Characters, conjugacy classes and centrally large subgroups of p-groups of small rank

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Abstract

In this paper, we investigate the irreducible characters and conjugacy classes of a metacyclic p-group where p is an odd prime number. We show that, if $|G'| = p^n$, then G has precisely $|G : G'|(p-1)/p^{k+1}$ irreducible characters of degree p^k, for $k = 1, \ldots, n$. We also show that G contains precisely $|Z(G)|(p^k - p^{k-2})$ conjugacy classes of length p^k, for $k = 1, \ldots, n$. Moreover, we have $|G| = |Z(G)| \cdot |G'|^2$. We also investigate the concept of a centrally large subgroup, as introduced by Glauberman. In addition, we show that these results are, in general, false for $p = 2$. Furthermore, we consider p-groups of small rank, for $p > 2$.

1 Introduction

The motivaton of this paper is twofold. In 1963, Richard Brauer [5] formulated a list of open problems in group theory. One of these, Problem 7, asks for an investigation of the irreducible characters of finite p-groups. In the present paper, we study this problem in the case of p-groups of small rank where p is an odd prime number.

The other motivation also dates back to the early sixties. Already in the famous odd-order paper, Feit and Thompson had to consider the structure and embedding of solvable (in particular nilpotent) groups of odd order that have p-rank at most 2 (see Chapter 4 in [2]). In his celebrated N-group paper, describing the simple groups whose local subgroups are solvable, Thompson introduced a division of these groups according to the rank of the Sylow p-subgroups ($p > 2$) of its 2-local subgroups. For a group where the rank of its Sylow p-subgroups is at most 2 the signalizer functor theorems are not applicable. So other special arguments are needed. The finite groups G with $e(G) \leq 2$ where $e(G)$ is the maximum of the ranks of the Sylow p-subgroups of the 2-local subgroups of G where p is odd are now termed quasithin.The classification of simple quasithin groups of even characteristic has now been completed in [1].

Blackburn classified p-groups of rank at most 2 where p is an odd prime in his paper [4]. He showed that there are three classes of such p-groups for $p \geq 5$, and a fourth class for $p = 3$. In this paper, we first discuss the metacyclic case.

Suppose that $|G'| = p^n$. We show that G has irreducible characters of degree $1, p, p^2, \ldots, p^n$. Also, for $k = 1, \ldots, n$, G has precisely $|G : G'|(p-1)/p^{k+1}$ irreducible characters of degree p^k. Moreover, an irreducible character of G has degree p^k if and only if $\Omega_{n-k}(G)$ is contained in its kernel.

We use character theory in order to prove some interesting properties of metacyclic p-groups. Thus we show that $|G| = |Z(G)| \cdot |G'|^2$, and that a subgroup L of G is maximal abelian if and only if $|L| = |G : G'|$.

We also investigate the conjugacy classes of G. We show that an element $g \in G$ is contained in $\mho_k(G)$ if and only if the length of its conjugacy class is bounded above by p^{n-k}. In particular, we have $\mho_n(G) = Z(G)$. Moreover, we prove that G contains exactly $|Z(G)|(p^k - p^{k-2})$ conjugacy classes of length p^k, for $k = 1, \ldots, n$.

Several authors have studied metacyclic p-groups; see Section III.11 in [10] and §44 in [3], and the references given in these two books. A classification of the isomorphism types of metacyclic p-groups was obtained by Liedahl in [13]. In [9], the authors showed that $|KK^{-1}| \equiv 1 \pmod{p-1}$, for every conjugacy class K of a finite metacyclic p-group. In a similar way, the authors showed that the number of irreducible constituents of $\chi\overline{\chi}$ is congruent to 1 mod $p-1$, for $\chi \in \mathrm{Irr}(G)$.

We also investigate the centrally large subgroups of G, as defined by Glauberman in [7]. We show that the centrally large subgroups of G are precisely the nonabelian subgroups of G and the maximal abelian subgroups of G. In this respect metacyclic p-groups behave in exactly the same way as extraspecial p-groups (cf. Example 2.8 in [7]).

In the last chapter, we turn to the non-metacyclic case. When $p \geq 5$ there are two character degrees and at most three types of conjugacy classes. Their lengths differ from those in the metacyclic case. For $p = 3$, the fourth type of groups needs some special attention. It consists of 3-groups of maximal class with a metacyclic subgroup of index 3. We use our knowledge of the metacyclic case to determine conjugacy classes, character degrees and centrally large subgroups; the results in this case significantly differ from those in the metacyclic case.

Most of our results do not hold for $p = 2$, as will be shown by several examples.

2 Irreducible characters

Let p be an odd prime, and let G be a metacyclic p-group. We write $G = AB$ where A, B are cyclic subgroups of G and $A \trianglelefteq G$. Moreover, we set $C := C_G(A)$ and denote by $\mathrm{SCN}(G)$ the set of maximal abelian normal subgroups of G.

Lemma 2.1. *(i) C is abelian; in particular, $C \in \mathrm{SCN}(G)$;*

(ii) $|G : C| = |G'|$;

(iii) $|G : Z(G)| = |G'|^2$.

Proof. (i) Since A is abelian, we have $A \subseteq Z(C)$. Thus $C/Z(C)$ is cyclic, and C is abelian.

(ii) This is an easy consequence of Lemma 5.1 in [9].

(iii) Since C is abelian, Lemma 12.12 in [11] implies that $|C| = |G'| \cdot |Z(G)|$. Thus (iii) is a consequence of (ii). □

Proposition 2.2. *Suppose that $|G'| = p^n$. Then $\{\chi(1) : \chi \in \mathrm{Irr}(G)\} = \{1, p, p^2, \ldots, p^n\}$, and*

$$\{\chi \in \mathrm{Irr}(G) : \chi(1) \leq p^k\} = \{\chi \in \mathrm{Irr}(G) : \Omega_{n-k}(G') \subseteq \mathrm{Ker}(\chi)\}$$

for $k = 0, \ldots, n$.

