m=1}^{\varphi(p^k)} s_{lm} \sigma_m(\overline{t_j}) = \sum_{l=1}^{\varphi(p^k)} \sum_{\tau \in \mathcal{A}} \sigma_l(t_i \tau(\overline{t_j})) \\ &= \sum_{\tau \in \mathcal{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l((\zeta^{-i} - \zeta^{i'})_{\tau} (\zeta^j - \zeta^{-j'})) \\ &= \sum_{\tau \in \mathcal{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i} + \zeta^{i'-j'\tau} - \zeta^{-i-j'\tau} - \zeta^{i'+j\tau}). \end{aligned} \quad (4.3)$$

As in the proof of Lemma 4.3 we have

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i}) = \begin{cases} \varphi(p^k) & \text{if } p^k \mid j\tau - i, \\ 0 & \text{if } p^{k-1} \nmid j\tau - i, \\ -p^{k-1} & \text{otherwise.} \end{cases}$$

This can be combined to

$$\sum_{\tau \in \mathcal{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i}) = p^k |\{\tau \in \mathcal{A} : p^k \mid j\tau - i\}| - p^{k-1} |\{\tau \in \mathcal{A} : p^{k-1} \mid j\tau - i\}|.$$

We get similar expressions for the other numbers  $i' - j'\tau$ ,  $-i - j'\tau$  and  $i' + j\tau$ . Since  $i + i' \equiv j + j' \equiv 0 \pmod{p^{k-1}}$ , we have  $j\tau - i \equiv i' - j'\tau \equiv -i - j'\tau \equiv i' + j\tau \pmod{p^{k-1}}$ . Thus, the terms of the form  $p^{k-1} |\{\dots\}|$  in (4.3) cancel out each other. This proves the proposition.  $\square$

Since the group  $\text{Aut}(\langle u \rangle)$  is cyclic,  $\mathcal{A}$  is uniquely determined by its order. We introduce a notation.

**Definition 4.5.** Let  $\mathcal{A}$  be as in Proposition 4.4. Then we define  $\Gamma(k, |\mathcal{A}|)$  as the minimum of the expressions

$$\sum_{1 \leq i \leq j \leq \varphi(p^k)} q_{ij}(a_{i-1}^u, a_{j-1}^u),$$

where  $q$  ranges over all positive definite, integral quadratic forms. By Theorem 4.2 we have  $k_0(B) \leq \Gamma(k, |\mathcal{A}|)$ , and  $\sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \Gamma(k, |\mathcal{A}|)$  if  $(u, b_u)$  is major.

We will calculate  $\Gamma(k, |\mathcal{A}|)$  by induction on  $k$ . First we collect some easy facts.

**Lemma 4.6.** Let  $\mathcal{H} \leq (\mathbb{Z}/p^k\mathbb{Z})^\times$  where we regard  $\mathcal{H}$  as a subset of  $\{1, \dots, p^k\}$ . Then  $|\{\sigma \in \mathcal{H} : \sigma \equiv 1 \pmod{p^j}\}| = \gcd(|\mathcal{H}|, p^{k-j})$  for  $1 \leq j \leq k$ .

*Proof.* The canonical epimorphism  $(\mathbb{Z}/p^k\mathbb{Z})^\times \rightarrow (\mathbb{Z}/p^j\mathbb{Z})^\times$  has kernel  $\mathcal{K}$  of order  $p^{k-j}$ . Hence,  $|\{\sigma \in \mathcal{H} : \sigma \equiv 1 \pmod{p^j}\}| = |\mathcal{H} \cap \mathcal{K}| = \gcd(|\mathcal{H}|, p^{k-j})$ , since the  $p$ -subgroups of the cyclic group  $(\mathbb{Z}/p^k\mathbb{Z})^\times$  are totally ordered by inclusion.  $\square$

**Lemma 4.7.** Let  $|\mathcal{A}|_p$  be the order of a Sylow  $p$ -subgroup of  $\mathcal{A}$ . Then we have

$$(a_0^u, a_0^u) = (|\mathcal{A}| + |\mathcal{A}|_p) p^{d-k}$$

and

$$\frac{p^{k-d}}{\gcd(|\mathcal{A}|_p, j)} (a_i^u, a_j^u) \in \{0, \pm 1, \pm 2\}$$

for  $i + j > 0$ . If  $a_i^u \neq 0$  for some  $i \geq 1$ , then  $(a_i^u, a_i^u) = 2p^{d-k} \gcd(|\mathcal{A}|_p, j)$ . Moreover,  $(a_i^u, a_j^u) = 0$  whenever  $\gcd(i, p^{k-1}) \neq \gcd(j, p^{k-1})$ .

*Proof.* For  $i = j = 0$  we have  $i + j'\tau = p^{k-1}\tau \not\equiv 0 \pmod{p^k}$  and  $i' + j\tau = p^{k-1} \not\equiv 0 \pmod{p^k}$  for all  $\tau \in \mathcal{A}$ . Moreover, by Lemma 4.6 there are precisely  $|\mathcal{A}|_p$  elements  $\tau \in \mathcal{A}$  such that  $i' - j'\tau = p^{k-1}(1 - \tau) \equiv 0 \pmod{p^k}$ . The first claim follows from Proposition 4.4.

Now let  $i + j > 0$  and  $\tau \in \mathcal{A}$  such that  $i \equiv j\tau \pmod{p^k}$ . Then we have  $j \neq 0$ . Assume that also  $\tau_1 \in \mathcal{A}$  satisfies  $i \equiv j\tau_1 \pmod{p^k}$ . Then  $j(\tau - \tau_1) \equiv 0 \pmod{p^k}$  and  $\tau^{-1}\tau_1 \equiv 1 \pmod{p^k/\gcd(p^k, j)}$ . Thus, Lemma 4.6 implies

$$|\{\tau \in \mathcal{A} : p^k \mid i - j\tau\}| \in \{0, \gcd(|\mathcal{A}|_p, j)\}.$$

The same argument also works for the other summands in (4.2), since  $\gcd(|\mathcal{A}|_p, j) = \gcd(|\mathcal{A}|_p, j')$ . This gives

$$p^{k-d} (a_i^u, a_j^u) \in \{0, \pm \gcd(|\mathcal{A}|_p, j), \pm 2 \gcd(|\mathcal{A}|_p, j)\}$$

whenever  $i + j > 0$ .

