On vertices of simple modules for symmetric groups of small degrees

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Abstract

We present some algorithmic methods for the computation of vertices of indecomposable and simple modules over group algebras in prime characteristic. Furthermore, we apply these to the simple modules of the symmetric groups and determine the vertices of all simple modules of the symmetric groups of degree at most 14 and 15 in characteristic 2 and 3, respectively, with one exception.

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1 Introduction and notation

The representation theory of the symmetric groups has been known to be an active field of research since the beginning of the last century, and a powerful combinatorial machinery has been developed since. While the ordinary theory is quite well understood, the situation turns out to be much more complicated when it comes to the modular case. Given a field F of prime characteristic p, on the one hand one does have an explicit combinatorial parametrization of the isomorphism classes of simple $F\mathfrak{S}_n$-modules by the p-regular partitions of n.
However, still little seems to be known about several structural invariants of simple $F\mathfrak{S}_n$-modules. Among those are the vertices and sources. Currently, one is far from knowing what vertices of simple $F\mathfrak{S}_n$-modules look like in general. As a starting point it is therefore worthwhile to have available as many explicit examples as possible, in order to detect general patterns. Moreover, once knowing the vertices of simple $F\mathfrak{S}_n$-modules for certain "small" n, one then also obtains the vertices of various infinite series of simple modules for symmetric groups, via the Scopes equivalence [32] and the modular branching rules [22].

The purpose of this article is to present some algorithmic approaches towards the determination of vertices of indecomposable modules over group algebras, most of which do indeed work for arbitrary finite groups. Our computations have basically been carried out using the computer algebra system MAGMA [3] and are based on algorithms which have been developed by the third author in [35] and have later been extended by the first author in [8]. Furthermore, one of the computationally investigated simple modules has been treated with the C-MeatAxe [31] and GAP [14]. Further details will be given in the course of this article.

The main problem occuring when dealing with simple $F\mathfrak{S}_n$-modules on the computer is the fact that their dimensions soon become huge. As a consequence thereof, those parts of the vertex computation consisting of restricting a simple $F\mathfrak{S}_n$-module to several p-subgroups of $\mathfrak{S}_n$ and computing direct sum decompositions of such restrictions become extremely time and memory consuming. To overcome this problem at least to a considerable amount, we used randomized algorithms to "strip" indecomposable direct summands with cyclic vertices and, in particular, projective direct summands off FP-modules where P is a p-group, without computing a direct sum decomposition. Besides these algorithms, our computations involve

methods coming from group cohomology. Namely, given an indecomposable FG-module M and an elementary abelian p-group E, Theorem 3.14 can be applied to show that E has to be contained in some vertex of M, thereby shrinking the number of p-subgroups of G which can possibly occur as vertices of M. After all, we have been able to compute the vertices of all simple $F\mathfrak{S}_n$-modules in characteristic 2 and 3, for $n \leq 14$ and $n \leq 15$, respectively, with one exception.
As far as the vertex computation is concerned, the simple $F\mathfrak{S}_n$-modules for $n < p^2$ are rather uninteresting. Namely, the vertices of these modules are known to be precisely the defect groups of the corresponding blocks, by Knörr's Theorem [23]. However, our methods do involve the decomposition numbers of $F\mathfrak{S}_n$ which for $n \geq 18$ are so far unknown in general. Currently, we are therefore not able to provide any computational data for $p \geq 5$.

Throughout this paper, F will denote a field of prime characteristic p, any F-algebra is supposed to be finite dimensional, associative and unitary, and a group G is always understood to be a finite group. Furthermore, an FG-module is always supposed to be a right FG-module of finite F-dimension. Unless otherwise stated, given an FG-module M, we let both the group algebra FG and the endomorphism algebra $\mathrm{End}_{FG}(M)$ act on M from the right. If M is an FG-module and K is an extension field of F we denote the KG-module $K \otimes_F M$ by M^K. If M is indecomposable such that also M^K is indecomposable, for all extension fields K of F, then we call M *absolutely indecomposable*.

An extensive introduction to the representation theory of the symmetric groups can be found in [19] and [20] where we also take our notation from. As usual, we denote the simple $F\mathfrak{S}_n$-module corresponding to the p-regular partition λ of n by D^λ, and the $F\mathfrak{S}_n$-Specht module corresponding to some partition μ of n by S^μ. Since Specht modules can be defined over any field, we write S^μ_F, whenever we need to distinguish the coefficient field we work over explicitly.

Furthermore, by P_n and Q_n we always denote a Sylow p-subgroup of the symmetric group $\mathfrak{S}_n$ and the alternating group $\mathfrak{A}_n$, respectively. By [20], 4.1.22 and 4.1.24 the Sylow p-subgroups of $\mathfrak{S}_n$ are of the following shape: let $C_p := \langle (1, \ldots, p) \rangle$ denote the cyclic group of order p, set $P_1 := 1$, $P_p := C_p$, and $P_{p^i} := P_{p^{i-1}} \wr C_p$ for $i \geq 2$. When viewed as a subgroup of $\mathfrak{S}_{p^i}$ in the natural way, P_{p^i} is a Sylow p-subgroup of $\mathfrak{S}_{p^i}$. Moreover, consider the p-adic expansion $n = \sum_{i=0}^r \alpha_i p^i$ of n. Then the Sylow p-subgroups of $\mathfrak{S}_n$ are precisely the $\mathfrak{S}_n$-conjugates of the Sylow p-subgroups of the Young subgroup $\mathfrak{S}_p^{\alpha_1} \times \cdots \times \mathfrak{S}_{p^r}^{\alpha_r}$. For the sake of simplicity, in the following we will then just write $P_n = \prod_{i=1}^r (P_{p^i})^{\alpha_i}$.

The present paper is structured as follows. In the first section we briefly summarize the basic facts about vertices of indecomposable FG-modules to provide the theoretical background needed for our computations. Moreover, we prove a result on vertices of simple FG-modules which involves the theory of twisted group algebras. Afterwards, in Section 3, we present our algorithmic methods. To give an impression of how these methods actually apply in practice, we illustrate each by a concrete example. The tables in the appendix of this article display our results on vertices and sources of simple modules for the symmetric groups.

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2 Determining vertices - theoretically

2.1 Known results

In this section we summarize the basic facts about vertices of indecomposable and, in particular, simple FG-modules, for a group G. We begin by listing some of the well-known criteria for an FG-module being *relatively H-projective*, for some subgroup H of G. A proof can for instance be found in [7], Thm. 19.2.

Theorem 2.1. *Let M be an FG-module, and let $H \leq G$. Then the following are equivalent:*

(1) Whenever $\mathcal{E} : 0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is a short exact sequence of FG-modules such that the short exact sequence of FH-modules $0 \longrightarrow \mathrm{Res}_H^G(M) \longrightarrow \mathrm{Res}_H^G(N) \longrightarrow \mathrm{Res}_H^G(L) \longrightarrow 0$ splits then so does $\mathcal{E}$.

(2) We have $M | \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$.

*(3) **(Higman's criterion)** Given a transversal $\{g_1, \dots, g_n\}$ for the right cosets of H in G, there is an $f \in \mathrm{End}_{FH}(M)$ such that*

$$\mathrm{Tr}_H^G(f) := \sum_{i=1}^{n} f^{g_i} = \mathrm{id}_M .$$

Remark 2.2. (a) In the notation of the previous theorem, we have an F-linear map $\mathrm{Tr}_H^G : \mathrm{End}_{FH}(M) \longrightarrow \mathrm{End}_{FG}(M)$ which does not depend on the choice of the transversal for $H\backslash G$. This map Tr_H^G is called *relative trace* from H to G. An FG-module satisfying the conditions of the theorem is called *relatively H-projective.*

(b) Suppose now that M is an indecomposable FG-module. Then a subgroup P of G which is minimal subject to the condition that M is relatively P-projective is called a *vertex* of M. The vertices of M form a G-conjugacy class of p-subgroups of G. Given a fixed vertex P of M, there are a Sylow p-subgroup S of G and a defect group R of the block containing M such that $P \leq R \leq S$ and $|S : P| \mid \dim(M)$. Moreover, there exists an indecomposable FP-module L with vertex P such that $M | \mathrm{Ind}_P^G(L)$. Then L is called a *source* of M, and L is determined up to isomorphism and $N_G(P)$-conjugacy.

When it comes to simple FG-modules and their vertices, the following theorems are among the most important results:

Theorem 2.3 (Erdmann [12])**.** *Let D be a simple FG-module with cyclic vertex P. Then P is a defect group of the block of FG containing D.*

Theorem 2.4 (Knörr [23]). *Suppose that F is algebraically closed, let D be a simple FG-module with vertex P, and let B be the block of FG containing D. Then there exists a block b of $F[PC_G(P)]$ with defect group P such that $b^G = B$. Hence $C_R(P) \leq P \leq R$, for some defect group R of B.*

As a direct consequence of Knörr's Theorem, simple FG-modules belonging to blocks with abelian defect groups possess precisely those defect groups as vertices. As we are particularly interested in the case where $G = \mathfrak{S}_n$, we mention the result below which follows immediately from Knörr's Theorem and a result by Olsson (cf. [30], Prop. 1.4). For this we recall that the defect groups of any block B of $F\mathfrak{S}_n$ of weight w are precisely the $\mathfrak{S}_n$-conjugates of the Sylow p-subgroups of $\mathfrak{S}_{pw}$. A proof for this can, for instance, be found in [20], Thm. 6.2.45.

Lemma 2.5 ([9] L. 3.3, Cor. 3.4). *Let D be a simple $F\mathfrak{S}_n$-module belonging to a block B of weight w. Moreover, let $R \in \mathrm{Syl}_p(\mathfrak{S}_{pw})$ be a defect group of B, and let $P \leq R$ be a vertex of D. Then $C_{\mathfrak{S}_{pw}}(P) = Z(P)$. In particular, D is not relatively $\mathfrak{S}_{pw-1}$-projective.*

Remark 2.6. (a) As far as simple modules for the symmetric groups and their vertices are concerned, the modular branching rules due to A. Kleshchev (cf. [22]) and the Scopes equivalence (cf. [32]) are essential. Both concepts possess purely combinatorial descriptions, and they sometimes allow us to relate simple $F\mathfrak{S}_n$-modules to simple $F\mathfrak{S}_m$-modules, for certain $m < n$, such that this relation is vertex and source preserving. Details on how branching rules and the Scopes equivalence have actually been applied to our examples can for instance be found in [8] and [35] and will also appear in [10].

(b) Knörr's Theorem as well as some further results presented in the next section require an algebraically closed field. However, when it comes to explicit computations we will of course have to work over finite fields. To ensure that we draw correct conclusions from the computational data, two well-known facts shall be mentioned at this point. Firstly, consider an indecomposable FG-module M with vertex P, and let K be any extension field of F. Then P is a vertex of each indecomposable direct summand of the KG-module M^K. A proof for this can be found in [13], L. III. 4.14.
Secondly, by [20], Thm. 2.1.12, every field is a splitting field for the symmetric group $\mathfrak{S}_n$, so that it is indeed safe to construct a simple module for $\mathfrak{S}_n$ over a finite field $\mathbb{F}_q$, for some p-power q, (usually even the prime field $\mathbb{F}_p$) and then compute its vertices. When regarded as $F\mathfrak{S}_n$-module for an extension field F of $\mathbb{F}_q$, this module is still simple, and its vertices remain unchanged.

(c) In this context, we might also have to decide whether an indecomposable $\mathbb{F}_qG$-module is absolutely indecomposable. To settle this problem, we will make use of the following fact which is a direct consequence of Wedderburn's Theorem.

Proposition 2.7. *Let M be an FG-module and set $E := \mathrm{End}_{FG}(M)$. Suppose that the E-module M, up to isomorphism, possesses exactly one composition factor D , and suppose further that $\dim_F(D) = 1$. Then the FG-module M is absolutely indecomposable.*

2.2 Twisted group algebras

In the following, we will present a result on vertices of simple FG-modules which involves the theory of twisted group algebras, and which will essentially be derived from part of the results and proofs given by Knörr in [23]. First of all we briefly collect the necessary notation. For convenience, in this subsection we assume the field F to be algebraically closed. Moreover, given an FG-module M, in this subsection we regard M as a left E-module where $E := \mathrm{End}_{FG}(M)$. For a positive integer n we denote the highest p-power dividing n by n_p.

Remark 2.8. (a) If M is an FG-module with $E := \mathrm{End}_{FG}(M)$ then it becomes an E-FG-bimodule in the obvious way. Moreover, there is a bijection between the set of isomorphism classes of indecomposable projective E-modules and the set of isomorphism classes of indecomposable direct summands of the FG-module M. This bijection is called *Fitting correspondence*, and is obtained via

$$eE \longmapsto eEM = eM = e(M),$$

for any primitive idempotent e in E.