Proof. We argue by induction on n. If $n = 0$ then G is abelian, and the result is trivial. Thus let $n > 0$, so that $|\Omega_1(G')| = p$ since $G' \subseteq A$. Then $|\overline{G}'| = p^{n-1}$ where $\overline{G} := G/\Omega_1(G')$. By induction, we have $\{\chi(1) : \chi \in \mathrm{Irr}(\overline{G})\} = \{1, p, p^2, \ldots, p^{n-1}\}$ and

$$\{\chi \in \mathrm{Irr}(\overline{G}) : \chi(1) \leq p^k\} = \{\chi \in \mathrm{Irr}(\overline{G}) : \Omega_{n-1-k}(\overline{G}') \subseteq \mathrm{Ker}(\chi\})$$

for $k = 0, \ldots, n-1$. Certainly $\Omega_1(G') \nsubseteq \mathrm{Ker}(\chi)$ for some $\chi \in \mathrm{Irr}(G)$. Since G' is cyclic this implies that $G' \cap \mathrm{Ker}(\chi) = 1$. Let $\alpha \in \mathrm{Irr}(A)$ such that $(\chi_A|\alpha)_A \neq 0$, and let $T := I_G(\alpha)$ denote the inertial group of α in G. Clifford theory yields that $\chi = \tau^G$ for some $\tau \in \mathrm{Irr}(T|\alpha)$. Since T/A is cyclic, we have $\tau_A = \alpha$; in particular, $\tau(1) = \alpha(1) = 1$, so that $T' \subseteq \mathrm{Ker}(\tau)$. Since $T' \trianglelefteq G$ we conclude that $T' \subseteq G' \cap \mathrm{Ker}(\chi) = 1$. Thus T is abelian, so that $C = C_G(A) \subseteq I_G(\alpha) = T \subseteq C$. Hence $T = C$, and $\chi(1) = \tau^G(1) = |G : T|\tau(1) = |G : C| = |G'| = p^n$ by Lemma 2.1. We conclude that $\{1, p, p^2, \ldots, p^{n-1}, p^n\} \subseteq \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$.
Conversely, let $\phi \in \mathrm{Irr}(G)$. Since C is an abelian normal subgroup of index p^n in G, by Lemma 2.1, Itô's Theorem implies that $\phi(1) \in \{1, p, p^2, \ldots, p^n\}$. This shows that

$$\{\chi(1) : \chi \in \mathrm{Irr}(G)\} = \{1, p, p^2, \ldots, p^n\}.$$

The second assertion for $k = n$ follows. So let $k \in \{0, 1, \ldots, n-1\}$ and $\chi \in \mathrm{Irr}(G)$. If $\chi(1) \leq p^k$ then $\chi(1) < p^n$, so the argument above shows that $\Omega_1(G') \subseteq \mathrm{Ker}(\chi)$. Thus χ can be viewed as an element in $\mathrm{Irr}(\overline{G})$. By induction, we have $\Omega_{n-1-k}(\overline{G}') \subseteq \mathrm{Ker}(\chi)$. Hence $\Omega_{n-k}(G') \subseteq \mathrm{Ker}(\chi)$. Conversely, suppose that $\Omega_{n-k}(G') \subseteq \mathrm{Ker}(\chi)$; in particular, $\Omega_1(G') \subseteq \mathrm{Ker}(\chi)$. Thus χ can be viewed as an element in $\mathrm{Irr}(\overline{G})$, with $\Omega_{n-1-k}(\overline{G}') \subseteq \mathrm{Ker}(\chi)$. Hence $\chi(1) \leq p^k$, and the result follows. □

Corollary 2.3. *Let $|G'| = p^n$, and let $k \in \{1, \ldots, n\}$. Then*

$$|\{\chi \in \mathrm{Irr}(G) : \chi(1) = p^k\}| = |G : G'|\frac{p-1}{p^{k+1}}.$$

Proof. We have $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$ and, by Proposition 2.2,

$$\sum_{\chi \in \mathrm{Irr}(G),\, \chi(1)<p^n} \chi(1)^2 = \sum_{\chi \in \mathrm{Irr}(G/\Omega_1(G'))} \chi(1)^2 = |G/\Omega_1(G')| = \frac{|G|}{p}.$$

Thus $|\{\chi \in \mathrm{Irr}(G) : \chi(1) = p^n\}|p^{2n} = |G| - \frac{|G|}{p} = |G|\frac{p-1}{p}$ by Proposition 2.2, so that

$$|\{\chi \in \mathrm{Irr}(G) : \chi(1) = p^n\}| = |G : G'|\frac{p-1}{p^{n+1}}.$$

The rest follows by induction, using Proposition 2.2 again. □

Remark 2.4. Corollary 2.3 implies that

$$|\{\chi \in \mathrm{Irr}(G) : \chi(1) = p^k\}| = p|\{\chi \in \mathrm{Irr}(G) : \chi(1) = p^{k+1}\}|$$

for $k = 1, \ldots, n-1$. Thus a short computation, making use of Lemma 2.1 (iii), shows that

$$|\,\mathrm{Irr}(G)| = p^{l-1}(p^{n+1} + p^n - 1) \quad \text{where} \quad p^l = |Z(G)|.$$

In Chapter 12 of [11], Isaacs investigates the *vanishing-off subgroup* $\mathrm{V}(\chi) := \langle x \in X : \chi(x) \neq 0\rangle$, for an irreducible character χ of a finite group X. In our situation, we have $\mathrm{V}(\chi) = Z(\chi)$ for $\chi \in \mathrm{Irr}(G)$; in particular, the elements $g \in G$ such that $\chi(g) \neq 0$ form a subgroup of G. Moreover, we have $|G : \mathrm{V}(\chi)| = \chi(1)^2$ for $\chi \in \mathrm{Irr}(G)$. (Note that the wreath product $C_3 \wr C_3$ of order 81 gives an example of a finite 3-group P where the elements x with $\chi(x) \neq 0$ do not form a subgroup, for suitable $\chi \in \mathrm{Irr}(P)$.)

In the following, we denote by $\mathrm{Max}(G)$ the set of maximal subgroups of G.

Lemma 2.5. *Suppose that $G' \neq 1$. Then the following holds, for $M \in \mathrm{Max}(G)$:*

(i) $|G' : M'| = p$, *so that* $|G : G'| = |M : M'|$.

(ii) $Z(G) \subseteq Z(M)$ *and* $|Z(M) : Z(G)| = p$.

(iii) *If* $A \subseteq M$ *then* $|A \cap Z(M) : A \cap Z(G)| = p$.