Suppose  $i \geq 1$  and  $i \equiv i\tau \pmod{p^k}$  for some  $\tau \in \mathcal{A}$ . Then  $\tau \equiv 1 \pmod{p}$  and thus  $i \equiv i\tau - (i + i')(\tau - 1) \equiv -i'\tau + i + i' \pmod{p^k}$ . Hence  $i' \equiv i'\tau \pmod{p^k}$ . This shows  $|\{\tau \in \mathcal{A} : p^k \mid i - i\tau\}| = |\{\tau \in \mathcal{A} : p^k \mid i' - i'\tau\}|$ . Moreover, we have  $|\{\tau \in \mathcal{A} : p^k \mid i + i'\tau\}| = |\{\tau \in \mathcal{A} : p^k \mid i\tau^{-1} + i'\}| = |\{\tau \in \mathcal{A} : p^k \mid i' + i\tau\}|$ . This shows  $a_i^u = 0$  or  $(a_i^u, a_i^u) = 2p^d \gcd(|\mathcal{A}|_p, j)/p^k$ .

Finally suppose that  $\gcd(i, p^{k-1}) \neq \gcd(j, p^{k-1})$ . Then  $i \not\equiv j\tau \pmod{p^{k-1}}$  and thus  $p^k \nmid i - j\tau$  for all  $\tau \in \mathcal{A}$ . The same holds for the other terms in (4.2), since  $i + i' \equiv j + j' \equiv 0 \pmod{p^{k-1}}$ . The last claim follows.  $\square$

**Proposition 4.8.** *We have*

$$\Gamma(1, |\mathcal{A}|) = (|\mathcal{A}| + (p-1)/|\mathcal{A}|)p^{d-1}.$$

*Proof.* Since  $|\mathcal{A}| \mid p-1$ , we have  $|\mathcal{A}|_p = 1$ . Hence,  $(a_0^u, a_0^u) = (|\mathcal{A}| + 1)p^{d-1}$  and  $(a_i^u, a_j^u) \in \{0, \pm p^{d-1}, \pm 2p^{d-1}\}$  for  $i + j > 0$  by Lemma 4.7. First we determine the indices  $i$  such that  $a_i^u = 0$ . Observe that we always have  $i' = 1$ . In particular for all  $i, j$  we have  $p \mid i' - j'\tau$  for  $\tau = 1$ . It follows that  $a_i^u = 0$  if and only if  $-i \equiv \tau \pmod{p}$  for some  $\tau \in \mathcal{A}$ . We write this condition in the form  $-i \in \mathcal{A}$ . This gives exactly  $|\mathcal{A}| - 1$  vanishing rows and columns. Thus, all the scalar products  $(a_i^u, a_j^u)$  with  $-i \in \mathcal{A}$  or  $-j \in \mathcal{A}$  vanish. Hence, assume that  $-i \notin \mathcal{A}$  and  $-j \notin \mathcal{A}$ . Then  $(a_i^u, a_j^u) \in \{p^{d-1}, 2p^{d-1}\}$  for  $i + j > 0$ . In case  $(a_i^u, a_j^u) = 2p^{d-1}$  we have  $a_i^u = a_j^u$ . This happens exactly when  $j \neq 0$  and  $ij^{-1} \in \mathcal{A}$ . Since  $-i \notin \mathcal{A}$ , the coset  $i\mathcal{A}$  in  $\mathcal{G}$  does not contain  $-1$ . Hence, there are precisely  $|\mathcal{A}|$  choices for  $j$  such that  $ij^{-1} \in \mathcal{A}$ .

Hence, we have shown that the rows  $a_i^u$  for  $i = 1, \dots, p-2$  split in  $|\mathcal{A}| - 1$  zero rows and  $(p-1)/|\mathcal{A}| - 1$  groups consisting of  $|\mathcal{A}|$  equal rows each. If we replace the matrix  $A$  by  $UA$  for a suitable matrix  $U \in \text{GL}(p-1, \mathbb{Z})$ , we get a new matrix with exactly  $(p-1)/|\mathcal{A}|$  non-vanishing rows (this is essentially the same as taking another (positive definite) quadratic form in (4.1), see [30]). After leaving out the zero rows we get a  $(p-1)/|\mathcal{A}| \times (p-1)/|\mathcal{A}|$  matrix

$$AA^T = p^{d-1} \begin{pmatrix} |\mathcal{A}| + 1 & 1 & \dots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}.$$

Now we can apply the quadratic form  $q$  corresponding to the Dynkin diagram  $A_{(p-1)/|\mathcal{A}|}$  in Equation (4.1). This gives

$$\Gamma(1, |\mathcal{A}|) \leq (|\mathcal{A}| + (p-1)/|\mathcal{A}|)p^{d-1}.$$

On the other hand  $p^{1-d}AA^T$  is the square of the matrix

$$\begin{pmatrix} 1 & \dots & 1 & & \\ 1 & & & 1 & \\ \vdots & & & & \ddots \\ 1 & & & & & 1 \end{pmatrix}$$

which has exactly  $|\mathcal{A}| + (p-1)/|\mathcal{A}|$  columns. This shows that  $\Gamma(1, |\mathcal{A}|)$  cannot be smaller.  $\square$

The next proposition gives an induction step.

**Proposition 4.9.** *If  $|\mathcal{A}|_p \neq 1$ , then*

$$\Gamma(k, |\mathcal{A}|) = \Gamma(k-1, |\mathcal{A}|/p).$$

*Proof.* Since  $|\mathcal{A}|_p \neq 1$ , we have  $k \geq 2$ . Let  $i \in \{1, \dots, \varphi(p^k) - 1\}$  such that  $\gcd(i, p) = 1$ . We will see that  $(a_i^u, a_i^u) = 0$  and thus  $a_i^u = 0$ . By Lemma 4.7 and Equation (4.2) it suffices to show that there is some  $\tau \in \mathcal{A}$  such that  $p^k \mid i' + i\tau$ . We can write this in the form  $-i^{-1}i' \in \mathcal{A}$ , since  $i$  represents an element of  $(\mathbb{Z}/p^k\mathbb{Z})^\times$ . Now let  $-i' = i + \alpha p^{k-1}$  for some  $\alpha \in \mathbb{Z}$ . Then  $-i^{-1}i' = 1 + i^{-1}\alpha p^{k-1}$  is an element of order  $p$  in  $\mathcal{G}$ . Since  $\mathcal{G}$  has only one subgroup of order  $p$ , it follows that  $-i^{-1}i' \in \mathcal{A}$ .

Hence, in order to apply Theorem 4.2 it remains to consider the indices which are divisible by  $p$ . Let  $\overline{\mathcal{A}}$  be the image of the canonical map  $(\mathbb{Z}/p^k\mathbb{Z})^\times \rightarrow (\mathbb{Z}/p^{k-1}\mathbb{Z})^\times$  under  $\mathcal{A}$ . Then  $|\overline{\mathcal{A}}| = |\mathcal{A}|/p$  (cf. Lemma 4.6). If  $i$  and  $j$  are divisible by  $p$ , we have

$$|\{\tau \in \mathcal{A} : p^k \mid i + j\tau\}| = p \cdot |\{\tau \in \overline{\mathcal{A}} : p^{k-1} \mid (i/p) + (j/p)\tau\}|.$$

A similar equality holds for the other summands in (4.2). Here observe that  $(i/p)' = i'/p$ , where the dash on the left refers to the case  $p^{k-1}$ . Thus, the remaining matrix is just the matrix in case  $p^{k-1}$ . Hence  $\Gamma(k, |\mathcal{A}|) = \Gamma(k-1, |\overline{\mathcal{A}}|) = \Gamma(k-1, |\mathcal{A}|/p)$ .  $\square$

Now we are in a position to prove the main theorem of this section.