(b) Let Q be a subgroup of G, and let W be an indecomposable FQ-module with inertial group $I := \{g \in N_G(Q) | W^g \cong W\}$ in $N_G(Q)$. Setting $E := \mathrm{End}_{FI}(\mathrm{Ind}_Q^I(W))$ and $\overline{I} := I/Q$, we deduce that E is an $\overline{I}$-graded F-algebra

$$E = \bigoplus_{\bar{x} \in \overline{I}} E_{\bar{x}},$$

where the components are defined as $E_{\bar{x}} = \{f \in E | f(W \otimes 1) \subseteq W \otimes x\}$, for $x \in I$ and $\bar{x} = xQ \in \overline{I}$. Furthermore, for $x \in I$ there is an isomorphism of F-vector spaces $E_{\bar{x}} \cong \mathrm{Hom}_{FQ}(W, W^x) \cong \mathrm{Hom}_{FQ}(W \otimes 1, W \otimes x)$, by [29], L. 4.6.4. Thus, in particular, $E_1 \cong \mathrm{End}_{FQ}(W)$ with $1_E \in E_1$, and this is an isomorphism of F-algebras.

In fact, E is a crossed product of $\overline{I}$ with E_1, i.e. for each $\bar{x} \in \overline{I}$, there is some $u_{\bar{x}} \in \mathcal{U}(E) \cap E_{\bar{x}}$. Moreover, $E_{\bar{x}} = u_{\bar{x}} E_1 = E_1 u_{\bar{x}}$, for $\bar{x} \in \overline{I}$. In particular, $\dim(E_{\bar{x}}) = \dim(E_1)$, for $\bar{x} \in \overline{I}$, and E is thus a free E_1-module of rank $|\overline{I}|$.

Finally, consider the Jacobson radical $\mathbf{J}(E_1)$ of E_1. Given $\bar{x} \in \overline{I}$, the respective unit $u_{\bar{x}}$ yields an automorphism

$$E_1 \longrightarrow E_1,\ a \longmapsto u_{\bar{x}} a u_{\bar{x}}^{-1}.$$

Consequently, $u_{\bar{x}} \mathbf{J}(E_1) = \mathbf{J}(E_1) u_{\bar{x}}$, and $J := \mathbf{J}(E_1)E = E\mathbf{J}(E_1)$ is a nilpotent ideal in E, since $E_1 u_{\bar{x}} = u_{\bar{x}} E_1 = E_{\bar{x}}$. Furthermore, we have

$$J = \bigoplus_{\bar{x} \in \overline{I}} \mathbf{J}(E_1) u_{\bar{x}} \subseteq \bigoplus_{\bar{x} \in \overline{I}} (J \cap E_{\bar{x}}) \subseteq J,$$

and thus also

$$\overline{E} := E/J = \bigoplus_{\bar{x} \in \overline{I}} \overline{E_{\bar{x}}}$$

is an $\overline{I}$-graded F-algebra where $\overline{E_{\bar{x}}} = (E_{\bar{x}} + J)/J \cong E_{\bar{x}}/E_{\bar{x}}\mathbf{J}(E_1)$, for $\bar{x} \in \overline{I}$. Since E is a crossed product so is $\overline{E}$. Since W is indecomposable, E_1 is a local F-algebra, and hence $\overline{E_1} \cong E_1/\mathbf{J}(E_1) \cong F$. This shows that $\dim(\overline{E_1}) = \dim(\overline{E_{\bar{x}}}) = 1$, for $\bar{x} \in \overline{I}$, and $\overline{E}$ is thus a twisted group algebra.

The assertion of the next lemma follows immediately from the proof of [23], Prop. 4.2.

Lemma 2.9. *In the notation of the previous remark, let N be a simple projective $\overline{E}$-module. Then we have $|\overline{I}|_p \,\big|\, \dim(N) \,\big|\, |\overline{I}|$.*

Theorem 2.10. *Let D be a simple FG-module with vertex Q and source W. Furthermore, let V be the Green correspondent of D in $H := N_G(Q)$. Then the following holds:*

$$|H:Q|_p \,\left|\, \frac{\dim(V)}{\dim(W)} \,\right|\, |H:Q|.$$

Proof. Firstly, by [18], Thm. VII.9.3 we have

$$\mathrm{Res}_Q^H(V) \cong k(W_1 \oplus \cdots \oplus W_m).$$

Here k is a positive integer, and $W = W_1, \ldots, W_m$ are pairwise non-isomorphic indecomposable FQ-modules such that W and W_i are conjugate in H, for $i = 1, \ldots, m$. In analogy to Remark 2.8, we denote the inertial group of W in H by I. Then $m = |H:I|$.

Since W is a source of V, we have $V | \mathrm{Ind}_Q^H(W)$. More precisely, by [18], Thm. VII.9.6, there exists an indecomposable direct summand U of $\mathrm{Ind}_Q^I(W)$ such that $V \cong \mathrm{Ind}_I^H(U)$. Thus, in particular, $\dim(U) = k\dim(W)$. We now set $E := \mathrm{End}_{FI}(\mathrm{Ind}_Q^I(W))$. Via Fitting correspondence, U, up to isomorphism, then corresponds to a unique indecomposable projective E-module eE. Here e is a primitive idempotent in E, and U is isomorphic to $e\,\mathrm{Ind}_Q^I(W)$ as FI-modules.

For $\overline{I} := I/Q$, in the notation of the above remark, E is an $\overline{I}$-graded F-algebra of F-dimension $|\overline{I}|\dim(E_1)$, and $\overline{E} := E/\mathbf{J}(E_1)E$ is a twisted group algebra of F-dimension $|\overline{I}|$. Furthermore, the $\overline{E}$-module $\overline{eE} := eE/eE\mathbf{J}(E_1)$ is also indecomposable and projective. Since V is actually the Green correspondent of a simple FG-module, the proofs of [23], Prop. 3.1, Thm. 3.3 yield that the $\overline{E}$-module $\overline{eE}$ is simple. Consequently, $|\overline{I}|_p \,\big|\, \dim(\overline{eE}) \,\big|\, |\overline{I}|$, by Lemma 2.9. Since $U \cong e\,\mathrm{Ind}_Q^I(W)$, we may now apply [23], Prop. 1.1, Cor. 1.2, to see that

$$k\dim(W) = \dim(U) = \mathrm{rk}_{E_1}(eE)\dim(W) = \mathrm{rk}_{\overline{E_1}}(\overline{eE})\dim(W).$$

Since E_1 is a local F-algebra, k therefore equals the F-dimension of the simple $\overline{E}$-module $\overline{eE}$. Hence $|I:Q|_p \,\big|\, k \,\big|\, |I:Q|$, and multiplication with $|H:I|$ finally yields the assertion

$$|H:Q|_p \,\left|\, k|H:I| = \frac{\dim(V)}{\dim(W)} \,\right|\, |H:Q|.$$

□

Corollary 2.11. *In the notation of the previous theorem, let $Q \leq H_1 \leq H$ such that $|H_1 : Q| \not\equiv 0 \pmod p$. Moreover, let $\mathrm{Res}_{H_1}^H(V)$ be an indecomposable FH_1-module. Then Q is a Sylow p-subgroup of G.*

Proof. Retaining the notation from the above proof, we again have

$$\mathrm{Res}_Q^{H_1}(\mathrm{Res}_{H_1}^H(V)) = \mathrm{Res}_Q^H(V) \cong k(W_1 \oplus \cdots \oplus W_m),$$

and since $\mathrm{Res}_{H_1}^H(V)$ is indecomposable and relatively Q-projective, the modules $W_1, \ldots, W_m$ are actually conjugate in H_1, by [18], Thm. VII.9.3. Moreover, $m = |H_1 : L|$ where L denotes the inertial group of W in H_1. We now set $E := \mathrm{End}_{FL}(\mathrm{Ind}_Q^L(W))$ which is an L/Q-graded F-algebra, and we further set $\overline{E} := E/\mathbf{J}(E_1)E$ where E_1 is the 1-component of E in the sense of Remark 2.8. As in the proof of the previous theorem, we then obtain that k is equal to the F-dimension of an indecomposable projective module for the twisted group algebra $\overline{E}$. By assumption, $p \nmid |H_1 : Q|$ so that $p \nmid |L : Q|$ either, and $\overline{E}$ is thus semisimple, by [29], Thm. 2.8.17. Hence k is equal to the F-dimension of a simple projective $\overline{E}$-module, and Lemma 2.9 yields $k \,\big|\, |L : Q|$. Consequently, neither $|L : Q|$ nor $|H_1 : L|k = \dim(V)/\dim(W)$ is divisible by p so that $|H : Q|$ is not divisible by p, by Theorem 2.10. Thus Q is a Sylow p-subgroup of $H = N_G(Q)$ and therefore a Sylow p-subgroup of G as well. □

Example 2.12. Consider the simple $\mathbb{F}_3\mathfrak{S}_{15}$-module $D := D^{(9,3^2)}$ of dimension 1716. Then D is contained in the principal block of $\mathbb{F}_3\mathfrak{S}_{15}$, and is isomorphic to the 6th exterior power $\bigwedge^6 D^{(14,1)}$ of the natural simple $\mathbb{F}_3\mathfrak{S}_{15}$-module. Furthermore, $\mathrm{Res}_{\mathfrak{S}_{14}}^{\mathfrak{S}_{15}}(D)$ is isomorphic to the simple Specht module $S^{(8,1^6)}$ which has vertex $(P_3)^4$, by [34], Thm. 2. Now let $P \leq P_{15} = P_9 \times (P_3)^2$ be a vertex of D. By Lemma 2.5, we know that $Z(P) = C_{\mathfrak{S}_{15}}(P)$, and so either $P = P_{15}$ or $P = (P_3)^5$, since these are the only subgroups of P_{15} whose centralizers in $\mathfrak{S}_{15}$ are equal to $Z(P)$ and which contain a subgroup conjugate to $(P_3)^4$. We assume that the latter holds true. Then $\mathfrak{S}_{15}$ possesses a subgroup H_1 isomorphic to $\mathfrak{S}_3 \wr C_5$ such that $P \trianglelefteq H_1$ and $|H_1 : P| = 2^5 \cdot 5 \not\equiv 0 \pmod 3$. Our computations show that $\mathrm{Res}_{H_1}^{\mathfrak{S}_{15}}(D)$ is absolutely indecomposable. Thus $\mathrm{Res}_{H_1}^{\mathfrak{S}_{15}}(D)$ is also indecomposable when regarding D as $\overline{\mathbb{F}_3}\mathfrak{S}_{15}$-module. Hence $(P_3)^5$ has to be a Sylow 3-subgroup of $\mathfrak{S}_{15}$, by Corollary 2.11, a contradiction. Consequently, D has vertex P_{15}.

3 Determining vertices - algorithmically

In the following, we will explain our algorithmic approaches towards the computation of vertices of indecomposable and simple FG-modules. In order to compute a vertex of an indecomposable FG-module M, the basic idea is to restrict M to some subgroup H of G such that M is relatively H-projective, determine the indecomposable direct summands of $\mathrm{Res}_H^G(M)$ and compute a vertex of each of these summands. The maximal among these vertices are then also vertices of M. Mostly, H will be a Sylow p-subgroup of G or a defect group of the block B containing M or, in case $G = \mathfrak{S}_n$, some Young subgroup of $\mathfrak{S}_n$ containing a defect group of the relevant block B. However, as the dimensions of the modules under consideration grow and as the group H is mostly a p-group, the computation of direct summands requires both a lot of time and memory. The methods presented in this section are basically used to shrink the dimensions of the modules one actually has to work with.

3.1 Testing for relative projectivity

Given a subgroup H of G, among the criteria for an FG-module being relatively H-projective given in Theorem 2.1, Higman's criterion turns out to be most efficient in practice. One of the reasons for this is the fact that the relative trace map carries a transitivity property in the following sense:

Proposition 3.1. *Let M be an FG-module and $K \leq H \leq G$. Then we have* $\mathrm{Tr}_K^G = \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H$.

We now assume the FG-module M to be indecomposable with $d := \dim(M)$, and fix a subgroup H of G. Since $\mathrm{Tr}_H^G(\mathrm{End}_{FH}(M))$ is an ideal of the local F-algebra $E := \mathrm{End}_{FG}(M)$, the module M is relatively H-projective if and only if $\mathrm{Tr}_H^G(\mathrm{End}_{FH}(M)) \not\subseteq \mathbf{J}(E)$. This in turn is the case if and only if $\mathrm{rk}(\mathrm{Tr}_H^G(f_i)) = d$, for some $i \in \{1, \ldots, m\}$, where $\{f_1, \ldots, f_m\}$ denotes an F-basis of $\mathrm{End}_{FH}(M)$.