Proof. (i) Assume first that $1 \neq G' = M'$, and choose G of minimal order with this property. If $|G'| > p$ then $\Omega_1(G') < G' \leq \Phi(G) \leq M$. Thus $1 \neq \overline{G}' = \overline{M}'$ where $\overline{G} := G/\Omega_1(G')$ and $\overline{M} := M/\Omega_1(G')$. This contradicts the minimality of G.
So we may assume that $|G'| = p$. Then, by Lemma 2.1, we have $|G : Z(G)| \leq p^2$. Also $Z(G) \subseteq \Phi(G)$ since otherwise $G = \langle Z(G), \Phi(G), g\rangle = \langle Z(G), g\rangle = Z(G)$ for some $g \in G$. Since $|G : \Phi(G)| = p^2$ this implies $Z(G) = \Phi(G)$. Thus $|M : Z(G)| = p$, so that M is abelian. Hence $G' = M' = 1$, a contradiction.
This shows that $M' < G'$. We need to prove that $|G' : M'| = p$. For this we may assume

that $M' = 1$. Then M is abelian, so the irreducible characters of G have degree 1 or p. Thus $|G'| \leq p$ by Proposition 2.2. Since $G' \neq M' = 1$ the assertion follows.

(ii) Assume that $Z(G) \not\subseteq M$. Then $G = Z(G)M$ and $G' = M'$, a contradiction.
Thus $Z(G) \subseteq M$, so that $Z(G) \subseteq Z(M)$. By Lemma 2.1, we have $|G : Z(G)| = |G'|^2$ and, similarly, $|M : Z(M)| = |M'|^2$. Thus $|Z(M)| = p|Z(G)|$ by (i).

(iii) Suppose that $A \subseteq M$. Then Lemma 12.12 in [11] implies that $|A| = |G'| \cdot |A \cap Z(G)|$ and, similarly, $|A| = |M'| \cdot |A \cap Z(M)|$. Thus $|A \cap Z(M) : A \cap Z(G)| = p$ by (i). □

Remark 2.6. The result above implies that there exists a metacyclic p-group G which cannot be embedded in a bigger metacyclic p-group F. In fact, suppose that $|Z(G)| = p$ and, w.l.o.g., $|F : G| = p$. Then the result above would give the contradiction $Z(F) = 1$.

Remark 2.7. In the situation of Lemma 2.5 we have, by Corollary 2.3,

$$|\{\chi \in \mathrm{Irr}(M) : \chi(1) = p^k\}| = |\{\chi \in \mathrm{Irr}(G) : \chi(1) = p^k\}|,$$

for $k = 0, \ldots, n-1$ (where $|G'| = p^n$ and, therefore, $|M'| = p^{n-1}$). This implies, by induction, that

$$|\{\chi \in \mathrm{Irr}(L) : \chi(1) = p^k\}| = |\{\chi \in \mathrm{Irr}(G) : \chi(1) = p^k\}|$$

whenever L is a nonabelian or a maximal abelian subgroup of G and $p^k \leq |L'|$. In particular, every maximal abelian subgroup of G has order $|G : G'|$. We now prove the converse.

Lemma 2.8. *Let $L \leq G$. Then L is abelian if and only if $|L| \leq |G : G'|$. Moreover, L is a maximal abelian subgroup of G if and only if $|L| = |G : G'|$.*

Proof. Suppose first that L is abelian. Then L is contained in a maximal abelian subgroup K of G. Thus, by Remark 2.7, we have $|L| \leq |K| = |G : G'|$.
Now suppose, conversely, that $|L| \leq |G : G'|$. We show by induction on $|G|$ that L is abelian. Thus we may assume that $G' \neq 1$. If $|G : L| = p$ then $p = |G'| = |G : C|$, by Lemma 2.1. Thus Lemma 2.5 implies that $|L : L'| = |G : G'| = |C : C'|$, so that $|L'| = |C'| = 1$, and we are done in this case.
Now suppose that $|G : L| \geq p^2$, and let $L < M \in \mathrm{M}ax(G)$. Then Lemma 2.5 implies that $|M : L| = \frac{|G:L|}{p} \geq \frac{|G'|}{p} = |M'|$, i.e. $|L| \leq |M : M'|$. Thus L is abelian by induction. □

In [12], Janko studied p-groups P such that whenever A is a maximal subgroup of any minimal nonabelian subgroup Q of P then A is also a maximal abelian subgroup of P.

Remark 2.9. If G is nonabelian then G/G' is noncyclic. Thus G/G' contains at least $p+1$ subgroups of order $|Z(G)|$. Hence G contains at least $p+1$ normal subgroups of order $|G'| \cdot |Z(G)| = |G : G'|$. Therefore $|\mathrm{SCN}(G)| \geq p+1$.

Proposition 2.10. *Suppose that $A \neq 1$. Then G has nilpotency class $\lceil \frac{n}{r} \rceil + 1$ where $|G'| = p^n$ and $|C_A(G)| = p^r$.*

Proof. We argue by induction on $|G|$. Note that $r \neq 0$ since $A \neq 1$ and G is a p-group. If G is abelian then $n = 0$, and the assertion is true in this case.
So suppose that G is nonabelian. Then $G' \subseteq A$ and $C_A(G) \subseteq A$. Since A is cyclic, there are two possibilities. Suppose first that $G' \subseteq C_A(G)$. Then G has nilpotency class 2, and $n \leq r$, so that also $\lceil \frac{n}{r} \rceil + 1 = 2$.
Thus we may assume that $C_A(G)$ is properly contained in G'. In this case Lemma 12.12 in [11] implies that $|A : G'| = |C_A(G)|$. We set $\overline{G} := G/C_A(G)$ and $\overline{A} := A/C_A(G)$. Then $|\overline{A} : \overline{G}'| = |A : G'|$ and $|C_{\overline{A}}(\overline{G})| = |\overline{A} : \overline{G}'|$, again by Lemma 12.12 in [11].
Assume that $\overline{A} = 1$. Then $A = C_A(G) \subseteq Z(G)$, so that $G/Z(G)$ is cyclic. But then G is abelian, a contradiction.
Thus we must have $\overline{A} \neq 1$. Then, by induction, $\overline{G}$ has nilpotency class $\lceil \frac{n-r}{r} \rceil + 1 = \lceil \frac{n}{r} \rceil$. Suppose that G has nilpotency class c. Then $K_c(G) \neq 1 = K_{c+1}(G)$. Since $K_c(G) \subseteq Z(G) \cap A = C_A(G)$ we must have $K_c(\overline{G}) = 1$, so that $\overline{G}$ has nilpotency class $c - 1 = \lceil \frac{n}{r} \rceil$. The result follows. □

3 Conjugacy classes

We keep the hypotheses and notation of the preceding section. Then G is a regular p-group by Satz III.10.2 in [10].