**Theorem 4.10.** *Let  $B$  be a  $p$ -block of a finite group  $G$  where  $p$  is an odd prime, and let  $(u, b_u)$  be a  $B$ -subsection such that  $l(b_u) = 1$  and  $b_u$  has defect  $d$ . Moreover, let  $\mathcal{F}$  be the fusion system of  $B$  and  $|\text{Aut}_{\mathcal{F}}(\langle u \rangle)| = p^s r$ , where  $p \nmid r$  and  $s \geq 0$ . Then we have*

$$k_0(B) \leq \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle| \cdot r} p^d. \quad (4.4)$$

If (in addition)  $(u, b_u)$  is major, we can replace  $k_0(B)$  by  $\sum_{i=0}^{\infty} p^{2i} k_i(B)$  in (4.4).

*Proof.* As before let  $|\langle u \rangle| = p^k$ . We will prove by induction on  $k$  that

$$\Gamma(k, p^s r) = \frac{p^k + p^s(r^2 - 1)}{p^{kr}} p^d.$$

By Proposition 4.8 we may assume  $k \geq 2$ . By Proposition 4.9 we can also assume that  $s = 0$ . As before we consider the matrix  $A$ . Like in the proof of Proposition 4.9 it is easy to see that the indices divisible by  $p$  form a block of the matrix  $AA^T$  which contributes  $\Gamma(k-1, r)/p$  to  $\Gamma(k, r)$ . It remains to deal with the matrix  $\tilde{A} := (a_i^u : \gcd(i, p) = 1)$ . By Lemma 4.7 the entries of  $p^{k-d} \tilde{A} \tilde{A}^T$  lie in  $\{0, \pm 1, \pm 2\}$ . Moreover, if  $\gcd(i, p) = 1$  we have  $(a_i^u, a_i^u) = 2p^{d-k}$  (see proof of Proposition 4.9).

With the notation of the proof of Proposition 4.4 we have  $VA = W$ . In particular  $\text{rk } AA^T = \text{rk } A = \text{rk } W = |\mathcal{G} : \mathcal{A}|$ . If we set  $A_1 := (a_i^u : \gcd(i, p) > 1)$ , it also follows that  $\text{rk } A_1 A_1^T = \text{rk } A_1 = \varphi(p^{k-1})/r$ . Since the rows of  $\tilde{A}$  are orthogonal to the rows of  $A_1$  (see Lemma 4.7), we see that  $\text{rk } \tilde{A} = (\varphi(p^k) - \varphi(p^{k-1}))/r = p^{k-2}(p-1)^2/r$ .

Now we will find  $p^{k-2}(p-1)^2/r$  linearly independent rows of  $\tilde{A}$ . For this observe that  $\mathcal{A}$  acts on  $\Omega := \{i : 1 \leq i \leq p^{k-1}, \gcd(i, p) = 1\}$  by  ${}^\tau i := \tau \cdot i \pmod{p^{k-1}}$  for  $\tau \in \mathcal{A}$ . Since  $p \nmid r$ , every orbit has length  $r$  (see Lemma 4.6). We choose a set of representatives  $\Delta$  for these orbits. Then  $|\Delta| = p^{k-2}(p-1)/r$ . Finally for  $i \in \Delta$  we set  $\Delta_i := \{i + jp^{k-1} : j = 0, \dots, p-2\}$ . We claim that the rows  $a_i^u$  with  $i \in \bigcup_{j \in \Delta} \Delta_j$  are linearly independent. We do this in two steps.

**Step 1:**  $(a_i^u, a_j^u) = 0$  for  $i, j \in \Delta, i \neq j$ .

We will show that all summands in (4.2) vanish. First assume that  $i \equiv j\tau \pmod{p^k}$  for some  $\tau \in \mathcal{A}$ . Then of course we also have  $i \equiv j\tau \pmod{p^{k-1}}$  which contradicts the choice of  $\Delta$ . Exactly the same argument works for the other summands. For the next step we fix some  $i \in \Delta$ .

**Step 2:**  $a_j^u$  for  $j \in \Delta_i$  are linearly independent.

It suffices to show that the matrix  $A' := p^{k-d}(a_l^u, a_m^u)_{l, m \in \Delta_i}$  is invertible. We already know that the diagonal entries of  $A'$  equal 2. Now write  $m = l + jp^{k-1}$  for some  $j \neq 0$ . We consider the summands in (4.2). Assume that there is some  $\tau \in \mathcal{A}$  such that  $l \equiv m\tau \equiv (l + jp^{k-1})\tau \pmod{p^k}$ . Then we have  $\tau \equiv 1 \pmod{p^{k-1}}$  which implies  $\tau = 1$ . However, this contradicts  $j \neq 0$ . On the other hand we have  $l' \equiv m'\tau \equiv l'\tau \pmod{p^k}$  for  $\tau = 1 \in \mathcal{A}$ . Now assume  $-l \equiv m'\tau \pmod{p^k}$ . Then the argument above implies  $\tau = 1$  and  $l + l' \equiv 0 \pmod{p^k}$  which is false. Similarly the last summand in (4.2) equals 0. Thus, we have shown that  $A' = (1 + \delta_{lm})_{l, m \in \Delta_i}$  is invertible.

This implies that the rank of  $\tilde{A}$  is  $p^{k-2}(p-1)^2/r$ . Hence, there exists an integral matrix  $U \in \text{GL}(p^{k-2}(p-1)^2, \mathbb{Q})$  such that the only non-zero rows of  $U\tilde{A}$  are  $a_i^u$  for  $i \in \bigcup_{j \in \Delta} \Delta_j$ . Then we can leave out the zero rows and obtain a matrix (still denoted by  $\tilde{A}$ ) of dimension  $p^{k-2}(p-1)^2/r$ . Moreover, the two steps above show that  $p^{k-d} \tilde{A} \tilde{A}^T$  consists of  $p^{k-2}(p-1)/r$  blocks of the form  $(1 + \delta_{ij})_{1 \leq i, j \leq p-1}$ . Thus, an application of the quadratic form  $q$  corresponding to the Dynkin diagram  $A_{p^{k-2}(p-1)^2/r}$  in Equation (4.1) gives

$$\Gamma(k, r) \leq \frac{\Gamma(k-1, r)}{p} + \frac{p^{k-1}(p-1)}{p^{kr}} p^d = \frac{p^k + r^2 - 1}{p^{kr}} p^d.$$