One last thing to mention at this point is the following: the naive computation of $\mathrm{Tr}_H^G(f)$ for a fixed $f \in \mathrm{End}_{FH}(M)$ requires a time proportional to the index $|G : H|$. Here one can achieve a speed up by taking an appropriate chain of subgroups $H = H_0 < H_1 < \ldots < H_k = G$ and successively compute $\mathrm{Tr}_{H_{k-1}}^{H_k} \circ \cdots \circ \mathrm{Tr}_{H_0}^{H_1}$. The required time is then only proportional to $|H_k : H_{k-1}| + \cdots + |H_1 : H_0|$ instead. Moreover, often it happens to be the case that $\mathrm{Tr}_H^{H_j}(f) = 0$ for some $j < k$, so that the computation of $\mathrm{Tr}_H^G(f)$ can then be stopped at that point.

The MAGMA system provides the function `EndomorphismAlgebra` which we used to compute the endomorphism algebra of an FG-module.

3.2 Stripping off projective direct summands

Remark 3.2. Consider a p-group P and an indecomposable FG-module M with vertex $Q \leq P$. Among the indecomposable direct summands of the restriction $N := \mathrm{Res}_P^G(M)$ we can in principle ignore the projective ones. The algorithm below is probably well-known, and is a very useful tool for concrete vertex computations. It is applied in order to strip indecomposable projective direct summands off N, without computing a direct sum decomposition of N. That algorithm is randomized and based on the following fact: suppose that we have an $x \in N$ such that $\dim(xFP) = |P| = \dim(FP)$. Then xFP is a projective submodule of N, and since a projective FP-module is also injective, it is even a direct summand of N. Conversely, if N has any indecomposable projective submodule L then there also is an element $x \in N$ generating L.

The algorithm "ProjSummands" below is a Monte Carlo algorithm in the sense that it might not detect all projective indecomposable direct summands of the input module N. In the following, we estimate the probability of success of the algorithm.

Proposition 3.3. *Suppose that the FP-module N possesses exactly l indecomposable projective direct summands, for some $l \in \mathbb{N}$. After $b \in \mathbb{N}$ random trials, a generator $x \in N$ for an indecomposable projective direct summand of N is then found with probability at least*

$$1 - \left(\frac{1}{|F|^l}\right)^b \geq 1 - \left(\frac{1}{2^b}\right).$$

Proof. Consider a fixed decomposition

$$N = N_1 \oplus \cdots \oplus N_l \oplus Y,$$

where $N_1, \ldots, N_l$ are indecomposable projective, and Y is projective-free. For $i \in \{1, \ldots, l\}$, each vector in $N_i \setminus \mathrm{Rad}(N_i)$ is a generator for N_i. Let now $x := \sum_{i=1}^{l} x_i + y \in N$ such that $x_i \in N_i$, for $i = 1, \ldots, l$, and $y \in Y$. Suppose first that $x \notin \mathrm{Rad}(N_1) \oplus \cdots \oplus \mathrm{Rad}(N_l) \oplus Y$.

Then $x_j \notin \mathrm{Rad}(N_j)$, for some $j \in \{1, \dots, l\}$ so that $x_j FP = N_j$, and we thus obtain an FP-epimorphism

$$xFP \longrightarrow x_j FP = N_j \cong FP.$$

On the other hand, xFP is clearly an epimorphic image of FP so that $xFP \cong FP$.

Conversely, suppose that $x \in \mathrm{Rad}(N_1) \oplus \cdots \oplus \mathrm{Rad}(N_l) \oplus Y$. If $xFP \cong FP$ then xFP would be a projective submodule and thus a projective direct summand of $\mathrm{Rad}(N_1) \oplus \cdots \oplus \mathrm{Rad}(N_l) \oplus Y$ which is impossible.

Therefore, a randomly chosen vector $x \in N$ is a generator for some indecomposable projective direct summand of N if and only if it is contained in $N \setminus (\mathrm{Rad}(N_1) \oplus \cdots \oplus \mathrm{Rad}(N_l) \oplus Y)$. In other words, $x \in N$ generates an indecomposable projective direct summand of N with probability

$$\frac{|F|^{\dim(Y)+l|P|} - |F|^{\dim(Y)+l|P|-l}}{|F|^{\dim(Y)+l|P|}} = 1 - \frac{1}{|F|^l}.$$

Hence the probability of finding a generating vector for an indecomposable projective direct summand of N after $b \in \mathbb{N}$ trials is

$$1 - \left(\frac{1}{|F|^l}\right)^b \geq 1 - \frac{1}{2^b}.$$

This proves the proposition. □

Remark 3.4. In all our explicit computations we have actually chosen $b := 10$ such that a generating vector for an indecomposable projective direct summand of N, if existing, is then found by the algorithm "ProjSummands" with probability greater than 99.9%.

Input: a p-group P, an FP-module N having indecomposable projective direct summands $N_1, \dots, N_l$ for some $l \in \mathbb{N}_0$, and an integer $b \geq 1$
Output: an FP-module isomorphic to $N/(N_1 \oplus \cdots \oplus N_k)$ for some $k \leq l$

$i \leftarrow 0$
while $i < b$ **do**
  choose $x \in N$ randomly, and compute $xFP \leq N$
  if $\dim(xFP) = |P|$ **then** $\{xFP \cong FP\}$
    $N \leftarrow N/(xFP)$
    $i \leftarrow 0$
  else
    $i \leftarrow i + 1$
  end if
end while
return N

Algorithm 1: ProjSummands

Example 3.5. (a) Consider the simple $\mathbb{F}_2\mathfrak{S}_{13}$-module $D := D^{(5,4,3,1)}$ of dimension 8448 which is contained in the principal block of $\mathbb{F}_2\mathfrak{S}_{13}$. Moreover, D is relatively $\mathfrak{A}_{12}$-projective, since $(5,4,3,1)$ is an S-partition in the sense of [1]. The algorithm "ProjSummands" detects all of the seven indecomposable projective direct summands of $\mathrm{Res}^{\mathfrak{S}_{13}}_{P_{12}}(D)$. The remaining quotient module N has dimension 1280, and its indecomposable direct summands could then be determined by applying the MAGMA function `IndecomposableSummands` to N directly. It turned out that N has exactly two indecomposable direct summands, both of which are actually absolutely indecomposable. One of them has dimenion 1024 and vertex Q_{12}. Hence also D has vertex Q_{12}.

(b) As a second example we consider the simple $\mathbb{F}_2\mathfrak{S}_{14}$-module $D := D^{(5,4,3,2)}$ of dimension 35840 which is contained in the block with 2-core $(3,2,1)$ and 2-weight 4. By [15], Cor. 3.21, we have $D \cong D^{(8,6)} \otimes D^{(9,5)}$. Both of the tensor factors belong to the principal block of $\mathbb{F}_2\mathfrak{S}_{14}$ and have vertex P_{14}. Restricting both of them to $\mathfrak{S}_8$ first, we obtain $\mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_8}(D^{(8,6)}) \cong 4N_1$ and $\mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_8}(D^{(9,5)}) \cong 6N_2 \oplus 4N_3$ where N_1, N_2, N_3 are indecomposable with $\dim(N_1) = 16$, $\dim(N_2) = 48$ and $\dim(N_3) = 68$. Furthermore, application of "ProjSummands" shows that $\mathrm{Res}^{\mathfrak{S}_8}_{P_8}(N_1 \otimes N_2)$ is projective, and $\mathrm{Res}^{\mathfrak{S}_8}_{P_8}(N_1 \otimes N_3)$ has seven indecomposable projective direct summands. Denote the remaining quotient module by N. Then N has dimension 192, is indecomposable over $\mathbb{F}_2$, and has vertex E_8 where E_8 denotes an elementary abelian group of order 8 acting regularly on the set $\{1, \ldots, 8\}$. From this we deduce that D has vertex E_8.

3.3 Stripping off indecomposable direct summands with cyclic vertices

As in the previous subsection, let P be a p-subgroup of the group G such that the indecomposable FG-module M has a vertex $Q \leq P$. Again set $N := \mathrm{Res}^G_P(M)$. Provided that the FG-module M is simple and the defect groups of the block containing M are non-cyclic, also the vertices of M cannot be cyclic, by Theorem 2.3. For $G = \mathfrak{S}_n$ this happens to be the case whenever M belongs to a block of weight at least 2. For that reason, we now develop an algorithm which strips indecomposable direct summands with cyclic vertices off a given FP-module. This will be a Monte Carlo algorithm in that it might not detect all indecomposable direct summands of the input module N which have cyclic vertices. For that reason, we will also give an estimate of its probability of success.

First of all, we recall the following:

Remark 3.6. (a) Consider a cyclic group $C = \langle c \rangle$ of order $q := p^n$, for some $n \in \mathbb{N}_0$. Then there are exactly q isomorphism classes of indecomposable FC-modules. More precisely, for each $i \in \{1, \ldots, q\}$, up to isomorphism, $T_i := FC/\mathbf{J}(FC)^i$ is the uniquely determined indecomposable FC-module of dimension i. Moreover, T_i affords a matrix representation Δ_i of C over F such that $\Delta_i(c)$ equals a Jordan block of size i, for $i \in \{1, \ldots, q\}$. A proof for this can, for instance, be found in [18], Thm. VII.5.3.

(b) Suppose, we are given a short exact sequence of FP-modules

$$\mathcal{E} : 0 \longrightarrow L \xrightarrow{\iota} N \xrightarrow{\nu} N/L \longrightarrow 0,$$

where L is a submodule of N. Furthermore, ι and ν denote the inclusion map and the canonical epimorphism, respectively. Then the sequence $\mathcal{E}$ splits if and only if $N \cong L \oplus N/L$, by [4], L. 6.12.

Keeping this notation, we get:

Proposition 3.7. *Let $C = \langle c \rangle$ be a subgroup of P of order q. Furthermore, let $x \in N$ and $s \in \{1, \dots, q\}$ such that $x(c-1)^s = 0$. Then the following hold:*

(i) There is an FC-epimorphism $\psi : T_s \longrightarrow xFC$.

(ii) The map $\varphi : \mathrm{Ind}_C^P(T_s) \longrightarrow xFP$, $a \otimes g \longmapsto a\psi \cdot g$, where $a \in T_s$ and $g \in P$, is an FP-epimorphism. In particular, $\dim(xFP) \leq s|P : C|$.

(iii) The map φ is an FP-isomorphism if and only if $\dim(xFP) = s|P : C|$.

(iv) Suppose that $\dim(xFP) = s|P : C|$. Then xFP is a direct summand of N if and only if

$$\dim(N(c-1)^i) = \dim(xFP(c-1)^i) + \dim((N/xFP)(c-1)^i),$$

for all $i \in \{1, \dots, q\}$.

Proof. Assertions (i)–(iii) are obvious, by what we have mentioned in Remark 3.6. In order to prove (iv), suppose that $\dim(xFP) = s|P : C|$ so that $xFP \cong \mathrm{Ind}_C^P(xFC) \cong \mathrm{Ind}_C^P(T_s)$, by (iii). In particular, xFP is then absolutely indecomposable, and is relatively C-projective. Now xFP is a direct summand of N if and only if the short exact sequence of FP-modules

$$\mathcal{E} : 0 \longrightarrow xFP \xrightarrow{\iota} N \xrightarrow{\nu} N/(xFP) \longrightarrow 0$$

splits. Here, ι and ν again denote the inclusion map and the canonical epimorphism, respectively. Since xFP is relatively C-projective, the sequence $\mathcal{E}$ splits if and only if the corresponding short exact sequence of FC-modules

$$\mathcal{E}' : 0 \longrightarrow \mathrm{Res}_C^P(xFP) \longrightarrow \mathrm{Res}_C^P(N) \longrightarrow \mathrm{Res}_C^P(N/(xFP)) \longrightarrow 0$$

splits, by Theorem 2.1. As mentioned in Remark 3.6, this in turn holds if and only if $\mathrm{Res}_C^P(N) \cong \mathrm{Res}_C^P(xFP) \oplus \mathrm{Res}_C^P(N/(xFP))$, by [4], L. 6.12. If Δ and Γ are matrix representations of P over F afforded by N and $xFP \oplus N/(xFP)$, respectively, then $\mathrm{Res}_C^P(N) \cong \mathrm{Res}_C^P(xFP) \oplus \mathrm{Res}_C^P(N/(xFP))$ if and only if $\Delta(c)$ and $\Gamma(c)$ have the same Jordan canonical form which is equivalent to

$$\dim(N(c-1)^i) = \dim(xFP(c-1)^i) + \dim((N/xFP)(c-1)^i),$$

for $i = 1, \dots, q$. This finally proves the proposition. □

Proposition 3.8. *Let $C = \langle c \rangle$ be a subgroup of P of order q, and let $s \in \{1, \dots, q\}$. We set $N(s) := \{x \in N | x(c-1)^s = 0\}$. Let further $N = N_1 \oplus N_2 \oplus \cdots \oplus N_l \oplus Y$ be a fixed decomposition such that $N_j \cong \mathrm{Ind}_C^P(T_s)$, for $j = 1, \dots, l$, and Y has no indecomposable direct summand isomorphic to $\mathrm{Ind}_C^P(T_s)$.*

(i) There exists an $x_j \in N_j$ such that $x_j(c-1)^s = 0 \neq x_j(c-1)^{s-1}$ and $x_jFP = N_j$, for all $j \in \{1, \dots, l\}$. In particular, $x_j \notin \mathrm{Rad}(N_j)$, for all $j \in \{1, \dots, l\}$.