Lemma 3.1. *Let $N := \Omega_1(G') \neq 1$ and $\overline{G} := G/N$. Moreover, let $g \in G \setminus Z(G)$ and $\overline{g} := gN$. Then $|G : C_G(g)| = p|\overline{G} : C_{\overline{G}}(\overline{g})|$.*

Proof. We argue by induction on $|G : C_G(g)|$. Suppose first that $|G : C_G(g)| = p$. We claim that $\overline{g} \in Z(\overline{G})$. In fact, if $h \in G$ then $h^p \in C_G(g)$ since $C_G(g) \in \mathrm{Max}(G)$. Thus $[g, h^p] = 1$. Since G is a regular p-group this implies that $[g, h]^p = 1$, by Satz III.10.6 in [10]. Hence $[g, h] \in \Omega_1(G') = N$, and the claim is proved.
Now suppose that $|G : C_G(g)| > p$, and write $C_{\overline{G}}(\overline{g}) = H/N$ where $N \leq H \leq G$.
If $H = G$ then $\overline{g} \in Z(\overline{G})$ and thus $|G : C_G(g)| \leq |N| = p$, a contradiction.
Hence $H \neq G$, i.e. $H \leq M$ for some $M \in \mathrm{Max}(G)$. Then $C_G(g) = C_M(g)$ and $|G : C_G(g)| = p \cdot |M : C_M(g)|$. Similarly, we have $C_{\overline{G}}(\overline{g}) = C_{\overline{M}}(\overline{g})$ where $\overline{M} := M/N$. Thus $|\overline{G} : C_{\overline{G}}(\overline{g})| = p \cdot |\overline{M} : C_{\overline{M}}(\overline{g})|$.
If $\Omega_1(M') = 1$ then $M' = 1$, so that M is abelian. This implies that $M \subseteq C_G(g)$ and therefore $|G : C_G(g)| \leq p$, a contradiction.
Thus $\Omega_1(M') \neq 1$, so that $\Omega_1(M') = \Omega_1(G') = N$. Since $g \in M \setminus Z(M)$ we have $|M : C_M(g)| = p \cdot |\overline{M} : C_{\overline{M}}(\overline{g})|$ by induction, and the result follows. □

For $i \in \mathbb{N}_0$, we set $\mathcal{K}_i(G) := \{g \in G : |G : C_G(g)| = p^i\}$. Since $|G : C_G(g)| \leq |G'| =: p^n$ for $g \in G$, we have $\mathcal{K}_i(G) = \emptyset$ if $i > n$.

Proposition 3.2. *Suppose that $|G'| = p^n \neq 1$. Then, for $i = 1, \ldots, n$, we have*

$$|\mathcal{K}_i(G)| = |Z(G)|(p^{2i} - p^{2i-2});$$

thus G contains precisely $|Z(G)|(p^i - p^{i-2})$ conjugacy classes of length p^i.

Proof. We argue by induction on i. Suppose first that $i = 1$. Then

$$\mathcal{K}_i(G) = \{g \in G : C_G(g) \in \mathrm{Max}(G)\} = \bigcup_{M \in \mathrm{Max}(G)} Z(M) \setminus Z(G)$$

where the union is disjoint. By Lemma 2.5, we have $|Z(M)| = p \cdot |Z(G)|$ for $M \in \mathrm{Max}(G)$. Since $|\mathrm{Max}(G)| = p + 1$ this implies

$$|\mathcal{K}_1(G)| = (p+1)|Z(G)|(p-1) = |Z(G)|(p^2 - 1),$$

and the result follows in this case.
We set $N := \Omega_1(G')$ and $\overline{G} := G/N$. The canonical map $G \longrightarrow \overline{G}$, $g \longmapsto \overline{g} := gN$, restricts to a surjection $\mathcal{K}_0(G) \cup \mathcal{K}_1(G) \longrightarrow Z(\overline{G})$, by Lemma 3.1. Moreover, $\mathcal{K}_0(G) \cup \mathcal{K}_1(G)$ consists of complete cosets of N, and $|\mathcal{K}_0(G) \cup \mathcal{K}_1(G)| = p^2|Z(G)|$. This implies that $|Z(\overline{G})| = p \cdot |Z(G)|$.
Now suppose that $i > 1$. By Lemma 3.1 the canonical map $G \longrightarrow \overline{G}$ induces a surjection $\mathcal{K}_i(G) \longrightarrow \mathcal{K}_{i-1}(\overline{G})$. Moreover, $\mathcal{K}_i(G)$ consists of complete cosets of N. Thus, by induction, we obtain

$$|\mathcal{K}_i(G)| = p \cdot |\mathcal{K}_{i-1}(\overline{G})| = p \cdot |Z(\overline{G})|(p^{2i-2} - p^{2i-4}) = |Z(G)|(p^{2i} - p^{2i-2}),$$

and the result is proved. □

Proposition 3.3. *Suppose that $|G'| = p^n$, $g \in G$ and $s \in \{0, \dots, n\}$. Then $g \in \mho_s(G)$ if and only if $|G : C_G(g)| \leq p^{n-s}$. In particular, we have $\mho_n(G) = Z(G)$.*