The minimality of  $\Gamma(k, r)$  is not so clear as in the proof of Proposition 4.8, since here we do not know if  $\det U \in \{\pm 1\}$ . However, it suffices to give an example where  $k_0(B) = \Gamma(k, r)$ . For this let  $G = \langle u \rangle \rtimes C_r$  and  $B$  be the principal block of  $G$ . Then it is easy to see that the hypothesis of the theorem is satisfied. By Dade's theorem on blocks with cyclic defect groups we have

$$k_0(B) = k(B) = \frac{|D| - 1}{r} + r = \Gamma(k, r).$$

Hence, the proof is complete.  $\square$

We add some remarks. It is easy to see that the right hand side of (4.4) is always an integer. Indeed by construction it suffices to show that  $k \leq d$ , where  $|\langle u \rangle| = p^k$ . Since  $u \in Z(C_G(u))$ ,  $u$  is contained in every defect group of  $b_u$ . Hence,  $k \leq d$ . Moreover, if  $\mathcal{A} = \mathcal{G}$  (i. e.  $s = k - 1$  and  $r = p - 1$ ) or  $\mathcal{A}$  is a  $p$ -group (i. e.  $r = 1$ ), we get the same bound as in Proposition 2.1 and Proposition 2.2. In all other cases Theorem 4.10 really improves Proposition 2.1 and Proposition 2.2. For  $k \geq 2$  the case  $s = 0$  and  $r = p - 1$  gives the best bound for  $k_0(B)$ . If  $k$  tends to infinity,  $\Gamma(k, p - 1)$  goes to  $p^d/(p - 1)$ .

Coming back to our intended task, i. e. to prove Olsson's Conjecture (in some cases), we have to say (in contrast to the case  $p = 2$ ) that Olsson's Conjecture does not follow from Theorem 4.10 if it does not already follow from Proposition 2.1, since the right hand side of (4.4) is always larger than  $p^{d-1}$ .

In the proof we already saw that Inequality (4.4) is sharp for blocks with cyclic defect groups. Perhaps it is possible that this can provide a more elementary proof of Dade's theorem (observe that we only used Dade's theorem for the minimality). For this it would be sufficient to bound  $l(B)$  from below, since the difference  $k(B) - l(B)$  is locally determined.

As an application of Theorem 4.10 we give a concrete example. Let  $B$  be an 11-block with defect group  $D \cong C_{11} \times C_{11}$  (for smaller primes results by Usami and Puig give more complete informations, e. g. [57, 43]). Then it is known that Brauer's  $k(B)$ -Conjecture and thus Olsson's Conjecture hold. However, the precise values for  $k(B)$  and  $l(B)$  are unknown. Since  $D$  is abelian, the fusion system  $\mathcal{F}$  is controlled by  $\text{Aut}_{\mathcal{F}}(D)$ . Assume that  $\text{Aut}_{\mathcal{F}}(D)$  has order 5 and acts only on one factor  $C_{11}$  non-trivially. Then there are two non-trivial (major) subsections  $(u, b_u)$  and  $(v, b_v)$  such that  $l(b_u) = 1$  and  $l(b_v) = 5$ . Moreover,  $|\mathcal{A}| = 5$  and Theorem 4.10 implies  $k(B) \leq 77$  which is better than Brauer's  $k(B)$ -Conjecture. On the other hand the block  $b_v$  dominates a block of  $C_G(v)/\langle v \rangle$  with cyclic defect group  $D/\langle v \rangle \cong C_{11}$ . Hence, the Cartan matrix of  $b_v$  has the form  $11(2 + \delta_{ij})_{1 \leq i, j \leq 5}$  up to basic sets (see [13, 46]). Now Theorem 2.4 applied to  $(v, b_v)$  gives exactly the same bound on  $k(B)$ . Under the action of  $\text{Aut}_{\mathcal{F}}(D)$  the group  $D$  splits in 11 orbits of length 1 and 22 orbits of length 5. Hence, by Theorem 5.9.4 in [35] we get  $k(B) - l(B) = 72$ . Since  $B$  is centrally controlled, Theorem 1.1 in [29] implies  $l(B) \geq l(b_v) = 5$ . This shows  $k(B) = 77$  and  $l(B) = 5$  (this can also be obtained from Corollary 2 in [58]). By Theorem IV.4.18 in [17] we also have  $k_0(B) = k(B)$ , because  $B$  has defect 2.

Now assume that  $\text{Aut}_{\mathcal{F}}(D)$  acts diagonally (and thus fixed point freely) on both factors  $C_{11}$ . Then we have  $l(b_u) = 1$  for all non-trivial subsections  $(u, b_u)$ . Thus, Theorem 2.4 and Theorem 1 in [58] become useless in this situation, but Theorem 4.10 still implies  $k(B) \leq 77$ . However, for the principal block  $B$  of  $G = D \rtimes \text{Aut}_{\mathcal{F}}(D)$  we have  $k(B) = k_0(B) = 29$  and  $l(B) = 5$ .

As was pointed out earlier, for odd primes  $p$  and  $l(b_u) > 1$  there is not always a stable character in  $\text{IBr}(b_u)$  under  $N_G(\langle u \rangle, b_u)$ , even for  $l(b_u) = 2$  (see Proposition (2E)(ii) and the example following it in [26]). However, the situation is better if we consider the principal block.

**Proposition 4.11.** *Let  $B$  be the principal  $p$ -block of  $G$  for an odd prime  $p$ , and let  $(u, b_u)$  a  $B$ -subsection such that  $l(b_u) = 2$ , and  $b_u$  has defect  $d$  and Cartan matrix  $C_u = (c_{ij})$ . Then we may replace  $C_u$  by an equivalent matrix such that  $p^d c_{11} / \det C_u$  is divisible by  $p$ . Moreover, let  $\mathcal{F}$  be the fusion system of  $B$  and  $|\text{Aut}_{\mathcal{F}}(\langle u \rangle)| = p^s r$ , where  $p \nmid r$  and  $s \geq 0$ . Then we have*

$$k_0(B) \leq \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle| \cdot r} c_{11}.$$