(ii) If $x \in N \setminus (\mathrm{Rad}(N_1) \oplus \cdots \oplus \mathrm{Rad}(N_l) \oplus Y)$ such that $x(c-1)^s = 0$ then xFP is a direct summand of N isomorphic to $\mathrm{Ind}_C^P(T_s)$.

(iii) A randomly chosen vector $x \in N(s)$ generates an indecomposable direct summand of N isomorphic to $\mathrm{Ind}_C^P(T_s)$ with probability at least $1 - 1/(|F|^l)$.

Proof. In order to prove (i), let $j \in \{1, \dots, l\}$. Since $N_j \cong \mathrm{Ind}_C^P(T_s)$, we may choose an FP-isomorphism $\alpha : \mathrm{Ind}_C^P(T_s) \longrightarrow N_j$, and set $x_j := ((1 + \mathbf{J}(FC)^s) \otimes 1)\alpha$.

Next, let $x \in N \setminus (\mathrm{Rad}(N_1) \oplus \cdots \oplus \mathrm{Rad}(N_l) \oplus Y)$ with $x(c-1)^s = 0$. Then $x = \sum_{j=1}^l x_j + y$, for appropriate $x_1 \in N_1, \dots, x_l \in N_l$, and $y \in Y$. Moreover, $x_k \notin \mathrm{Rad}(N_k)$, for some $k \in \{1, \dots, l\}$. Since N_k is a cyclic FP-module, this particularly implies that $N_k = x_k FP$.

Consequently, denoting the projection from N onto N_k by π_k, we obtain an FP-epimorphism

$$\pi_k | xFP : xFP \longrightarrow N_k.$$

Thus $\dim(xFP) \geq \dim(N_k) = \dim(\mathrm{Ind}_C^P(T_s)) = s|P : C|$. On the other hand, $\dim(xFP) \leq s|P : C|$, by Proposition 3.7 (ii). Consequently, $\pi_k|xFP : xFP \longrightarrow N_k$ is an FP-isomorphism. Let $\iota : xFP \longrightarrow N$ be the inclusion map, and let $\nu : N \longrightarrow N/(xFP)$ be the canonical epimorphism so that we obtain the short exact sequence of FP-modules

$$\mathcal{E} : 0 \longrightarrow xFP \xrightarrow{\iota} N \xrightarrow{\nu} N/(xFP) \longrightarrow 0.$$

Furthermore, we set $\rho := \pi_k \cdot (\pi_k|xFP)^{-1} \in \mathrm{Hom}_{FP}(N, xFP)$. Then we have

$$(x\iota)\rho = x\rho = (x\pi_k)(\pi_k|xFP)^{-1} = (x_k)(\pi_k|xFP)^{-1} = x.$$

Thus $\iota \cdot \rho = \mathrm{id}_{xFP}$ so that the sequence $\mathcal{E}$ actually splits, and xFP is a direct summand of N.

It remains to prove (iii). According to the decomposition of N, we also have a decomposition of vector spaces

$$N(s) = N_1(s) \oplus \cdots \oplus N_l(s) \oplus Y(s).$$

By (ii), each vector in $N(s) \setminus (N_1(s) \cap \mathrm{Rad}(N_1)) \oplus \cdots \oplus (N_l(s) \cap \mathrm{Rad}(N_l)) \oplus Y(s))$ generates a direct summand of N isomorphic to $\mathrm{Ind}_C^P(T_s)$. Since, by (i), $N_j(s) \nsubseteq \mathrm{Rad}(N_j)$, and

$$N_j(s)/(N_j(s) \cap \mathrm{Rad}(N_j)) \cong (N_j(s) + \mathrm{Rad}(N_j))/\mathrm{Rad}(N_j) \subseteq N_j/\mathrm{Rad}(N_j) \cong F,$$

we obtain that $N_j(s) \cap \mathrm{Rad}(N_j)$ has codimension 1 in $N_j(s)$, for $j = 1, \dots, l$. That is $\dim((N_1(s) \cap \mathrm{Rad}(N_1)) \oplus \cdots \oplus (N_l(s) \cap \mathrm{Rad}(N_l)) \oplus Y(s)) = \dim(N(s)) - l$, and therefore a randomly chosen vector $x \in N(s)$ generates an indecomposable direct summand of N isomorphic to $\mathrm{Ind}_C^P(T_s)$ with probability at least

$$\frac{|F|^{\dim(N(s))} - |F|^{\dim(N(s)) - l}}{|F|^{\dim(N(s))}} = 1 - \frac{1}{|F|^l}.$$

□

Remark 3.9. (a) To summarize, suppose that $C = \langle c \rangle \leq P$ with $|C| = q$. If N possesses an indecomposable direct summand with vertex C then there exist some $s \in \{1, \dots, q\}$ and some $x \in N(s) \setminus N(s-1)$ generating that summand. Once having found a vector $x \in N(s)$ such that $\dim(xFP) = s|P : C|$, we know that $xFP \cong \mathrm{Ind}_C^P(T_s)$, by Proposition 3.7 (iii). Due to Proposition 3.7 (iv), the test whether xFP is actually a direct summand of N then reduces to the computation of the Jordan canonical forms of $\Delta(c)$ and $\Gamma(c)$ where Δ and Γ are the matrix representations of P over F afforded by N and $xFP \oplus N/(xFP)$, respectively. In this way, step 13 of the algorithm is carried out in practice.

(b) As already mentioned above, the algorithm "CyclicVertexSummands" may miss indecomposable direct summands of N with cyclic vertices. However, Proposition 3.8 shows that after $b \in \mathbb{N}$ random trials a generating vector for some indecomposable direct summand of N with cyclic vertex C, if existing, is found with probability at least

$$1 - \left(\frac{1}{|F|}\right)^b \geq 1 - \frac{1}{2^b}.$$

Again, in all applications we have chosen $b := 10$ so that the probability of success of the algorithm "CyclicVertexSummands" is then greater than 99.9%.

Input: a p-group P, an FP-module N with corresponding matrix representation Δ having $l \in \mathbb{N}_0$ indecomposable direct summands $N_1, \ldots, N_l$ with cyclic vertices, and an integer $b \geq 1$

Output: an FP-module isomorphic to $N/(N_1 \oplus \cdots \oplus N_k)$ for some $k \leq l$

1: choose a transversal $\{C_1, \ldots, C_m\}$ for the conjugacy classes of non-trivial cyclic subgroups of P
2: $j \leftarrow 1$
3: **while** $j \leq m$ **do**
4:    $C \leftarrow C_j$
5:    $q := |C|$
6:    **for** $q \geq s \geq 1$ **do**
7:       $i \leftarrow 0$
8:       **while** $i < b$ and $\dim(N) \geq s|P : C|$ **do**
9:          **if** $i = 0$ **then**
10:             $N(s) \leftarrow \ker(\Delta((c-1)^s))$
11:          **end if**
12:          choose $x \in N(s)$ randomly, and compute $xFP \leq N$
13:          **if** $\dim(xFP) = s|P : C|$ and $\mathrm{Res}_C^P(N) \cong \mathrm{Res}_C^P(xFP) \oplus \mathrm{Res}_C^P(N/xFP)$ **then**
14:             $N \leftarrow N/(xFP)$
15:             $\Delta \leftarrow$ matrix representation of G over F afforded by N
16:             $i \leftarrow 0$
17:          **else**
18:             $i \leftarrow i + 1$
19:          **end if**
20:       **end while**
21:    **end for**
22:    $j \leftarrow j + 1$
23: **end while**
24: **return** N

Algorithm 2: CyclicVertexSummands

Example 3.10. Consider the simple $\mathbb{F}_3\mathfrak{S}_{14}$-module $D := D^{(8,3,2,1)}$ of dimension 6369. Then D is contained in the principal block of $\mathbb{F}_3\mathfrak{S}_{14}$ whose defect groups are conjugate to P_{12}. Restricting D to $\mathfrak{S}_{12}$ we obtain

$$\mathrm{Res}_{\mathfrak{S}_{12}}^{\mathfrak{S}_{14}}(D) \cong 2D^{(7,2^2,1)} \oplus D^{(8,2,1^2)} \oplus M,$$

where M is indecomposable of dimension 1968. Since the simple direct summands belong to blocks with abelian defect group $(P_3)^2$, we can ignore them and investigate M in more detail. For this, set $N := \mathrm{Res}^{\mathfrak{S}_{12}}_{P_{12}}(M)$. Then the algorithms "ProjSummands" and "CyclicVertex-Summands" detect all of the 11 indecomposable direct summands with cyclic vertices, five of which are actually projective, and the remaining six have vertices of order 3. The remaining quotient module has dimension 267 and possesses two non-isomorphic indecomposable direct summands of dimension 12 both of which have vertex $(P_3)^4$. The other indecomposable direct summands have smaller vertices so that also D has vertex $(P_3)^4$.

3.4 Rank varieties of elementary abelian p-groups

Next we explain a method which can be used to decide whether, for an indecomposable FG-module M, a given elementary abelian subgroup E of G has to be contained in some vertex of M. For convenience, we now assume the field $F \supset \mathbb{F}_p$ to be algebraically closed. The results presented below require some further notation coming from group cohomology.

Remark 3.11. Consider an elementary abelian group $E := \langle g_1, \dots, g_n \rangle \leq G$ of order p^n, for some $n \in \mathbb{N}$.

(a) Given $\alpha = (\alpha_1, \dots, \alpha_n) \in F^n \setminus \{0\}$, we set

$$u_\alpha := 1 + \sum_{i=1}^{n} \alpha_i (g_i - 1) \in 1 + \mathbf{J}(FE) \subseteq \mathcal{U}(FE).$$

The cyclic group of order p generated by u_α is called a *shifted cyclic subgroup* of FE.

(b) For an FG-module M, its *rank variety* with respect to $g_1, \dots, g_n$ is defined as

$$V_E^r(M) := \{\alpha \in F^n \setminus \{0\} | \, \mathrm{Res}^G_{\langle u_\alpha \rangle}(M) \text{ is not free } \} \cup \{0\}.$$

This is a homogeneous affine F-variety, by [5], Thm. 9.3.1. Thus if $(\alpha_1, \dots, \alpha_n) \in F^n$ is contained in $V_E^r(M)$ then so is $\beta(\alpha_1, \dots, \alpha_n)$, for all $\beta \in F$.

Keeping the notation from the previous remark, we will make use of the following important result on rank varieties.

Theorem 3.12 ([5] Prop. 9.7.2, Thm. 9.6.4)**.** *Let N, N_1 and N_2 be FE-modules. Then $V_E^r(N_1 \oplus N_2) = V_E^r(N_1) \cup V_E^r(N_2)$ and $V_E^r(N_1 \otimes_F N_2) = V_E^r(N_1) \cap V_E^r(N_2)$.*

Lemma 3.13. *Let V be an $\mathbb{F}_p$-vector space of dimension $n \in \mathbb{N}$ with basis $\{b_1, \dots, b_n\}$, and let $\alpha_1, \dots, \alpha_n \in F$ be linearly independent over $\mathbb{F}_p$. Let further U be a proper $\mathbb{F}_p$-subspace of V. Then*

$$x := \alpha_1 \otimes b_1 + \cdots + \alpha_n \otimes b_n \notin F \otimes_{\mathbb{F}_p} U.$$

Proof. Suppose that U is a proper $\mathbb{F}_p$-subspace of V such that $F \otimes_{\mathbb{F}_p} U$ contains the element x defined above. Furthermore, let $\dim_{\mathbb{F}_p}(U) = m$, and consider an $\mathbb{F}_p$-basis $\{u_1, \dots, u_m\}$ of U. Then there are elements $\beta_1, \dots, \beta_m \in F$ such that $x = \beta_1 \otimes u_1 + \cdots + \beta_m \otimes u_m$. Moreover, we may write $u_i = \sum_{j=1}^{n} \gamma_{ij} b_j$ with $\gamma_{ij} \in \mathbb{F}_p$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. From this we get

$$\sum_{j=1}^{n} \alpha_j \otimes b_j = \sum_{i=1}^{m} \beta_i \otimes u_i = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_i \otimes \gamma_{ij} b_j = \sum_{j=1}^{n} \sum_{i=1}^{m} \gamma_{ij} \beta_i \otimes b_j,$$

and consequently $\alpha_j = \sum_{i=1}^{m} \gamma_{ij} \beta_i \in \sum_{i=1}^{m} \mathbb{F}_p \beta_i$, for $j \in \{1, \ldots, n\}$. Since $m < n$, the elements $\alpha_1, \ldots, \alpha_n$ thus have to be linearly dependent over $\mathbb{F}_p$, a contradiction to the hypothesis, and the assertion of the lemma follows. □