Proof. We argue by induction on $|G|$. If G is abelian then $n = 0 = s$, and the result is trivial. Thus we may assume that G is nonabelian. We set $N := \Omega_1(G')$, $\overline{G} := G/N$ and $\overline{g} := gN$. Note that $|\overline{G}'| = p^{n-1}$.
Suppose first that $g \in \mho_s(G)$, so that $\overline{g} \in \mho_s(\overline{G})$.
If $s \leq n-1$ then $|\overline{G} : C_{\overline{G}}(\overline{g})| \leq p^{n-1-s}$ by induction. Hence $|G : C_G(g)| \leq p^{n-s}$ by Lemma 3.1.
Suppose that $s = n$. For $x, y \in G$, we have $[x, y]^{p^n} = 1$. Since G is regular, Satz III.10.6 in [10] implies that $[x^{p^n}, y] = 1$. Thus $\mho_n(G) \subseteq Z(G)$; in particular, $g \in Z(G)$ and $|G : C_G(g)| = 1 = p^{n-s}$. This finishes one direction.
Now suppose, conversely, that $|G : C_G(g)| \leq p^{n-s}$.
If $s < n$ then $|\overline{G} : C_{\overline{G}}(\overline{g})| \leq p^{n-1-s}$ by Lemma 3.1. Thus $\overline{g} \in \mho_s(\overline{G})$ by induction. We write $g = x^{p^s} y$ where $x \in G$ and $y \in \Omega_1(G') = \mho_{n-1}(G') \subseteq \mho_s(G') \subseteq \mho_s(G)$. Thus $g \in \mho_s(G)$, and we are done in this case.
It remains to deal with the case $s = n$, i.e. $g \in Z(G)$. In the first part of the proof, we have shown that $\mho_n(G) \subseteq Z(G)$. On the other hand, $G/\mho_n(G)$ is a product of two cyclic groups whose order is bounded above by p^n. Thus $|G/\mho_n(G)| \leq p^{2n} = |G : Z(G)|$, by Lemma 2.1. Hence $\mho_n(G) = Z(G)$, and the result follows. □

Remark 3.4. Proposition 3.3 implies that

$$\mho_{n-s}(G) = \bigcup_{i=0}^{s} \mathcal{K}_i(G) \quad \text{for} \quad s = 0, \dots, n.$$

Thus $|\mho_{n-s}(G)| = |Z(G)|p^{2s}$ for $s = 0, \dots, n$.

Recall that a finite p-group P is called *powerful* if $P/\mho_1(P)$ is abelian (since $p \neq 2$). Thus P is powerful if and only if $\mho_1(P) = \Phi(P)$.
Since $G/\mho_1(G)$ is metacyclic of exponent p or 1, we obtain $|G/\mho_1(G)| \leq p^2$; in particular, G is powerful.

Proposition 3.5. *Let $g \in G$. If $g \notin \Phi(G)$ then $C_G(g)$ is abelian. The converse holds provided that G is nonabelian.*

Proof. If $g \notin \Phi(G) = \mho_1(G)$ then $|G : C_G(g)| = p^n = |G'|$ by Proposition 3.3. Thus $C_G(g)$ is abelian by Lemma 2.8.
Now suppose, conversely, that $C_G(g)$ is abelian and that G is nonabelian. Then $C_G(g)$ is a maximal abelian subgroup of G, so that $|C_G(g)| = |G : G'|$ by Remark 2.7. Hence $|G : C_G(g)| = |G'| = p^n$, so Proposition 3.3 implies that $g \notin \mho_1(G) = \Phi(G)$. □

We note that G is not necessarily a semidirect product of two cyclic subgroups (cf. [13]). However, we can show the following.

Proposition 3.6. *There are cyclic subgroups A_1, B_1 of G such that $A_1B_1 = G$ and $A_1 \cap B_1 = 1$.*

Proof. We argue by induction on $|G|$ and may assume that G is noncyclic. Then $|G/\Phi(G)| = p^2$ and $\Phi(G) = \mho_1(G)$.
Suppose first that $\Phi(G) = \mho_1(G)$ is cyclic. Then there is $a \in G$ such that $\Phi(G) = \langle a^p \rangle$. Thus $\langle a \rangle$ is cyclic of index p in G, so the result follows from Theorem 5.4.3 in [8].
Now suppose that $\Phi(G) = \mho_1(G)$ is noncyclic. Then, by induction, there are cyclic subgroups $A_2 = \langle a_2 \rangle$, $B_2 = \langle b_2 \rangle$ of G such that $A_2B_2 = \Phi(G)$ and $A_2 \cap B_2 = 1$. By Proposition 2.6 in [6], there are $a_1, b_1 \in G$ such that $a_1^p = a_2$ and $b_1^p = b_2$. Let $A_1 := \langle a_1 \rangle$ and $B_1 := \langle b_1 \rangle$.
Assume that $A_2 \cap B_1 \neq 1$. Then $A_2 \cap B_1 \supseteq \Omega_1(B_1) = \Omega_1(B_2)$, so that $A_2 \cap B_2 \supseteq \Omega_1(B_2) \neq 1$, a contradiction.
Thus $A_2 \cap B_1 = 1$, and $|A_2B_1| = |A_2| \cdot |B_1| = p \cdot |A_2| \cdot |B_2| = p \cdot |\Phi(G)|$. Since $A_2B_1 \supseteq A_2B_2 = \Phi(G)$ we conclude that $A_2B_1 \trianglelefteq G$.
Assume that $A_1 \cap B_1 \neq 1$. Then $A_1 \cap B_1 \supseteq \Omega_1(A_1) = \Omega_1(A_2)$, so that $A_2 \cap B_1 \supseteq \Omega_1(A_2) \neq 1$, a contradiction.
Thus $A_1 \cap B_1 = 1$ and $|A_1B_1| = |A_1| \cdot |B_1| = |G|$, so that $A_1B_1 = G$. □

This proposition generalizes a result of Huppert which says that a nonabelian metacyclic p-group ($p > 2$) G is the product of two disjoint cyclic subgroups provided that G admits a nontrivial action of a p'-group of operators (see [2], Theorem 4.12).

4 Centrally large subgroups

In [7], Glauberman introduced the notion of centrally large subgroups of finite p-groups. In this section we use some of our results to investigate this notion for a finite metacyclic p-group G where p is an odd prime.
Following [7], we set

$$f(P) := \max\{|R| \cdot |Z(R)| : R \leq P\},$$

for any finite p-group P. Then Proposition 2.4 in [7] implies that

$$f(P) = \max\{|R| \cdot |C_P(R)| : R \le P\}.$$

In our situation, we prove:

Lemma 4.1. *We have $f(G) = |G : G'|^2$.*

Proof. By Lemma 2.8 , G contains a maximal abelian subgroup L of order $|G : G'|$. Thus $f(G) \ge |L| \cdot |Z(L)| = |L|^2 = |G : G'|^2$.
Now let R be a subgroup of G such that $|R| \cdot |Z(R)| = f(G)$. If R is abelian then $|R| \le |G : G'|$ by Remark 2.7, and $|R| \cdot |Z(R)| \le |G : G'|^2$.
Thus we may asssume that R is nonabelian. Then $|R| \ge |G : G'|$ by Lemma 2.8, and R contains a maximal abelian subgroup L of G of order $|G : G'|$. Let M be a maximal subgroup of R containing L. Then Lemma 2.5 implies that $|R| \cdot |Z(R)| = |M| \cdot |Z(M)|$. Continuing in this way we obtain $|M| \cdot |Z(M)| = \ldots = |L| \cdot |Z(L)| = |G : G'|^2$. $\square$

The following result may be of independent interest.