*Proof.* By Brauer's third main theorem  $b_u$  is the principal block of  $C_G(u)$  and so  $\text{IBr}(b_u)$  contains the trivial Brauer character. Hence, both characters of  $\text{IBr}(b_u)$  are stable under  $N_G(\langle u \rangle)$ . As in the proof of Theorem 3.1,  $\frac{p^d}{\det C_u} C_u \pmod{p}$  has rank 1. Hence, we can replace  $C_u$  by an equivalent matrix (still denoted by  $C_u = (c_{ij})$ ) such that  $p^d c_{11} / \det C_u$  and  $p^d c_{12} / \det C_u$  are divisible by  $p$ . As in the proof of Theorem 3.1, the rows  $d_i^u$  and  $a_j^i$  become  $\widehat{d}_i^u$  and  $\widehat{a}_j^i$  for  $i = 1, 2$  and  $j = 0, \dots, \varphi(|\langle u \rangle|) - 1$ . Write  $p^d C_u^{-1} = (\widetilde{c}_{ij})$ . For  $\chi \in \text{Irr}_0(B)$  we have

$$0 \neq p^d m_{\chi\chi}^{(u, b_u)} \equiv \widetilde{c}_{11} (\widehat{d}_{\chi\varphi_1}^u)^2 \pmod{(\pi)};$$

in particular  $\widehat{a}_j^1(\chi) \neq 0$  for some  $j \in \{0, \dots, \varphi(p^k) - 1\}$ . Now since

$$(\widehat{d}_1^u, \gamma(\widehat{d}_1^u)) = \begin{cases} c_{11} & \text{if } \gamma \in \mathcal{A}, \\ 0 & \text{if } \gamma \in \mathcal{G} \setminus \mathcal{A}, \end{cases}$$

the proof works as in case  $l(b_u) = 1$ . □

## 5 Controlled blocks

In this section we will use Proposition 2.1 to show that Olsson's Conjecture is satisfied for controlled block with certain defect groups. Here a block  $B$  of  $G$  with defect group  $D$  is *controlled* if  $N_G(D, b_D)$  controls the fusion system  $\mathcal{F}$  of  $B$  (see section 2 for notations). Recall that in this situation all subgroups of  $D$  are fully  $\mathcal{F}$ -normalized. In particular for a subsection  $(u, b_u)$  the block  $b_u$  has defect group  $C_D(u)$  (cf. Lemma 2.3). Our strategy will be to find a subsection  $(u, b_u)$  such that  $l(b_u) = 1$  and  $|C_D(u)| = |D : D'|$ . Then Olsson's Conjecture follows from Proposition 2.1. Observe that the inequality  $|D : C_D(u)| \leq |D'|$  always holds by elementary group theory. The next proposition gives a general criterion for this situation.

**Proposition 5.1.** *Let  $B$  be a controlled block of  $G$  with defect group  $D$ . Suppose that there exists an element  $u \in D$  such that  $|D : C_D(u)| = |D'|$  and  $N_G(D, b_D) \cap C_G(u) \subseteq C_D(u) C_G(C_D(u))$ . Then Olsson's Conjecture holds for  $B$ .*

*Proof.* By Proposition 2.1(b) in [3], also  $b_u$  is a controlled block and it suffices to show that  $b_u$  has inertial index 1, since then  $b_u$  is nilpotent and  $l(b_u) = 1$ . Observe that  $(C_D(u), b_{C_D(u)})$  is a maximal  $b_u$ -subpair. Hence, Proposition 2.2 in [3] implies

$$N_G(C_D(u), b_{C_D(u)}) = [N_G(D, b_D) \cap N_G(C_D(u), b_{C_D(u)})] C_G(C_D(u)) = [N_G(D, b_D) \cap N_G(C_D(u))] C_G(C_D(u)).$$

Thus,

$$N_{C_G(u)}(C_D(u), b_{C_D(u)}) = [N_G(D, b_D) \cap N_G(C_D(u)) \cap C_G(u)] C_G(C_D(u)) = C_D(u) C_G(C_D(u)),$$

and the claim follows. □

Since  $|N_G(D, b_D) : D C_G(D)| \not\equiv 0 \pmod{p}$ , Proposition 5.1 implies the following:

**Proposition 5.2.** *Let  $D$  be a finite  $p$ -group. Suppose that, for every  $p'$ -subgroup  $L$  of  $\text{Aut}(D)$ , there exists a regular orbit of  $L$  on  $\{u \in D : |D : C_D(u)| = |D'|\}$ . Then Olsson's Conjecture holds for all controlled blocks with defect group  $D$ .*

If  $u$  is an element of  $D$  such that  $|D : C_D(u)| = |D'|$ , then  $D' = \{[u, v] : v \in D\}$ ; in particular, every element in  $D'$  is a commutator. Thus, one cannot expect to prove Olsson's Conjecture for all possible defect groups in this way (see for example [21]). In the following, we will verify the conditions in Proposition 5.2 under several hypotheses.

**Theorem 5.3.** *Let  $D$  be a finite  $p$ -group, where  $p$  is an odd prime, and suppose that one of the following holds:*

- (i)  $D$  has maximal class,
- (ii)  $D$  has class 2 and  $|D : \Phi(D)| = p^2$ ,
- (iii)  $D'$  is cyclic and  $|D : \Phi(D)| = p^2$ ,
- (iv)  $D$  has  $p$ -rank 2.

*Then every  $p'$ -subgroup of  $\text{Aut}(D)$  has a regular orbit on  $\{u \in D : |D : C_D(u)| = |D'|\}$ . In particular, Olsson's Conjecture holds for all controlled blocks with defect group  $D$ .*

*Proof.*

- (i) Suppose first that  $D$  is non-abelian of order  $p^3$  and exponent  $p$ . Then, by Proposition 2.2 in [23],  $\text{Aut}(D)$  is isomorphic to a semidirect product of  $\text{Inn}(D)$  and  $\text{GL}(2, p)$ . It is easy to see that the action of  $\text{GL}(2, p)$  on  $D$  has an orbit of length  $p^3 - p^2 - p + 1 = |\text{GL}(2, p)|/p$ , i. e. the stabilizer in  $\text{GL}(2, p)$  of every element in this orbit has order  $p$ . Moreover, every element in this orbit has a centralizer in  $D$  of order  $p^2 = |D : D'|$ . Thus, the result follows in this case.

Suppose next that  $D$  is non-abelian of order  $p^3$  and exponent  $p^2$ . Then, by Proposition 3.3 in [22],  $\text{Out}(D)$  is a semidirect product of  $C_p$  and  $C_{p-1}$ . Let  $L$  be a non-trivial  $p'$ -subgroup of  $\text{Aut}(D)$ . Then  $L$  is cyclic and  $|L| \mid p-1$ . Moreover,  $L$  acts faithfully on  $D/\Phi(D) = D/Z(D)$ , a 2-dimensional vector space over the field  $\mathbb{F}_p$ . Since  $\mathbb{F}_p$  is a splitting field for  $L$ , it is easy to see that  $L$  has a regular orbit on  $D/\Phi(D)$  and thus on  $D \setminus Z(D)$ . If  $u$  is an element in this orbit, then  $|C_D(u)| = p^2 = |D : D'|$ . Thus, the result follows in this case.