Theorem 3.14 (Benson [2]). *Let G be a group and M an indecomposable FG-module with vertex P. Moreover, let $E := \langle g_1, \ldots, g_n \rangle \leq G$ be an elementary abelian group of order p^n, for some $n \in \mathbb{N}$ such that $E \not\leq_G P$, and let $\alpha_1, \ldots, \alpha_n \in F$ be linearly independent over $\mathbb{F}_p$. Set $u := 1 + \sum_{i=1}^{n} \alpha_i (g_i - 1)$. Then $\mathrm{Res}^G_{\langle u \rangle}(M)$ is free, or equivalently, $\dim_F(M \cdot (u-1)^{p-1}) = \dim_F(M)/p$.*

Proof. Let L be an FP-module which is a source of M. Then $M | \mathrm{Ind}_P^G(L)$ and thus

$$\mathrm{Res}^G_E(M) | \mathrm{Res}^G_E(\mathrm{Ind}^G_P(L)) \cong \bigoplus_{PgE \in P\backslash G/E} \mathrm{Ind}^E_{E \cap g^{-1}Pg}(\mathrm{Res}^{g^{-1}Pg}_{E \cap g^{-1}Pg}(L^g)),$$

by Mackey's Decomposition Theorem. Since $E \not\leq_G P$, there exist maximal subgroups $E_1, \ldots, E_r$ of E and FE_i-modules N_i, for $i = 1, \ldots, r$, such that

$$\mathrm{Res}^G_E(M) = \bigoplus_{i=1}^{r} \mathrm{Ind}^E_{E_i}(N_i), \qquad (*)$$

for some $r \in \mathbb{N}$. The map

$$f : F \otimes_{\mathbb{F}_p} E \longrightarrow \mathbf{J}(FE)/\mathbf{J}(FE)^2,\ \beta \otimes g \longmapsto \beta(g-1) + \mathbf{J}(FE)^2,$$

is an isomorphism of F-vector spaces, and the previous lemma shows that

$$\alpha_1 \otimes g_1 + \cdots + \alpha_n \otimes g_n \notin F \otimes_{\mathbb{F}_p} U,$$

for every proper subgroup U of E. Hence, if $i \in \{1, \ldots, r\}$ and if $E_i = \langle x_1, \ldots, x_{n-1} \rangle$ is one of the maximal subgroups of E mentioned above, we obtain

$$(u-1) + \mathbf{J}(FE)^2 = \alpha_1(g_1 - 1) + \cdots + \alpha_n(g_n - 1) + \mathbf{J}(FE)^2 \notin \sum_{j=1}^{n-1} F[(x_j - 1) + \mathbf{J}(FE)^2].$$

Consequently, $\{(x_1 - 1) + \mathbf{J}(FE)^2, \ldots, (x_{n-1} - 1) + \mathbf{J}(FE)^2, (u-1) + \mathbf{J}(FE)^2\}$ is an F-basis for $\mathbf{J}(FE)/\mathbf{J}(FE)^2$ so that $FE = F\widetilde{E}$ where $\widetilde{E} := \langle x_1, \ldots, x_{n-1}, u \rangle$ is an elementary abelian subgroup of $1 + \mathbf{J}(FE)$. Thus, in particular, $|\widetilde{E}| = p^n$ and $\widetilde{E} = E_i \langle u \rangle$. From this we deduce

$$\mathrm{Ind}^E_{E_i}(N_i) = N_i \otimes_{FE_i} FE = N_i \otimes_{FE_i} F\widetilde{E} = \mathrm{Ind}^{\widetilde{E}}_{E_i}(N_i)$$

and

$$\mathrm{Res}^E_{\langle u \rangle}(\mathrm{Ind}^E_{E_i}(N_i)) = \mathrm{Res}^{\widetilde{E}}_{\langle u \rangle}(\mathrm{Ind}^{\widetilde{E}}_{E_i}(N_i)) \cong \mathrm{Ind}^{\langle u \rangle}_{\langle u \rangle \cap E_i}(\mathrm{Res}^{E_i}_{\langle u \rangle \cap E_i}(N_i)) = \mathrm{Ind}^{\langle u \rangle}_1(\mathrm{Res}^{E_i}_1(N_i)).$$

This is a free $F\langle u\rangle$-module, and thus also $\mathrm{Res}^G_{\langle u\rangle}(M)$ is free, by $(*)$. Hence we have shown that $\mathrm{Res}^G_{\langle u\rangle}(M) \cong t \cdot F\langle u\rangle$ with $t = \dim_F(M)/p$.

Finally, consider the F-linear map

$$\varphi : F\langle u\rangle \longrightarrow F\langle u\rangle,\ x \longmapsto x(u-1)^{p-1} = x\sum_{k=0}^{p-1} u^k,$$

with image $\mathbf{J}(F\langle u\rangle)^{p-1}$. This map has thus rank 1 and induces an F-linear map $t \cdot F\langle u\rangle \longrightarrow t \cdot F\langle u\rangle$ of rank t.

Note further that, given any $F\langle u\rangle$-module N, the map φ gives rise to an $F\langle u\rangle$-epimorphism

$$\varphi_N : N \longrightarrow \mathrm{Rad}(N)^{p-1},$$

which is the zero map whenever N is projective-free. This proves the theorem. □

Remark 3.15. (a) By the previous theorem, the test whether $\mathrm{Res}^G_{\langle u\rangle}(M)$ is free thus reduces to the computation of $\mathrm{rk}(\Delta((u-1)^{p-1}))$ where Δ is the matrix representation of G over F afforded by M. Moreover, in practice we have to replace F by a sufficiently large finite field.

(b) In the notation of Remark 3.11, the previous theorem shows that the elementary abelian group E has to be contained in some vertex of M if there exist elements $\alpha_1, \ldots, \alpha_n \in F$ which are linearly independent over $\mathbb{F}_p$ such that $\alpha = (\alpha_1, \ldots, \alpha_n) \in V^r_E(\mathrm{Res}^G_E(M))$.

Example 3.16. In the following, we consider the simple $\mathbb{F}_2\mathfrak{S}_{14}$-module $D := D^{(7,4,2,1)}$ which has dimension 19240 and is contained in the principal block of $\mathbb{F}_2\mathfrak{S}_{14}$. Determining its vertices requires two basic steps the first of which we explain now. As usual, given a partition $\lambda := (\lambda_1, \ldots, \lambda_s)$ of n, we denote the corresponding Young subgroup $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_s}$ of $\mathfrak{S}_n$ by $\mathfrak{S}_\lambda$.

Let now $P \leq P_{14} = P_8 \times P_4 \times P_2$ be a vertex of D, and set $D_1 := D^{(8,4,2)}$ and $D_2 := D^{(13,1)}$. We consider all of these modules over the prime field $\mathbb{F}_2$ first. Computations show that $D_1 \otimes D_2 \cong D \oplus D^{(7,4,3)}$. The simple module $D^{(7,4,3)}$ belongs to a block of 2-weight 4 and has vertex $P_4 \times Q_4$, by [10]. The restriction $\mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_{(8,6)}}(D_2)$ is absolutely indecomposable and $\mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_{(8,6)}}(D_1) = M_1 \oplus M_2$ where M_1 and M_2 are absolutely indecomposable of dimension 1486 and 1024, respectively. Moreover, M_2 is relatively $\mathfrak{S}_{(8,4)}$-projective whereas D_1 has vertex P_{14}. Therefore, P_{14} has to be a vertex of M_1 as well, and P is thus a vertex of some indecomposable direct summand of $M_1 \otimes \mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_{(8,6)}}(D_2)$. As our computations show, there exist indecomposable direct summands N_1 and N_2 of $\mathrm{Res}^{\mathfrak{S}_{(8,6)}}_{\mathfrak{S}_{(8,4)}}(M_1)$ and $\mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_{(8,4)}}(D_2)$, respectively, such that $N_1 \otimes N_2$ has an indecomposable direct summand with vertex $P_8 \times P_2$. Consequently, the same holds true for $\mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_{(8,4)}}(D_1 \otimes D_2)$ and $\mathrm{Res}^{\mathfrak{S}_{14}}_{\mathfrak{S}_{(8,4)}}(D)$ as well so that we get $P_8 \times P_2 \leq_{\mathfrak{S}_{14}} P \leq P_8 \times P_4 \times P_2$. Furthermore, $C_{\mathfrak{S}_{14}}(P) = Z(P)$, by Lemma 2.5, so that P actually has to be one of the following groups:

$$P_8 \times P_4 \times P_2, \qquad P_8 \times (P_2)^3, \qquad P_8 \times \langle (9,11,10,12)\rangle \times P_2, \qquad P_8 \times Q_4 \times P_2.$$

Next, we consider the elementary abelian subgroup $E := \langle g_1, \ldots, g_7 \rangle = (P_2)^7$ of P_{14} where $g_i := (2i-1, 2i)$, for $i = 1, \ldots, 7$. Moreover, we can find a normal basis $\alpha_1, \ldots, \alpha_7$ of $\mathbb{F}_{128}$ over $\mathbb{F}_2$ such that neither $\mathrm{Res}^{\mathbb{F}_{128}\mathfrak{S}_{14}}_{\mathbb{F}_{128}\langle u \rangle}(D_2^{\mathbb{F}_{128}})$ nor $\mathrm{Res}^{\mathbb{F}_{128}\mathfrak{S}_{(8,6)}}_{\mathbb{F}_{128}\langle u \rangle}(M_1^{\mathbb{F}_{128}})$ is free, for $u := 1 + \sum_{i=1}^{7} \alpha_i (g_i - 1)$. We may now regard D_1 and D_2 as modules over $F := \overline{\mathbb{F}_{128}}$. In the notation of Remark 3.11 and Theorem 3.12, we then deduce that there exist indecomposable direct summands X and Y of $\mathrm{Res}^{F\mathfrak{S}_{14}}_{FE}(D_1)$ and $\mathrm{Res}^{F\mathfrak{S}_{14}}_{FE}(D_2)$, respectively, such that $(\alpha_1, \ldots, \alpha_7) \in V_E^r(X) \cap V_E^r(Y) = V_E^r(X \otimes_F Y)$. Hence there exists an indecomposable direct summand Z of $X \otimes Y$ whose restriction to $F\langle u \rangle$ is not free, by Theorem 3.12 and Remark 3.11 (b). We can now apply Theorem 3.14, and conclude that $E = (P_2)^7$ has to be contained in some vertex of Z. But then the vertex P of D has to contain a subgroup which is in $\mathfrak{S}_{14}$ conjugate to $(P_2)^7$. Consequently, either $P = P_8 \times P_4 \times P_2$ or $P = P_8 \times (P_2)^3$ holds true.

In fact, D has vertex $P_8 \times P_4 \times P_2$. To obtain this, we constructed the component of M of $\mathrm{Res}^{\mathbb{F}_2\mathfrak{S}_{14}}_{\mathbb{F}_2\mathfrak{S}_{(8,6)}}(D)$ belonging to the principal block of $\mathbb{F}_2\mathfrak{S}_{(8,6)}$. Then $\dim(M) = 9000$. Moreover, we considered $L := \mathrm{Res}^{\mathbb{F}_2\mathfrak{S}_{(8,6)}}_{\mathbb{F}_2 H}(M)$ where $\mathfrak{S}_8 \times (P_2 \wr C_3) =: H \trianglelefteq G := \mathfrak{S}_8 \times (P_2 \wr \mathfrak{S}_3)$ and $|G : H| = 2$. Then $L = L_1 \oplus L_2$ with absolutely indecomposable modules L_1 and L_2 of dimension 4392 and 4608, respectively. The somewhat lengthy computation involves the technique of fixed point condensation, so we skip the details here, and refer the reader to [8]. An extensive introduction to condensation methods is given in [25], and further applications can for instance be found in [26] and [27].

Now assume that $P = P_8 \times (P_2)^3$ which implies that M has to be relatively H-projective. However, due to the decomposition of L, application of Green's Indecomposability Theorem (cf.[16]) immediately leads to a contradiction. Hence D has vertex $P_8 \times P_4 \times P_2 = P_{14}$.

4 Some open questions and a proposition

We close by stating some open questions arising from our computational data. Moreover, we will show that the simple $F\mathfrak{S}_n$-modules with trivial sources are precisely the simple $F\mathfrak{S}_n$-Specht modules.

Question 4.1. Suppose that $p = 2$, and let D be a simple $F\mathfrak{S}_n$-module with non-trivial source L. Is then $\dim(L)$ even?