Lemma 4.2. *Let K be an abelian subgroup of G containing $Z(G)$. Then $K = Z(C_G(K))$.*

Proof. We argue by induction on $|G|$ and may assume that $Z(G) < K$. Then $C_G(K) < G$; in particular, G is nonabelian. We set $\overline{G} := G/Z(G)$. Then $|\Omega_1(\overline{G})| = p^2$; for otherwise $\overline{G}$ is cyclic and G is abelian, a contradiction. For $M \in \mathrm{Max}(G)$, we have $|Z(M) : Z(G)| = p$ by Lemma 2.5, so $\overline{Z(M)} := Z(M)/Z(G) \subseteq \Omega_1(\overline{G})$. Moreover, we have $M = C_G(Z(M))$. Thus the $p+1$ maximal subgroups M of G give rise to $p+1$ distinct minimal subgroups $\overline{Z(M)}$ of $\overline{G}$. Hence every minimal subgroup of $\overline{G}$ arises in this way.
Since $1 \ne \overline{K} := K/Z(G)$, we must have $\overline{Z(M)} \subseteq \overline{K}$ for some $M \in \mathrm{Max}(G)$. Then $Z(M) \subseteq K \subseteq C_G(K) \subseteq C_G(Z(M)) = M$, and $C_G(K) = C_M(K)$. By induction, we get $K = Z(C_M(K)) = Z(C_G(K))$. $\square$

Following [7], we set

$$\mathcal{F}(P) := \{R \le P : |R| \cdot |Z(R)| = f(P)\}$$

and

$$\mathcal{F}_1(P) := \{R \le P : |R| \cdot |C_P(R)| = f(P)\},$$

for any finite p-group P. The groups in $\mathcal{F}(P)$ are called *centrally large* subgroups of P. By Proposition 2.4 in [7], we have $\mathcal{F}(P) \subseteq \mathcal{F}_1(P)$. Moreover, $\mathcal{F}(P)$ contains a unique maximal element P_{CL}, and P_{CL} is also the unique maximal element in $\mathcal{F}_1(P)$. By Corollary 2.2 in [7], $\mathcal{F}_1(P)$ also contains a unique minimal element.

Proposition 4.3. *We have $\mathcal{F}_1(G) = \{Q \le G : Z(G) \le Q\}$.*

Proof. If $Q \in \mathcal{F}_1(G)$ then $Q = C_G(C_G(Q)) \supseteq Z(G)$ by Proposition 2.3 in [7].
Conversely, let $Z(G) \le Q \le G$. If Q is nonabelian then $Q \in \mathcal{F}(G)$ by the argument in the proof of Lemma 4.1. Hence $Q \in \mathcal{F}_1(G)$ by Proposition 2.4 (b) in [7].
Thus we may assume that Q is abelian. If $R := C_G(Q)$ is nonabelian then $R \in \mathcal{F}_1(G)$ by what we have just proved. Then also $C_G(R) \in \mathcal{F}_1(G)$ by Proposition 2.3 in [7]. Now $Q \subseteq R$ implies

that $C_G(R) \subseteq C_G(Q) = R$, i.e. $C_G(R) = Z(R) = Q$ by Lemma 4.2, so that $Q \in \mathcal{F}_1(G)$ in this case.
However, if $R := C_G(Q)$ is abelian then R is a maximal abelian subgroup of G. Then $|R| = |G : G'|$ and $|R| \cdot |C_G(R)| = f(G)$, so that $R \in \mathcal{F}_1(G)$. Thus we can argue in a similar way to obtain $Q \in \mathcal{F}_1(G)$. □

We have the following consequence.

Corollary 4.4. *$\mathcal{F}(G)$ consists of all nonabelian and all maximal abelian subgroups of G.*

Proof. If Q is a nonabelian subgroup of G then $Q \in \mathcal{F}(G)$ by the argument in the proof of Lemma 4.1.
If Q is a maximal abelian subgroup of G then $|Q| \cdot |Z(Q)| = |G : G'|^2 = f(G)$ by Lemma 2.8 and Lemma 4.1. Hence, $Q \in \mathcal{F}(G)$.
Conversely, let $Q \in \mathcal{F}(G)$ be abelian. Then $|Q| \cdot |Z(Q)| = f(G)$. Using the fact that Q is abelian and that $f(G) = |G : G'|^2$ we immediately obtain that $|Q| = |G : G'|$. Hence, by Lemma 2.8 we have Q is maximal abelian. □

Since $\mathcal{F}(G) \subseteq \mathcal{F}_1(G)$ by Proposition 2.4 in [7], every nonabelian subgroup of G contains $Z(G)$. Also, by Theorem 2.1 of [7], the product of any two nonabelian subgroups of G is again a subgroup of G.

Most of our results do not carry over to the case where $p = 2$. Suppose, for example, that G is dihedral, semidihedral or quaternion and that the order of G is at least 16. Then $|G| = |G'| \cdot |Z(G)|^2 \neq |G'|^2 \cdot |Z(G)|$. Moreover, the cyclic subgroup of index 2 is the unique maximal and minimal element in $\mathcal{F}_1(G)$. In particular, not every maximal abelian subgroup of G is centrally large in G.

5 Non-metacyclic groups

N. Blackburn has proved the following (see [4] or [10], III.12.4 and III.12.5):

Theorem 5.1. *Let G be a group of order p^n where p is an odd prime, and suppose that G has rank $d \leq 2$. Then one of the following holds:*

(i) G *is metacyclic.*

(ii) $G = \langle x, y, z \rangle$ *where* $x^p = y^p = z^{p^{n-2}} = [x, z] = [y, z] = 1$, $x^{-1}yx = yz^{p^{n-3}}$ *and* $n \geq 3$.