Finally, suppose that  $D$  is of maximal class and  $|D| \geq p^4$ . We denote the terms of the lower central series of  $D$  by  $D^1 = D$ ,  $D^2 = D'$ ,  $D^3 = [D^2, D]$ , etc. Then  $D_1 := C_D(D^2/D^4)$  is a characteristic maximal subgroup of  $D$ ; in particular,  $D_1$  is  $L$ -stable. Since  $L$  acts faithfully on  $D/\Phi(D)$ , a 2-dimensional vector space over  $\mathbb{F}_p$ , Maschke's Theorem implies that there exists another  $L$ -stable maximal subgroup  $Q$  of  $D$ . Thus,  $L$  acts via diagonal matrices on  $D/\Phi(D)$ ; in particular,  $L$  is abelian and  $|L| \mid (p-1)^2$ . Since  $D$  has  $p+1 \geq 4$  maximal subgroups, there is a maximal subgroup  $M$  of  $D$  such that  $D_1 \neq M \neq Q$  and  $M \neq C_G(Z_2(D))$ . Let  $u \in M$  such that  $M = \langle u, \Phi(D) \rangle$ . By Hilfssatz III.14.13 in [24], we have  $|C_D(u)| = p^2 = |D : D'|$ . Then  $K := C_L(u)$  stabilizes  $D_1$ ,  $Q$  and  $M$ , so that  $K$  acts via scalar matrices on  $D/\Phi(D)$ . Since  $K$  fixes a non-zero vector in  $D/\Phi(D)$ , we must have  $K = 1$ . This shows that the  $L$ -orbit of  $u$  is regular, and the result follows.

- (ii) Let  $D = \langle a, b \rangle$ . Then  $D' = \langle [a, b] \rangle = \{[a, b^n] : n \in \mathbb{Z}\} = \{[a, x] : x \in D\}$ . Thus, it suffices to show that  $L$  has a regular orbit on  $D/\Phi(D)$ . Arguing by induction on  $|D|$ , we may assume that  $|D'| = p$ . Then  $|D : C_D(a)| = p = |D : C_D(b)|$  and  $|D : Z(D)| = p^2$ . Thus,  $Z(D) = \Phi(D)$  and  $D' \subseteq \Omega_1(Z(D))$ . Suppose first that  $D' \neq \Omega_1(Z(D))$ . Then  $\Omega_1(Z(D)) = D' \times Q$ , where  $Q$  is an  $L$ -stable subgroup of  $D$ . Since  $Q \subseteq \Phi(D)$ ,  $L$  acts faithfully on  $D/Q$ , and  $D/Q$  satisfies the same hypotheses as  $D$ . Arguing by induction on  $|D|$ , the result follows in this case.

Thus we may assume that  $D' = \Omega_1(Z(D))$ ; in particular,  $Z(D) = \Phi(D)$  is cyclic. If  $|\Phi(D)| = p$ , then  $|D| = p^3$ , and we are done by (i). So we may assume that  $|\Phi(D)| > p$ . Then  $D' < \Phi(D) = D'\mathcal{U}_1(D)$ , so  $\mathcal{U}_1(D) \neq 1$  and  $\mathcal{U}_1(D) \cap Z(D) \neq 1$ . Hence,  $D' = \Omega_1(Z(D)) \subseteq \mathcal{U}_1(D)$ , and  $\Phi(D) = \mathcal{U}_1(D)$ . Moreover, we have  $|\Omega_1(D)| = |D : \mathcal{U}_1(D)| = p^2$ . Since  $\Omega_1(\Phi(D)) = \Omega_1(Z(D)) = D'$  is cyclic, we have  $\Omega_1(D) \not\subseteq \Phi(D)$ . Thus,  $\Omega_1(D)\Phi(D)$  is a maximal subgroup of  $D$  which is  $L$ -invariant. By Maschke's Theorem,  $L$  stabilizes another maximal subgroup  $Q$  of  $D$ , and so  $L$  acts via diagonal matrices on  $D/\Phi(D)$ . Let  $u \in D$  such that  $u \notin Q \cup \Omega_1(D)\Phi(D)$ . Then  $K := C_L(u)$  acts via scalar matrices on  $D/\Phi(D)$ . Since  $K$  fixes a non-zero vector in  $D/\Phi(D)$ , we must have  $K = 1$ , and the result follows.

- (iii) By (ii) we may assume that  $D$  has class at least 3. Let  $C := C_D(D')$ . Then  $D/C$  is isomorphic to a subgroup of  $\text{Aut}(D')$ . Since  $D'$  is cyclic and  $p \neq 2$ ,  $\text{Aut}(D')$  is cyclic. We write  $D = C\langle y \rangle$ . Then  $y \notin \Phi(D)$  and  $D = \langle c, y \rangle$  for some  $c \in C$ . Since  $D'$  is cyclic, we have  $\langle [c, y] \rangle \trianglelefteq D$ . Moreover,  $D/\langle [c, y] \rangle$  is abelian, so  $D' \leq \langle [c, y] \rangle \leq D'$ , i. e.  $D' = \langle [c, y] \rangle$ .

Now  $A := \langle c \rangle D'$  is an abelian normal subgroup of  $D$ , and  $D/A = \langle yA \rangle$  is cyclic. Thus, Aufgabe 2 on page 259 of [24] implies that

$$D' = \{[a, y] : a \in A\} = \{[u, y] : u \in D\};$$

in particular, we have  $|D'| = |D : C_D(y)|$ . Since  $D$  does not have class 2, we must have  $C < D$ . Thus,  $C\Phi(D)$  is a maximal subgroup of  $D$  which is characteristic in  $D$ . Hence,  $\overline{C} := C\Phi(D)/\Phi(D)$  is an  $L$ -invariant subgroup of  $\overline{D} := D/\Phi(D)$ . By Maschke's Theorem,  $\overline{C}$  has an  $L$ -invariant complement  $\overline{K}$  in  $\overline{D}$ . Hence, the action of  $L$  on  $\overline{D}$  induces an isomorphism between  $L$  and a group of diagonal matrices in  $\text{GL}(2, p)$ ; in particular,  $L$  is abelian and  $|L| \mid (p-1)^2$ . If  $L$  is isomorphic to a group of scalar matrices in  $\text{GL}(2, p)$ , then the  $L$ -orbit of  $\overline{y} := y\Phi(D)$  is certainly regular. Thus, the  $L$ -orbit of  $y$  is also regular. So we may assume that  $L$  does not act via scalar matrices on  $\overline{D}$ . We may choose  $y$  such that  $\overline{C} \neq \langle \overline{y} \rangle \neq \overline{K}$ . Then the three subgroups  $\overline{C}$ ,  $\overline{K}$  and  $\langle \overline{y} \rangle$  are  $C_L(\overline{y})$ -invariant, so  $C_L(\overline{y})$  acts via scalar matrices on  $\overline{D}$ . Since  $C_L(\overline{y})$  fixes  $\overline{y}$ , we must have  $C_L(\overline{y}) = 1$ . Thus, the  $L$ -orbit of  $\overline{y}$  is regular. Hence, the  $L$ -orbit of  $y$  is also regular.