Question 4.2. Let D be a simple $F\mathfrak{S}_n$-module belonging to the principal block of $F\mathfrak{S}_n$, and suppose $p \mid n$. Are then the vertices of D precisely the Sylow p-subgroups of $\mathfrak{S}_n$, or the Sylow p-subgroups of $\mathfrak{A}_n$?

Question 4.3. Suppose $n = p^k$, for some $k \in \mathbb{N}_0$, and let D be a simple $F\mathfrak{S}_n$-module belonging to the principal block of $F\mathfrak{S}_n$. Is then $\mathrm{Res}^{\mathfrak{S}_n}_{P_n}(D)$ always indecomposable? Provided that there is a positive answer to Question 4.2, in odd characteristic this restriction would then also be a source of D.

Question 4.4. Let $p = 2$ and $n \geq 3$, and consider the spin module $D(n)$ for $F\mathfrak{S}_n$, i.e. the simple module corresponding to the partition $(m+1, m)$ in case that $n = 2m+1$, or $(m+1, m-1)$ in case that $n = 2m$, for some $m \in \mathbb{N}$. Furthermore, let $n = \sum_{i=0}^{s} \alpha_i 2^i$ be the 2-adic expansion of n, and consider a vertex P of $D(n)$. Does then the following hold in general?

$$P \sim_{\mathfrak{S}_n} \begin{cases} Q_n, & \text{if } n \equiv 0 \pmod 4 \\ \prod_{i=0}^{s} (Q_{2^i})^{\alpha_i}, & \text{if } n \equiv 1, 3 \pmod 4 \\ P_n, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

According to our computations, there is a positive answer to this question, for $n \leq 27$.

Remark 4.5. Looking at the tables in the appendix, one observes that among the simple $F\mathfrak{S}_n$-modules which have been treated computationally, the ones with trivial sources are precisely the simple $F\mathfrak{S}_n$-Specht modules. In fact, this is generally true, as will be shown below. For this, we recall the following: let $\mu = (\mu_1, \ldots, \mu_s)$ be a partition of n, and consider the Young subgroup $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_s}$ of $\mathfrak{S}_n$. Then the indecomposable direct summands of the respective $F\mathfrak{S}_n$-permutation module $M^\mu := \mathrm{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_n}(F)$ are called *(indecomposable) Young modules*, and may also be parametrized by the partitions of n. Namely, fixing some indecomposable direct sum decomposition

$$M^\mu = Y_1 \oplus \cdots \oplus Y_m$$

of M^μ, there is precisely one $i \in \{1, \ldots, m\}$ such that $S^\mu \subseteq Y_i$. This $F\mathfrak{S}_n$-module $Y_i =: Y^\mu$ is unique up to isomorphism, and is called the *Young module* corresponding to μ. For a proof see [11].

Similarly, if $p > 2$ and if $n = a + b$, for some $a, b \in \mathbb{N}$, then one may consider partitions $\alpha := (\alpha_1, \ldots, \alpha_r)$ and $\beta := (\beta_1, \ldots, \beta_s)$ of a and b, respectively, and define the *signed permutation module*

$$M(\alpha|\beta) := \mathrm{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_\beta}^{\mathfrak{S}_n}(F \otimes S^{(1^b)}).$$

Here $F \otimes S^{(1^b)}$ is the outer tensor product of the trivial $F\mathfrak{S}_a$-module and the alternating $F\mathfrak{S}_b$-module. The indecomposable direct summands of a signed permutation module are called *(indecomposable) signed Young modules.* Note that both Young modules and signed Young modules have clearly trivial sources.

For the following, we fix a p-modular system (R, K, F). That is, R is a complete discrete valuation ring with quotient field K of characteristic 0 such that $R/(\pi) = F$, where (π) is the unique maximal ideal in R. Furthermore, (R, K, F) is then a splitting p-modular system for $\mathfrak{S}_n$, since every field is a splitting field for $\mathfrak{S}_n$. If M is any $R\mathfrak{S}_n$-module then we denote the $F\mathfrak{S}_n$-module $M/M\pi$ by $\overline{M}$.

Proposition 4.6. *The simple $F\mathfrak{S}_n$-modules with trivial sources are precisely the simple $F\mathfrak{S}_n$-Specht modules.*

Proof. Suppose first that S^μ is a simple $F\mathfrak{S}_n$-Specht module. If $p > 2$ then S^μ is a signed Young module, by [17]. Thus, in particular, S^μ has then trivial source. In case that $p = 2$ we either have $S^\mu \cong Y^\mu$ which has clearly trivial source, or $S^\mu = S^{(2^2)} \cong D^{(3,1)}$. This follows from [21]. In the latter case, S^μ has vertex Q_4 and trivial source, by [28].

Conversely, let D^λ be a simple $F\mathfrak{S}_n$-module with trivial source. Then D^λ is a direct summand of a permutation module and thus lifts to some $R\mathfrak{S}_n$-lattice S. Consequently, the $K\mathfrak{S}_n$-module $K \otimes_R S$ is simple so that there exists some partition μ of n with $K \otimes_R S \cong S^\mu_K$. By construction, S is an R-form of S^μ_K, and if S' is another R-form of S^μ_K then $\overline{S}$ and $\overline{S'}$ have the same composition factors, by [7], Prop. 16.16. Hence D^λ is isomorphic to the reduction modulo (π) of any R-form for S^μ_K. From [19], Cor. 8.11, we thus deduce that $D^\lambda \cong S^\mu_F$, and the proposition is proved. □

Remark 4.7. Note that the second argument actually applies to any group G. That is if (R, K, F) is a splitting p-modular system for G then each simple FG-module D with trivial sources lifts to an RG-lattice S. Moreover, as above, the KG-module $K \otimes_R S$ is simple, and D is isomorphic to the modular reduction of any R-form of $K \otimes_R S$.

A Tables

(1) The following tables display our results on vertices of simple $F\mathfrak{S}_n$-modules for $p = 2$ and $p = 3$, respectively. Here, blocks are labelled by their p-cores, and a defect group of each block is listed in column "def.". Moreover, simple modules are represented by the corresponding p-regular partitions, and the column "dim." contains the respective dimensions of the simple modules. Here neighbouring partitions are Mullineux conjugate so that the corresponding simple modules have the same vertices and sources. If a simple module is also isomorphic to Specht modules the corresponding partitions are listed in column "Specht". Finally, for each simple module, we list a vertex and the dimension of its sources, if computed.

If column "sce." contains two entries then the first one denotes the dimension of the sources of the respective simple module when realized over the prime field $\mathbb{F}_p$. Those sources are then not absolutely indecomposable but split when extending the coefficient field to $\mathbb{F}_{p^2}$ or $\mathbb{F}_{p^3}$. The second entry then denotes the dimensions of the sources over the bigger field where they turned out to be absolutely indecomposable. In order to give an impression on how far one can actually get by extensive application of modular branching rules, we have marked several partitions in bold face. A partition λ of n is marked if and only if the corresponding simple module D^λ satisfies neither of the following conditions:

- $D^\lambda = F$
- $D^\lambda \cong S^\mu$, for some partition μ of n, and D^λ belongs to a block with abelian defect groups
- the vertices and sources of D^λ can be obtained via reduction to smaller symmetric groups.

(2) A simple $\mathbb{F}_p\mathfrak{S}_n$-module D^μ has always been constructed as a composition factor of an appropriate $\mathbb{F}_p\mathfrak{S}_n$-Specht module. MAGMA provides the function `SymmetricRepresentation` which, for a given partition λ of n, allows us to construct a matrix representation of $\mathfrak{S}_n$ over $\mathbb{Q}$ afforded by the $\mathbb{Q}\mathfrak{S}_n$-Specht module $S^\lambda_{\mathbb{Q}}$. In fact, the computed matrices have integer entries so that we then obtain an $\mathbb{F}_p\mathfrak{S}_n$-module with the same composition factors as the $\mathbb{F}_p\mathfrak{S}_n$-Specht module $S^\lambda_{\mathbb{F}_p}$, simply by reducing these entries modulo p. The simple $F\mathfrak{S}_{15}$-module $D^{(7,5,3)}$

in characteristic 3 excepted, all computationally treated simple $F\mathfrak{S}_n$-modules have been constructed in that manner.

As a matter of memory request, the $\mathbb{F}_3\mathfrak{S}_{15}$-module $D^{(7,5,3)}$ of dimension 43497 has been constructed via the C-MeatAxe [31] and a GAP algorithm “SpechtModule” by J. Müller [24]. The module $D^{(7,5,3)}$ is contained in the block of $\mathbb{F}_3\mathfrak{S}_{15}$ with 3-core $(4,2)$ and 3-weight 3. Furthermore, let

$$\mathrm{Res}_{\mathfrak{S}_{10}}^{\mathfrak{S}_{15}}(D^{(7,5,3)}) = M \oplus N$$

such that M is the component of $\mathrm{Res}_{\mathfrak{S}_{10}}^{\mathfrak{S}_{15}}(D^{(7,5,3)})$ belonging to the principal block of $\mathbb{F}_3\mathfrak{S}_{10}$. Then $\dim(M) = 3402$, and $\mathrm{Res}_{P_9}^{\mathfrak{S}_{10}}(M)$ possesses exactly 36 indecomposable projective direct summands, all of which have been detected by the algorithm “ProjSummands”. The remaining quotient module has dimension 486, and splits into the direct sum of nine isomorphic modules of dimension 54. These are indecomposable over $\mathbb{F}_3$, and have vertex E_9. Here E_9 is understood to be an elementary abelian group of order 9, acting regularly on the set $\{1, \ldots, 9\}$. Consequently, also $D^{(7,5,3)}$ has vertex E_9.

(3) Note that, among all simple modules displayed below, the $\mathbb{F}_2\mathfrak{S}_{14}$-module $D^{(5,4,3,2)}$ with vertex E_8 and the $\mathbb{F}_3\mathfrak{S}_{15}$-module $D^{(7,5,3)}$ with vertex E_9 are the only ones whose vertices are not conjugate to a Sylow p-subgroup of some subgroup

$$\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k} \times \mathfrak{A}_{n_{k+1}} \times \cdots \times \mathfrak{A}_{n_l}$$

of $\mathfrak{S}_n$.

(4) The remaining simple module $D^{(6,4,3,2)}$ for $\mathfrak{S}_{15}$ in characteristic 3 has dimension 29106 and belongs to the principal 3-block of $\mathfrak{S}_{15}$. In consequence of too high memory request, we have not yet been able to construct any of the $\mathbb{F}_3\mathfrak{S}_{15}$-Specht modules having a composition factor isomorphic to $D^{(6,4,3,2)}$. According to Question 4.2, we heavily suspect that the vertices of $D^{(6,4,3,2)}$ are precisely the Sylow 3-subgroups of $\mathfrak{S}_{15}$.

A.1 Simple $F\mathfrak{S}_n$-modules in characteristic 2

n	block	def.	partitions	dim.	vtx.	sce.	Specht	
1	(1)	1	(1)	1	1	1	(1)	
2	$\emptyset$	P_2	(2)	1	P_2	1	(2)	(1^2)
3	(1)	P_2	(3)	1	P_2	1	(3)	(1^3)
	$(2,1)$	1	$(2,1)$	2	1	1	$(2,1)$	
4	$\emptyset$	P_4	(4)	1	P_4	1	(4)	(1^4)
			$(\mathbf{3},\mathbf{1})$	2	Q_4	1	(2^2)	
5	(1)	P_4	(5)	1	P_4	1	(5)	(1^5)
			$(\mathbf{3},\mathbf{2})$	4	Q_4	$4/2$	-	-
	$(2,1)$	P_2	$(4,1)$	4	P_2	1	$(4,1)$	$(2,1^3)$
6	$\emptyset$	P_6	(6)	1	P_6	1	(6)	(1^6)
			$(\mathbf{5},\mathbf{1})$	4	P_6	4	-	-
			$(\mathbf{4},\mathbf{2})$	4	P_6	4	-	-
	$(3,2,1)$	1	$(3,2,1)$	16	1	1	$(3,2,1)$	
7	(1)	P_6	(7)	1	P_6	1	(7)	(1^7)
			$(\mathbf{5},\mathbf{2})$	14	P_2^3	1	$(5,2)$	$(2^2,1^3)$
			$(4,2,1)$	20	P_6	4	-	-
	$(2,1)$	P_4	$(6,1)$	6	P_4	1	$(6,1)$	$(2,1^5)$
			$(4,3)$	8	Q_4	$4/2$	-	-
8	$\emptyset$	P_8	(8)	1	P_8	1	(8)	(1^8)
			$(\mathbf{7},\mathbf{1})$	6	P_8	6	-	-
			$(\mathbf{6},\mathbf{2})$	14	P_8	14	-	-
			$(\mathbf{5},\mathbf{3})$	8	Q_8	4	-	-
			$(\mathbf{4},\mathbf{3},\mathbf{1})$	40	Q_8	20	-	-
	$(3,2,1)$	P_2	$(5,2,1)$	64	P_2	1	$(5,2,1)$	$(3,2,1^3)$
9	(1)	P_8	(9)	1	P_8	1	(9)	(1^9)
			$(\mathbf{7},\mathbf{2})$	26	P_8	26	-	-
			$(6,2,1)$	78	P_8	14	-	-
			$(\mathbf{5},\mathbf{4})$	16	Q_8	8	-	-
			$(5,3,1)$	40	Q_8	20	-	-
	$(2,1)$	P_6	$(8,1)$	8	P_6	1	$(8,1)$	$(2,1^7)$
			$(6,3)$	48	P_2^3	1	$(6,3)$	$(2^3,1^3)$
			$(\mathbf{4},\mathbf{3},\mathbf{2})$	160	$Q_4\times P_2$	$4/2$	-	-