(iii) $G = \langle x, y, z \rangle$ *where* $x^p = y^p = z^{p^{n-2}} = [y, z] = 1$, $x^{-1}yx = yz^{sp^{n-3}}$, $x^{-1}zx = yz$ *and* $n \geq 4$. *Moreover, either $s = 1$, or s is a quadratic non-residue mod p.*

(iv) G *is a 3-group of maximal class.*

In this section we will discuss the conjugacy classes, irreducible characters and centrally large subgroups of the non-metacyclic groups in Theorem 5.1.

Proposition 5.2. *Suppose that $G = \langle x, y, z \rangle$ as in Theorem 5.1 (ii). Then the following holds:*

(i) G has precisely p^{n-1} irreducible characters of degree 1 and precisely $p^{n-2}-p^{n-3}$ irreducible characters of degree p.

(ii) G contains precisely p^{n-2} conjugacy classes of length 1 and precisely $p^{n-1}-p^{n-3}$ conjugacy classes of length p.

(iii) G is the unique maximal centrally large subgroup of G, and the minimal centrally large subgroups of G are the abelian subgroups of index p in G. Moreover, the unique minimal element in $\mathcal{F}_1(G)$ is the center $Z(G)$.

Proof. (i) By [10], Satz III.12.4, G has an abelian subgroup of index p. Thus the irreducible characters of G have degree 1 or p. It is easy to see that $G' = \langle z^{p^{n-3}} \rangle$. Hence the number of characters of G of degree 1 is $|G : G'| = p^{n-1}$, and (i) follows since $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$.

(ii) Since $|G'| = p$ the conjugacy classes of G have length 1 or p. It is easy to see that $Z(G) = \langle z \rangle$ has order p^{n-2}, so that (ii) is proved.

(iii) Certainly we have $f(G) \geq |G| \cdot |Z(G)| = p^{2n-2}$. Since $|H| \cdot |Z(H)| \leq p^{2n-2}$ for every proper subgroup H of G, we conclude that $f(G) = p^{2n-2}$. Thus G is the unique maximal centrally large subgroup of G.
Suppose that H is a maximal subgroup of G. Then H is a centrally large subgroup of G if and only if H is abelian. Moreover, every subgroup K of G with $|K| < p^{n-1}$ satisfies $|K| \cdot |Z(K)| < p^{2n-2}$, so it cannot be centrally large in G.
Since $|Z(G)| \cdot |G| = p^{2n-2}$ we have $Z(G) \in \mathcal{F}_1(G)$, and certainly every proper subgroup Z of $Z(G)$ satisfies $|Z| \cdot |G| < p^{2n-2}$. Thus $Z(G)$ is the unique minimal element in $\mathcal{F}_1(G)$. □

Next we turn to the groups in Theorem 5.1 (iii).

Proposition 5.3. *Suppose that $G = \langle x, y, z \rangle$ as in Theorem 5.1 (iii). Then the following holds:*

(i) G has precisely p^{n-2} irreducible characters of degree 1 and precisely $p^{n-2}-p^{n-4}$ irreducible characters of degree p.

(ii) G contains precisely p^{n-3} conjugacy classes of length 1, precisely $p^{n-2}-p^{n-4}$ conjugacy classes of length p, and precisely $p^{n-2}-p^{n-3}$ conjugacy classes of length p^2.

(iii) The only centrally large subgroup of G is the unique abelian subgroup A of index p in G. Moreover, we have $\mathcal{F}_1(G) = \{A\}$.

Proof. (i) By Satz III.12.4 in [10], G has an abelian subgroup A of index p. Thus the irreducible characters of G have degree 1 or p. It is easy to see that $G' = \langle y, z^{sp^{n-3}} \rangle$ has order p^2. Thus the number of characters of G of degree 1 is $|G : G'| = p^{n-2}$, and (i) follows since $\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$.

(ii) Since G is nonabelian, we have $Z(G) \subseteq A$. By Aufgabe III.1.2 in [10], this implies that $|A| = |G'| \cdot |Z(G)|$. Thus $|Z(G)| = p^{n-3}$, and G has precisely p^{n-3} conjugacy classes of length 1. The elements in $A \setminus Z(G)$ lie in conjugacy classes of length p each. This accounts for $p^{n-2}-p^{n-4}$ conjugacy classes of length p. Finally, for $g \in G \setminus A$, we have $G' = \{[a, g] : a \in A\}$, by Aufgabe III.1.2 in [10] again. Thus the conjugacy class of g has length p^2. This yields $p^{n-2} - p^{n-3}$ conjugacy classes of length p^2 in G.

(iii) We certainly have $f(G) \geq |A| \cdot |Z(A)| = p^{2n-2}$. Since $|G| \cdot |Z(G)| = p^{2n-3} < f(G)$ we conclude that $f(G) = p^{2n-2}$. Moreover, G is not a centrally large subgroup of G, and every centrally large subgroup R of G must satisfy $|R| = p^{n-1}$. Moreover, R has to be abelian. Since $|Z(G)| \neq p^{n-2}$, A is the only maximal subgroup of G which is abelian. Thus A is the only centrally large subgroup of G.
By Corollary 2.2 in [7], $\mathcal{F}_1(G)$ contains a unique minimal element B; in particular, we have $B \subseteq A$. If $B \subseteq Z(G)$ then $|B| \cdot |C_G(B)| \leq p^{n-3}p^n < f(G)$ which is impossible. Thus $B \not\subseteq Z(G)$ and $p^{2n-2} = |B| \cdot |C_G(B)| = |B| \cdot |A|$, so that $|B| = |A|$ and $B = A$. □

Now we turn to case (iv) in Theorem 5.1. Then $p = 3$, and G has maximal class. The structure of G is described in Satz III.14.17 of [10]: G is metabelian, and $G_1 := C_G(K_2(G)/K_4(G))$ is metacyclic of class $c \leq 2$. Moreover, G is not an exceptional group of maximal class, in the sense of Definition III.14.5 of [10]. We first consider the case where G_1 is abelian.

Proposition 5.4. *Let G be as in Theorem 5.1 (iv), with $n \geq 4$ (and $p = 3$). Moreover, suppose that G_1 is abelian. Then the following holds:*

- *(i) G has precisely p^2 irreducible characters of degree 1, and precisely $p^{n-2} - 1$ irreducible characters of degree p.*
- *(ii) G contains precisely p conjugacy classes of length 1, precisely $p^{n-2} - 1$ conjugacy classes of length p, and precisely $p^2 - p$ conjugacy classes of length p^{n-2}.*
- *(iii) G_1 is the only centrally large subgroup of G, and $\mathcal{F}_1(G) = \{G_1\}$.*

Proof. (i) Since G has an abelian subgroup of index p, the irreducible characters of G have degree 1 or p. Since G has maximal class, we have $|G : G'| = p^2$, and (i) follows.