(iv) By Theorem A.1 in [14], a result of Blackburn, there are four cases to consider. If  $D$  is metacyclic, then the result follows from (iii). If  $D$  has maximal class, then the result follows from (i). Now suppose that

$$D = \langle a, b, c \mid a^p = b^p = c^{p^{n-2}} = [a, c] = [b, c] = 1, [a, b] = c^{p^{n-3}} \rangle$$

for some  $n \geq 3$ . Then it is easy to see that  $D = \Omega_1(D) * Z(D)$ , where  $\Omega_1(D) = \langle a, b \rangle$  is a non-abelian group of order  $p^3$  and exponent  $p$ , and  $Z(D) = \langle c \rangle$  is cyclic of order  $p^{n-2}$ . Thus,  $|D'| = p$ , and  $L$  acts faithfully on  $\Omega_1(D)$ . By (i),  $L$  has a regular orbit on  $\Omega_1(D) \setminus Z(D)$ . The elements in this orbit satisfy  $|D : C_D(u)| = p = |D'|$ . Thus, the claim follows. Finally suppose that

$$D = \langle a, b, c \mid a^p = b^p = c^{p^{n-2}} = [b, c] = 1, [a, b^{-1}] = c^{\epsilon p^{n-3}}, [a, c] = b \rangle,$$

where  $n \geq 4$  and  $\epsilon$  is 1 or a fixed quadratic non-residue modulo  $p$ . Then it is easy to see that  $D'$  is elementary abelian of order  $p^2$ , and that  $C_D(D')$  is abelian of index  $p$  in  $D$ . One also checks that  $\Omega_2(D)$  is a group of order  $p^4$ , but otherwise of the same type as  $D$ . Since  $L$  acts faithfully on  $\Omega_2(D)$ , we may replace  $D$  by  $\Omega_2(D)$  and therefore assume that  $|D| = p^4$ . However, then  $D$  has maximal class, and therefore the result follows from (i).  $\square$

We observe that  $\text{GL}(2, p)$  contains a  $p'$ -subgroup  $L$  of order  $2(p-1)^2$  which is bigger than  $p^2$  for  $p > 3$ . Thus, when  $D$  is elementary abelian of order  $p^2$ , then there is no regular orbit of  $L$  on  $D$ . Nevertheless, Olsson's Conjecture holds in this case by Theorem VII.10.14 in [17]. As mentioned earlier in this paper, Olsson's Conjecture holds also for 2-blocks with maximal class defect groups. We also like to point out that Olsson's Conjecture for controlled blocks with maximal class defect groups follows easily from Proposition 2.5(i). However, we need the stronger result about the regular orbits in the proof of Theorem 5.3. In connection with (iii) in Theorem 5.3 we mention that by a result of Burnside,  $D'$  is already cyclic if  $Z(D')$  is (see Satz III.7.8 in [24]).

## 6 Defect groups of $p$ -rank 2

In this section we discuss Olsson's Conjecture for blocks which are not necessarily controlled. We begin with a special case for which the method of the previous section does not suffice. For this reason we use the classification of finite simple groups.

**Proposition 6.1.** *Let  $B$  be a block of a finite group  $G$  with a non-abelian defect group  $D$  of order  $5^3$  and exponent 5. Suppose that the fusion system  $\mathcal{F}$  of  $B$  is the same as the fusion system of the sporadic simple Thompson group  $Th$  for the prime 5. Then  $B$  is Morita equivalent to the principal 5-block of  $Th$ ; in particular, Olsson's Conjecture holds for  $B$ .*

*Proof.* By Fong reduction, we may assume that  $O_{5'}(G)$  is central and cyclic (cf. Section IV.6 in [7]). The ATLAS [12] shows that  $Th$  has a unique conjugacy class of elements of order 5. Thus, by our hypothesis, all non-trivial  $B$ -subsections are conjugate in  $G$ ; in particular, all  $B$ -subsections are major. Since  $O_5(G) \leq D$ , this implies that  $O_5(G) = 1$ . Thus  $F(G) = Z(G) = O_{5'}(G)$ .

Let  $N/Z(G)$  be a minimal normal subgroup of  $G/Z(G)$ . By Fong reduction, we may assume that  $B$  covers a unique block  $b$  of  $N$ . Then  $D \cap N$  is a defect group of  $b$ . By Fong reduction, we may also assume that  $D \cap N \neq 1$ . Since all non-trivial  $B$ -subsections are conjugate in  $G$  this implies that  $D \cap N = D$ , i.e.  $D \subseteq N$ . In particular,  $N/Z(G)$  is the only minimal normal subgroup of  $G/Z(G)$ . Hence  $N = F^*(G)$ , and  $E(G)$  is a central product of the components  $L_1, \dots, L_n$  of  $G$ .

For  $i = 1, \dots, n$ ,  $b$  covers a unique block  $b_i$  of  $L_i$ . Let  $D_i$  be a defect group of  $b_i$ . Then  $D_1 \times \dots \times D_n$  is a defect group of  $b$  (since  $O_5(G) = 1$ ). Thus  $D_1 \times \dots \times D_n \cong D$ . This shows that we must have  $n = 1$ . Hence  $E(G)$  is quasisimple, and  $S := E(G)/Z(E(G))$  is simple. Since  $F^*(G) = E(G)F(G) = E(G)Z(G)$ , we conclude that  $C_G(E(G)) = C_G(F^*(G)) = Z(F(G)) = Z(G)$ , so that  $G/Z(G)$  is isomorphic to a subgroup of  $\text{Aut}(E(G))$ .

Now we discuss the various possibilities for  $S$ , by making use of the classification of finite simple groups. In each case we apply [4].



If  $S$  is an alternating group then, by Section 2 in [4], the block  $b$  cannot exist. Similarly, if  $S$  is exceptional group of Lie type then, by Theorem 5.1 in [4], the block  $b$  cannot exist.

Now suppose that  $S$  is a classical group. Then, by Theorem 4.5 in [4],  $p = 5$  must be the defining characteristic of  $S$ . Moreover,  $S$  has to be isomorphic to  $\mathrm{PSL}(3, 5)$  or  $\mathrm{PSU}(3, 5)$ . Also,  $D$  is a Sylow 5-subgroup of  $E(G)$ . But now the ATLAS shows that  $S$  contains non-conjugate elements  $\bar{x}$  and  $\bar{y}$  of order 5 such that  $|C_S(\bar{x})| \neq |C_S(\bar{y})|$ . Thus there are elements  $x$  and  $y$  of order 5 in  $E(G)$  which are not conjugate in  $G$ . This contradicts the fact that all non-trivial  $B$ -subsections are conjugate in  $G$ .