n	block	def.	partitions	dim.	vtx.	sce.	Specht	
10	$\emptyset$	P_{10}	(10)	1	P_{10}	1	(10)	(1^{10})
			$(\mathbf{9},\mathbf{1})$	8	P_{10}	8	-	-
			$(\mathbf{8},\mathbf{2})$	26	P_{10}	26	-	-
			$(\mathbf{7},\mathbf{3})$	48	P_{10}	48	-	-
			$(\mathbf{6},\mathbf{4})$	16	P_{10}	16	-	-
			$(\mathbf{6},\mathbf{3},\mathbf{1})$	198	P_{10}	70	-	-
			$(\mathbf{5},\mathbf{3},\mathbf{2})$	200	P_{10}	200	-	-
	$(3,2,1)$	P_4	$(7,2,1)$	160	P_4	1	$(7,2,1)$	$(3,2,1^5)$
			$(5,4,1)$	128	Q_4	$4/2$	-	-
	$(4,3,2,1)$	1	$(4,3,2,1)$	768	1	1	$(4,3,2,1)$	
11	(1)	P_{10}	(11)	1	P_{10}	1	(11)	(1^{11})
			$(\mathbf{9},\mathbf{2})$	44	$P_4 \times P_2^3$	1	$(9,2)$	$(2^2,1^7)$
			$(8,2,1)$	186	P_{10}	26	-	-
			$(\mathbf{7},\mathbf{4})$	164	P_{10}	20	-	-
			$(7,3,1)$	198	P_{10}	70	-	-
			$(6,4,1)$	144	P_{10}	16	-	-
			$(\mathbf{5},\mathbf{4},\mathbf{2})$	416	P_{10}	96	-	-
	$(2,1)$	P_8	$(10,1)$	10	P_8	1	$(10,1)$	$(2,1^8)$
			$(8,3)$	100	P_8	26	-	-
			$(6,5)$	32	Q_8	8	-	-
			$(\mathbf{6},\mathbf{3},\mathbf{2})$	848	$P_4 \times Q_4$	$4/2$	-	-
			$(5,3,2,1)$	1168	Q_8	20	-	-
12	$\emptyset$	P_{12}	(12)	1	P_{12}	1	(12)	(1^{12})
			$(\mathbf{11},\mathbf{1})$	10	P_{12}	10	-	-
			$(\mathbf{10},\mathbf{2})$	44	P_{12}	44	-	-
			$(\mathbf{9},\mathbf{3})$	100	P_{12}	100	-	-
			$(\mathbf{8},\mathbf{4})$	164	P_{12}	164	-	-
			$(\mathbf{8},\mathbf{3},\mathbf{1})$	570	P_{12}	314	-	-
			$(\mathbf{7},\mathbf{5})$	32	Q_{12}	$32/16$	-	-
			$(\mathbf{7},\mathbf{3},\mathbf{2})$	1046	P_{12}	534	-	-
			$(\mathbf{6},\mathbf{5},\mathbf{1})$	288	Q_{12}	$288/144$	-	-
			$(\mathbf{6},\mathbf{4},\mathbf{2})$	416	P_{12}	416	-	-
			$(\mathbf{5},\mathbf{4},\mathbf{2},\mathbf{1})$	2368	Q_{12}	$1344/672$	-	-
	$(3,2,1)$	P_6	$(9,2,1)$	320	P_6	1	$(9,2,1)$	$(3,2,1^7)$
			$(7,4,1)$	1408	P_2^3	1	$(7,4,1)$	$(3,2^2,1^3)$
			$(5,4,3)$	1792	$Q_4 \times P_2$	$4/2$	-	-
	$(4,3,2,1)$	P_2	$(6,3,2,1)$	5632	P_2	1	$(6,3,2,1)$	$(4,3,2,1^3)$

n	block	def.	partitions	dim.	vtx.	sce.	Specht	
13	(1)	P_{12}	(13)	1	P_{12}	1	(13)	(1^{13})
			$(\mathbf{11},\mathbf{2})$	64	P_{12}	32	-	-
			$(10,2,1)$	364	P_{12}	44	-	-
			$(\mathbf{9},\mathbf{4})$	364	P_{12}	172	-	-
			$(9,3,1)$	570	P_{12}	314	-	-
			$(8,4,1)$	1572	P_{12}	164	-	-
			$(\mathbf{7},\mathbf{6})$	64	$Q_8 \times Q_4$	$32/16$	-	-
			$(7,5,1)$	288	Q_{12}	$288/144$	-	-
			$(\mathbf{7},\mathbf{4},\mathbf{2})$	2510	P_{12}	846	-	-
			$(6,4,3)$	2208	P_{12}	416	-	-
			$(\mathbf{5},\mathbf{4},\mathbf{3},\mathbf{1})$	8448	Q_{12}	512	-	-
	$(2,1)$	P_{10}	$(12,1)$	12	P_{10}	1	$(12,1)$	$(2,1^{11})$
			$(10,3)$	208	$P_4 \times P_2^3$	1	$(10,3)$	$(2^3,1^7)$
			$(8,5)$	560	P_{10}	20	-	-
			$(\mathbf{8},\mathbf{3},\mathbf{2})$	2848	$P_6 \times Q_4$	$4/2$	-	-
			$(\mathbf{6},\mathbf{5},\mathbf{2})$	1728	$Q_8 \times P_2$	8	-	-
			$(7,3,2,1)$	8008	P_{10}	70	-	-
			$(6,4,2,1)$	3200	P_{10}	96	-	-
14	$\emptyset$	P_{14}	(14)	1	P_{14}	1	(14)	(1^{14})
			$(\mathbf{13},\mathbf{1})$	12	P_{14}	12	-	-
			$(\mathbf{12},\mathbf{2})$	64	P_{14}	64	-	-
			$(\mathbf{11},\mathbf{3})$	208	P_{14}	208	-	-
			$(\mathbf{10},\mathbf{4})$	364	P_{14}	364	-	-
			$(\mathbf{10},\mathbf{3},\mathbf{1})$	1300	P_{14}	404	-	-
			$(\mathbf{9},\mathbf{5})$	560	P_{14}	560	-	-
			$(\mathbf{9},\mathbf{3},\mathbf{2})$	3418	P_{14}	1114	-	-
			$(\mathbf{8},\mathbf{6})$	64	P_{14}	64	-	-
			$(\mathbf{8},\mathbf{5},\mathbf{1})$	4576	P_{14}	992	-	-
			$(\mathbf{8},\mathbf{4},\mathbf{2})$	2510	P_{14}	1486	-	-
			$(\mathbf{7},\mathbf{5},\mathbf{2})$	2016	P_{14}	992	-	-
			$(\mathbf{7},\mathbf{4},\mathbf{2},\mathbf{1})$	19240	P_{14}		-	-
			$(\mathbf{6},\mathbf{5},\mathbf{3})$	4704	P_{14}	1120	-	-
			$(\mathbf{6},\mathbf{4},\mathbf{3},\mathbf{1})$	11648	Q_{14}		-	-
	$(3,2,1)$	P_8	$(11,2,1)$	560	P_8	1	$(11,2,1)$	$(3,2,1^9)$
			$(9,4,1)$	3808	P_8	26	-	-
			$(7,6,1)$	768	Q_8	8	-	-
			$(7,4,3)$	10880	$P_4 \times Q_4$	$4/2$	-	-
			$(\mathbf{5},\mathbf{4},\mathbf{3},\mathbf{2})$	35840	E_8	$12/4$	-	-
	$(4,3,2,1)$	P_4	$(8,3,2,1)$	23296	P_4	1	$(8,3,2,1)$	$(4,3,2,1^5)$
			$(6,5,2,1)$	13312	Q_4	$4/2$	-	-

A.2 Simple $F\mathfrak{S}_n$-modules in characteristic 3

n	blocks		def.	partitions		dim.	vtx.	sce.	Specht	
1	(1)		1	(1)		1	1	1	(1)	
2	(2)	(1^2)	1	(2)	(1^2)	1	1	1	(2)	(1^2)
3	$\emptyset$		P_3	(3)	$(2,1)$	1	P_3	1	(3)	(1^3)
4	(1)		P_3	(4)	(2^2)	1	P_3	1	(4)	(1^4)
	$(3,1)$	$(2,1^2)$	1	$(3,1)$	$(2,1^2)$	3	1	1	$(3,1)$	$(2,1^2)$
5	(2)	(1^2)	P_3	(5)	$(3,2)$	1	P_3	1	(5)	(1^5)
				$(2^2,1)$	$(4,1)$	4	P_3	1	$(2,1^3)$	$(4,1)$
	$(3,1^2)$		1	$(3,1^2)$		6	1	1	$(3,1^2)$	
6	$\emptyset$		P_3^2	(6)	(3^2)	1	P_3^2	1	(6)	(1^6)
				$\mathbf{(5,1)}$	$\mathbf{(3,2,1)}$	4	P_3^2	4	-	
				$\mathbf{(4,1^2)}$		6	P_3^2	6	-	
	$(4,2)$	$(2^2,1^2)$	1	$(4,2)$	$(2^2,1^2)$	9	1	1	$(4,2)$	$(2^2,1^2)$
7	(1)		P_3^2	(7)	$(4,3)$	1	P_3^2	1	(7)	(1^7)
				$(5,2)$	$(3,2,1^2)$	13	P_3^2	4	-	
				$(4,2,1)$		20	P_3^2	1	$(4,1^3)$	
	$(3,1)$	$(2,1^2)$	P_3	$(6,1)$	$(3^2,1)$	6	P_3	1	$(6,1)$	$(2,1^5)$
				$(3,2^2)$	$(5,1^2)$	15	P_3	1	$(3,1^4)$	$(5,1^2)$
8	(2)	(1^2)	P_3^2	(8)	(4^2)	1	P_3^2	1	(8)	(1^8)
				$(5,3)$	$(3,2^2,1)$	28	P_3^2	1	$(5,3)$	$(2^3,1^2)$
				$(5,2,1)$	$(4,2^2)$	35	P_3^2	1	$(5,1^3)$	$(4,1^4)$
				$(4,3,1)$	$(7,1)$	7	P_3^2	1	$(2,1^6)$	$(7,1)$
				$(3^2,1^2)$	$(6,2)$	13	P_3^2	4	-	-
	$(3,1^2)$		P_3	$(6,1^2)$	$(3^2,2)$	21	P_3	1	$(6,1^2)$	$(3,1^5)$
	$(4,2,1^2)$		1	$(4,2,1^2)$		90	1	1	$(4,2,1^2)$	
9	$\emptyset$		P_9	(9)	$(5,4)$	1	P_9	1	(9)	(1^9)
				$\mathbf{(8,1)}$	$\mathbf{(4^2,1)}$	7	P_9	7	-	
				$\mathbf{(7,1^2)}$	$\mathbf{(4,3,2)}$	21	P_9	21	-	
				$\mathbf{(6,3)}$	$\mathbf{(3^2,2,1)}$	41	P_9	41	-	
				$\mathbf{(6,2,1)}$	$\mathbf{(5,2^2)}$	35	P_9	35	-	
	$(4,2)$	$(2^2,1^2)$	P_3	$(7,2)$	$(4,3,1^2)$	27	P_3	1	$(7,2)$	$(2^2,1^5)$
				$(4,2^2,1)$	$(5,2,1^2)$	189	P_3	1	$(4,2,1^3)$	$(5,2,1^2)$
	$(5,3,1)$	$(3,2^2,1^2)$	1	$(5,3,1)$	$(3,2^2,1^2)$	162	1	1	$(5,3,1)$	$(3,2^2,1^2)$
10	(1)		P_9	(10)	(5^2)	1	P_9	1	(10)	(1^{10})
				$(8,2)$	$(4^2,1^2)$	34	P_9	7	-	
				$(7,3)$	$(4,3,2,1)$	41	P_9	41	-	
				$\mathbf{(7,2,1)}$	$\mathbf{(4,3^2)}$	84	P_3^3	1	$(7,1^3)$	$(4,1^6)$
				$(6,2,1^2)$	$(5,2^2,1)$	224	P_9	35	-	
	$(3,1)$	$(2,1^2)$	P_3^2	$(9,1)$	$(5,4,1)$	9	P_3^2	1	$(9,1)$	$(2,1^8)$
				$(6,4)$	$(3^2,2^2)$	90	P_3^2	1	$(6,4)$	$(2^4,1^2)$
				$(6,2^2)$	$(5,3,2)$	126	P_3^2	1	$(6,1^4)$	$(5,1^5)$
				$(4^2,2)$	$(8,1^2)$	36	P_3^2	1	$(3,1^7)$	$(8,1^2)$
				$(3^2,2,1^2)$	$(6,3,1)$	279	P_3^2	4	-	-
	$(5,3,1^2)$	$(4,2^2,1^2)$	1	$(5,3,1^2)$	$(4,2^2,1^2)$	567	1	1	$(5,3,1^2)$	$(4,2^2,1^2)$