(ii) Since G has maximal class, we have $|Z(G)| = p$. Thus G contains precisely p conjugacy classes of length 1. Moreover, the conjugacy classes of G contained in $G_1 \setminus Z(G)$ all have length p. This yields $p^{n-2} - 1$ conjugacy classes of G of length p. Let $g \in G \setminus G_1$. Then Aufgabe III.1.2 of [10] implies that $G' = \{[a, g] : a \in G_1\}$. Thus the conjugacy class of g has length $|G'| = p^{n-2}$. This yields $p^2 - p$ conjugacy classes of G of length p^{n-2}.

(iii) Since G_1 is abelian we have $f(G) \geq |G_1| \cdot |Z(G_1)| = p^{2n-2}$. Since $|G| \cdot |Z(G)| = p^{n+1} < p^{2n-2}$ this implies that $f(G) = p^{2n-2}$. Since G_1 is the only abelian subgroup of index p in G, G_1 is the only centrally large subgroup of G. Clearly, we have $G_1 \in \mathcal{F}_1(G)$. Since $\mathcal{F}_1(G)$ contains a unique minimal element we conclude, as above, that $\mathcal{F}_1(G) = \{G_1\}$. □

Finally, we turn to the remaining case where G_1 has nilpotency class 2. In this case we must have $|G| \geq p^5$; for otherwise $|G| \leq p^4$ and $|G_1| \leq p^3$. But $G_1 = C_G(Z_2(G))$ since G is not an exceptional group of maximal class, so that G_1 is then abelian.

Proposition 5.5. *Let G be as in Theorem 5.1 (iv), with $n \geq 5$ (and $p = 3$). Moreover, suppose that $G_1 = C_G(K_2(G)/K_4(G))$ has class 2. Then the following holds:*

- *(i) G has precisely p^2 irreducible characters of degree 1, precisely $p^{n-3} - 1$ irreducible characters of degree p, and precisely $p^{n-4} - p^{n-5}$ irreducible characters of degree p^2.*

(ii) G contains precisely p conjugacy classes of length 1, precisely $p^{n-4}-1$ conjugacy classes of length p, precisely $p^{n-3}-p^{n-5}$ conjugacy classes of length p^2 and precisely p^2-p conjugacy classes of length p^{n-2}.

(iii) If $n > 5$ then G_1 is the unique maximal centrally large subgroup of G. Moreover, the abelian subgroups of order p^{n-1} are the minimal centrally large subgroups of G. Furthermore, $Z(G_1)$ is the unique minimal element in $\mathcal{F}_1(G)$.

Proof. (ii) Since G is metabelian, G' is abelian, and $|G_1 : G'| = p$. Thus the irreducible characters of G_1 have degree 1 or p. By Proposition 2.2 this implies that $|G_1'| \le p$. Since G_1 is nonabelian we must have $|G_1'| = p$. Hence Lemma 2.1 shows that $|Z(G_1)| = p^{n-3}$. Moreover, G_1 contains precisely p^{n-3} conjugacy classes of length 1 and precisely $p^{n-2} - p^{n-4}$ conjugacy classes of length p.
Let $g \in G \setminus G_1$. Then $|C_G(g)| = p^2$ by Hilfssatz III.14.13 in [10], so the conjugacy class of g has length p^{n-2}. Furthermore, since $|G \setminus G_1| = p^n - p^{n-1}$, there are $p^2 - p$ such conjugacy classes. Since $|Z(G)| = p$, G contains precisely p conjugacy classes of length 1.
Now let $g \in G_1 \setminus Z(G)$. Then $C_G(g) \subseteq G_1$ by Hilfssatz III.14.13 in [10], so that $C_G(g) = C_{G_1}(g)$. Thus the $p^{n-3} - p$ elements in $Z(G_1) \setminus Z(G)$ fall into $p^{n-4} - 1$ conjugacy classes of length p, and the $p^{n-1} - p^{n-3}$ elements in $G_1 \setminus Z(G_1)$ fall into $p^{n-3} - p^{n-5}$ conjugacy classes of length p^2.

(i) Since G' is an abelian subgroup of index p^2 in G, the irreducible characters of G have degree 1, p or p^2. Moreover, G has precisely $|G : G'| = p^2$ irreducible characters of degree 1. We have seen above that G_1 has $|G_1 : G_1'| = p^{n-2}$ characters of degree 1. Thus G_1 has $p^{n-3} - p^{n-4}$ irreducible characters of degree p.
Let $g \in G \setminus G_1$. Then the action of $\langle g \rangle$ on the set of conjugacy classes of G_1 has only p fixed points, by what we saw above. By Brauer's Permutation Lemma, the action of $\langle g \rangle$ on $\mathrm{Irr}(G_1)$ has also p fixed points. These must be the elements of $\mathrm{Irr}(G_1/G')$. Thus $p^{n-2} - p$ characters of G_1 of degree 1 fall into G-orbits of length p. They induce to $p^{n-3} - 1$ irreducible characters of G of degree p. The $p^{n-3} - p^{n-4}$ irreducible characters of G_1 of degree p fall into G-orbits of length p each. This yields $p^{n-4} - p^{n-5}$ irreducible characters of G of degree p^2.

(iii) Let $n > 5$. Then $f(G) \ge |G_1| \cdot |Z(G_1)| = p^{n-1}p^{n-3} = p^{2n-4}$. Moreover, we have $|G| \cdot |Z(G)| = p^{n+1} < p^{2n-4}$. Since G has no abelian subgroup of index p, this implies that $f(G) = p^{2n-4}$.
The abelian subgroups of G of order p^{n-2} are therefore the minimal centrally large subgroups of G. Note that all maximal subgroups $M \ne G_1$ of G have maximal class, by Satz III.14.22 in [10], so they are not centrally large in G. Thus G_1 is the unique maximal centrally large subgroup of G.
By Proposition 2.4 in [7], G_1 is also the unique maximal element in $\mathcal{F}_1(G)$. It is easy to check that $Z(G_1)$ is the unique minimal element in $\mathcal{F}_1(G)$. □

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