The only remaining possibility is that  $S$  is a sporadic simple group. Then Table 1 in [4] implies that  $S \in \{HS, McL, Ru, Co_2, Co_3, Th\}$ . In all cases  $D$  is a Sylow 5-subgroup of  $S$ . In the first five cases we derive a contradiction as above, using the ATLAS. So we may assume that  $S = Th$ . Since  $Th$  has trivial Schur multiplier and trivial outer automorphism group, we must have  $G = S \times Z(G)$ . Thus  $B \cong b \otimes_{\mathbf{R}} \mathbf{R} \cong b$ , and  $b$  is the principal 5-block of  $Th$ , by [56]. Moreover, we have  $k_0(B) = k_0(b) = 20 \leq |D : D'|$ . This completes the proof.  $\square$

**Theorem 6.2.** *Let  $p > 3$ . Then Olsson's Conjecture holds for all  $p$ -blocks with defect groups of  $p$ -rank 2.*

*Proof.* Let  $B$  be a  $p$ -block with defect group  $D$  of  $p$ -rank 2 for  $p > 3$ . Then, by the Theorems 4.1, 4.2 and 4.3 in [14],  $B$  is controlled unless  $D$  is non-abelian of order  $p^3$  and exponent  $p$  (see also [55]). Hence, by Theorem 5.3(iv) we may assume that  $D$  is non-abelian of order  $p^3$  and exponent  $p$ .

If in addition  $p > 7$ , Hendren has shown that there is at least one non-major  $B$ -subsection. In this case the result follows easily from Proposition 2.5(i). Now let  $p = 7$ . Then the fusion system  $\mathcal{F}$  of  $B$  is one of the systems given in [47]. Kessar and Stancu showed using the classification of finite simple groups that three of them cannot occur for blocks (see [25]). In the remaining cases the number of  $\mathcal{F}$ -radical and  $\mathcal{F}$ -centric subgroups of  $D$  is always less than  $p + 1 = 8$ . In particular, there is an element  $u \in D \setminus Z(D)$  such that  $\langle u \rangle Z(D)$  is not  $\mathcal{F}$ -radical,  $\mathcal{F}$ -centric. Then by Alperin's fusion theorem  $\langle u \rangle$  is not  $\mathcal{F}$ -conjugate to  $Z(D)$ . Hence, the subsection  $(u, b_u)$  is non-major, and Olsson's Conjecture follows from Proposition 2.5(i).

In case  $p = 5$  the same argument shows that we can assume that  $\mathcal{F}$  is the fusion system of the principal 5-block of  $Th$ . However, in this case Olsson's Conjecture holds by Proposition 6.1.  $\square$

For  $p = 3$ , there are two fusion systems on the non-abelian group of order 27 and exponent 3 in [47], such that all subsections are major. These correspond to the simple groups  ${}^2F_4(2)'$  and  $J_4$ . However, Olsson's Conjecture holds for the 3-blocks of  ${}^2F_4(2)'$ ,  ${}^2F_4(2)$ ,  $J_4$ ,  $Ru$  and  $2.Ru$  (see [1, 2, 6, 5]; cf. Remark 1.3 in [47]). More generally, Olsson's Conjecture is known to hold for all principal blocks with a non-abelian defect group of order 27 and exponent 3, by Remark 64 in [36]. In addition to 3-blocks of defect 3, there are also non-controlled 3-blocks whose defect groups have maximal class and 3-rank 2. We plan to come back to this situation in a separate paper. On the other hand Brauer's  $k(B)$ -Conjecture is satisfied for all 3-blocks of defect 3 (see [53]).

We finish this paper with a similar result about minimal non-abelian defect groups.

**Theorem 6.3.** *Let  $p \neq 3$ . Then Olsson's Conjecture holds for all  $p$ -blocks with minimal non-abelian defect groups.*

*Proof.* By [51] we may assume  $p > 3$ . Let  $B$  be a block with minimal non-abelian defect group  $D$ . Then by Rédei's classification of minimal non-abelian groups (see [48]), we may assume that

$$D := \langle x, y \mid x^{p^r} = y^{p^s} = [x, y]^p = [x, x, y] = [y, x, y] = 1 \rangle$$

for  $r \geq s \geq 1$ . We set  $z := [x, y] \in Z(D)$ . Observe that  $\Phi(D) = Z(D) = \langle x^p, y^p, z \rangle$  and  $D' = \langle z \rangle$ . Let  $\mathcal{F}$  be the fusion system of  $B$ .

First assume  $s \geq 2$ . Then we show that  $B$  is controlled. By Alperin's Fusion Theorem it suffices to show that  $D$  does not contain  $\mathcal{F}$ -essential subgroups. By way of contradiction, assume that  $Q < D$  is  $\mathcal{F}$ -essential. Since  $C_D(Q) \subseteq Q$ ,  $Q$  is a maximal subgroup of  $D$ . Let  $a \in D$  be an element of order  $p$ . Then also  $aD' \in D/D' \cong C_{p^r} \times C_{p^s}$  has order  $p$ . Since  $r \geq s \geq 2$ , we see that  $a \in Z(D)$  and  $\Omega(D) \subseteq Z(D)$ . This shows that  $1 \neq D/Q = \mathrm{Aut}_D(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q)$  acts trivially on  $\Omega(Q)$ . On the other hand every  $p'$ -automorphism of  $\mathrm{Aut}_{\mathcal{F}}(Q)$  acts non-trivially on  $\Omega(Q)$  (see Theorem 5.2.4 in [19]). Hence,  $O_{p'}(\mathrm{Aut}_{\mathcal{F}}(Q)) \neq 1$  which contradicts the choice of  $Q$ . Thus, we have proved that  $B$  is a controlled block. Now the claim follows from Theorem 5.3(iii).

Now assume that  $s = 1$ . If also  $r = 1$ , then  $D$  is non-abelian of order  $p^3$  and exponent  $p$ . In this case we have seen in the proof of Theorem 6.2 that Olsson’s Conjecture holds for  $B$ , since  $p > 3$ . Thus, let  $r \geq 2$ . Since  $Z(D)$  has exponent  $p^{r-1}$ , we see that  $x$  is not  $\mathcal{F}$ -conjugate to an element in  $Z(D)$ . In particular  $(x, b_x)$  is a non-major  $B$ -subsection. Moreover,  $\langle x \rangle$  is fully  $\mathcal{F}$ -centralized, since  $C_D(x)$  is a maximal subgroup of  $D$ . Hence,  $C_D(x)$  is a defect group of  $b_x$  by Theorem 2.4(ii) in [31]. Now the claim follows from Proposition 2.5(i).  $\square$

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