n	blocks		def.	partitions		dim.	vtx.	sce.	Specht	
11	(2)	(1^2)	P_9	(11)	$(6,5)$	1	P_9	1	(11)	(1^{11})
				$\mathbf{(8,3)}$	$\mathbf{(4^2,2,1)}$	109	P_9	28	-	-
				$(8,2,1)$	$(4^2,3)$	120	P_3^3	1	$(8,1^3)$	$(4,1^7)$
				$(7,3,1)$	$(4,3,2,1^2)$	320	P_9	41	-	-
				$(6,3,1^2)$	$(5,2^2,1^2)$	791	P_9	35	-	-
				$(5^2,1)$	$(10,1)$	10	P_9	1	$(2,1^9)$	$(10,1)$
				$(5,4,1^2)$	$(9,2)$	34	P_9	7	-	-
				$(5,3^2)$	$(7,2^2)$	210	P_3^3	1	$(5,1^6)$	$(7,1^4)$
				$\mathbf{(5,3,2,1)}$	$\mathbf{(6,2^2,1)}$	714	P_3^3	4	-	-
				$(4,3,2^2)$	$(7,4)$	131	P_9	41	-	-
	$(3,1^2)$		P_3^2	$(9,1^2)$	$(5,4,2)$	45	P_3^2	1	$(9,1^2)$	$(3,1^8)$
				$(6,4,1)$	$(3^2,2^2,1)$	693	P_3^2	1	$(6,4,1)$	$(3,2^3,1^2)$
				$(6,3,2)$		252	P_3^2	1	$(6,1^5)$	
	$(4,2,1^2)$		P_3	$(7,2,1^2)$	$(4,3^2,1)$	594	P_3	1	$(7,2,1^2)$	$(4,2,1^5)$
12	$\emptyset$		P_{12}	(12)	$(6,6)$	1	P_{12}	1	(12)	(1^{12})
				$\mathbf{(11,1)}$	$\mathbf{(6,5,1)}$	10	P_{12}	10	-	
				$\mathbf{(10,1^2)}$	$\mathbf{(5^2,2)}$	45	P_{12}	45	-	
				$\mathbf{(9,3)}$	$\mathbf{(5,4,2,1)}$	143	P_{12}	62	-	
				$\mathbf{(9,2,1)}$	$\mathbf{(5,4,3)}$	120	P_{12}	120	-	
				$\mathbf{(8,4)}$	$\mathbf{(4^2,2^2)}$	131	P_{12}	131	-	
				$\mathbf{(8,2^2)}$	$\mathbf{(6,3^2)}$	210	P_{12}	210	-	
				$\mathbf{(7,4,1)}$	$\mathbf{(4,3,2^2,1)}$	1013	P_{12}	284	-	
				$\mathbf{(7,3,2)}$		252	P_{12}	126	-	
				$\mathbf{(6,4,1^2)}$	$\mathbf{(5,3,2,1^2)}$	1936	P_{12}	235	-	
				$\mathbf{(6,3,2,1)}$		1428	P_{12}	147	-	
	$(4,2)$	$(2^2,1^2)$	P_3^2	$(10,2)$	$(5^2,1^2)$	54	P_3^2	1	$(10,2)$	$(2^2,1^8)$
				$(7,5)$	$(4,3^2,2)$	297	P_3^2	1	$(7,5)$	$(2^5,1^2)$
				$(7,2^2,1)$	$(5,3^2,1)$	1728	P_3^2	1	$(7,2,1^3)$	$(5,2,1^5)$
				$(4^2,3,1)$	$(8,2,1^2)$	945	P_3^2	1	$(4,2,1^6)$	$(8,2,1^2)$
				$(4,3^2,1^2)$	$(7,3,1^2)$	1431	P_3^2	4	-	-
	$(5,3,1)$	$(3,2^2,1^2)$	P_3	$(8,3,1)$	$(4^2,2,1^2)$	891	P_3	1	$(8,3,1)$	$(3,2^2,1^5)$
				$(5,3,2^2)$	$(6,2^2,1^2)$	3564	P_3	1	$(5,3,1^4)$	$(6,2^2,1^2)$
	$(6,4,2)$	$(3^2,2^2,1^2)$	1	$(6,4,2)$	$(3^2,2^2,1^2)$	2673	1	1	$(6,4,2)$	$(3^2,2^2,1^2)$

n	block		def.	partitions		dim.	vtx.	sce.	Specht	
13	(1)		P_{12}	(13)	$(7,6)$	1	P_{12}	1	(13)	(1^{13})
				$(11,2)$	$(6,5,1^2)$	64	P_{12}	10	-	
				$(10,3)$	$(5^2,2,1)$	143	P_{12}	62	-	
				$(8,5)$	$(4^2,3,2)$	428	P_{12}	131	-	
				$(\mathbf{10,2,1})$	$(\mathbf{5^2,3})$	220	P_{12}	1	$(10,1^3)$	$(4,1^9)$
				$(\mathbf{7,5,1})$	$(\mathbf{4,3^2,2,1})$	2287	P_{12}	154	-	
				$(7,4,1^2)$	$(5,3^2,1^2)$	3367	P_{12}	235	-	
				$(9,2,1^2)$	$(5,4,3,1)$	1065	P_{12}	120	-	
				$(6,3^2,1)$	$(8,2^2,1)$	1938	P_{12}	210	-	
				$(\mathbf{7,3^2})$		924	$P_3{}^4$	1	$(7,1^6)$	
				$(7,3,2,1)$		1428	P_{12}	147	-	
	$(3,1)$	$(2,1^2)$	P_9	$(12,1)$	$(6^2,1)$	12	P_9	1	$(12,1)$	$(2,1^{11})$
				$(9,4)$	$(5,4,2^2)$	417	P_9	28	-	-
				$(9,2^2)$	$(5,4^2)$	495	$P_3{}^3$	1	$(9,1^4)$	$(5,1^8)$
				$(7,4,2)$	$(4,3,2^2,1^2)$	5082	P_9	41	-	-
				$(6,5,2)$	$(11,1^2)$	66	P_9	1	$(3,1^{10})$	$(11,1^2)$
				$(6,4,3)$	$(8,3,2)$	792	$P_3{}^3$	1	$(6,1^7)$	$(8,1^5)$
				$(\mathbf{6,4,2,1})$	$(\mathbf{5,3,2^2,1})$	10296	$P_3{}^3$	1	$(6,4,1^3)$	$(5,2^3,1^2)$
				$(6,3,2,1^2)$	$(6,3,2,2)$	8568	$P_3{}^3$	4	-	-
				$(4^2,2^2,1)$	$(8,4,1)$	1275	P_9	41	-	-
				$(5,4,2,1^2)$	$(9,3,1)$	1299	P_9	7	-	-
	$(5,3,1^2)$	$(4,2^2,1^2)$	P_3	$(5,3^2,2)$	$(7,2^2,1^2)$	7371	P_3	1	$(5,3,1^5)$	$(7,2^2,1^2)$
				$(8,3,1^2)$	$(4^2,3,1^2)$	4212	P_3	1	$(8,3,1^2)$	$(4,2^2,1^5)$
14	(2)	(1^2)	P_{12}	(14)	(7^2)	1	P_{12}	1	(14)	(1^{14})
				$(\mathbf{11,3})$	$(\mathbf{6,5,2,1})$	273	P_3^4	1	$(11,3)$	$(2^3,1^8)$
				$(11,2,1)$	$(6,5,3)$	286	P_{12}	1	$(11,1^3)$	$(4,1^{10})$
				$(10,3,1)$	$(5^2,2,1^2)$	1442	P_{12}	62	-	-
				$(9,3,1^2)$	$(5,4,3,1^2)$	5277	P_{12}	120	-	-
				$(\mathbf{8,6})$	$(\mathbf{4^2,3^2})$	1000	P_{12}	28	-	-
				$(8,5,1)$	$(4^2,3,2,1)$	3562	P_{12}	154	-	-
				$(8,4,1^2)$	$(6,3^2,1^2)$	3367	P_{12}	235	-	-
				$(8,3^2)$	$(7,4,3)$	1716	P_3^4	1	$(8,1^6)$	$(7,1^7)$
				$(\mathbf{8,3,2,1})$	$(\mathbf{6,4,3,1})$	6369	P_3^4	4	-	-
				$(7,6,1)$	$(13,1)$	13	P_{12}	1	$(2,1^{12})$	$(13,1)$
				$(7,3,2^2)$	$(7,3,2,1^2)$	9996	P_{12}	147	-	-
				$(6^2,1^2)$	$(12,2)$	64	P_{12}	10	-	-
				$(6,3^2,2)$	$(8,2^2,1^2)$	9309	P_{12}	210	-	-
				$(5^2,4)$	$(10,2^2)$	715	P_{12}	1	$(5,1^9)$	$(10,1^4)$
				$(5^2,2^2)$	$(10,4)$	560	P_{12}	62	-	-
				$(\mathbf{5,4^2,1})$	$(\mathbf{9,2^2,1})$	4213	P_{12}	7	-	-
				$(5,4,3,2)$	$(9,5)$	428	P_{12}	131	-	-
				$(\mathbf{5,3^2,2,1})$	$(\mathbf{7,4,2,1})$	20747	P_{12}		-	-
				$(4,3^2,2,1^2)$	$(7,5,2)$	13012	P_{12}	154	-	-
	$(3,1^2)$		P_9	$(12,1^2)$	$(6^2,2)$	78	P_9	1	$(12,1^2)$	$(3,1^{11})$
				$(9,4,1)$	$(5,4,2^2,1)$	4290	P_9	28	-	
				$(8,4,2)$	$(4^2,2^2,1^2)$	6357	P_9	41	-	
				$(6,4^2)$	$(9,3,2)$	1287	P_3^3	1	$(6,1^8)$	$(9,1^5)$
				$(6,4,2^2)$	$(6,3,2^2,1)$	28665	P_3^3	1	$(6,4,1^4)$	$(6,2^3,1^2)$
	$(4,2,1^2)$		P_3^2	$(10,2,1^2)$	$(5^2,3,1)$	2079	P_3^2	1	$(10,2,1^2)$	$(4,2,1^8)$
				$(7,5,1^2)$	$(4,3^2,2^2)$	15444	P_3^2	1	$(7,5,1^2)$	$(4,2^4,1^2)$
				$(7,3^2,1)$		9504	P_3^2	1	$(7,2,1^5)$	
	$(6,4,2,1^2)$	$(5,3,2^2,1^2)$	1	$(6,4,2,1^2)$	$(5,3,2^2,1^2)$	69498	1	1	$(6,4,2,1^2)$	$(5,3,2^2,1^2)$

$\mathbf{n = 15}$

blocks		def.	partitions		dim.	vtx.	sce.	Specht	
$\emptyset$		P_{15}	(15)	$(8,7)$	1	P_{15}	1	(15)	(1^{15})
			$\mathbf{(14,1)}$	$\mathbf{(7^2,1)}$	13	P_{15}	13	-	
			$\mathbf{(13,1^2)}$	$\mathbf{(7,6,2)}$	78	P_{15}	78	-	
			$\mathbf{(12,3)}$	$\mathbf{(6^2,2,1)}$	337	P_{15}	94	-	
			$\mathbf{(12,2,1)}$	$\mathbf{(6^2,3)}$	286	P_{15}	286	-	
			$\mathbf{(11,4)}$	$\mathbf{(6,5,2^2)}$	560	P_{15}	317	-	
			$\mathbf{(11,2^2)}$	$\mathbf{(6,5,4)}$	715	P_{15}	715	-	
			$\mathbf{(10,4,1)}$	$\mathbf{(5^2,2^2,1)}$	5732	P_{15}		-	
			$\mathbf{(10,3,2)}$	$\mathbf{(7,4^2)}$	1287	P_{15}	1287	-	
			$\mathbf{(9,6)}$	$\mathbf{(5,4,3^2)}$	1428	P_{15}	456	-	
			$\mathbf{(9,5,1)}$	$\mathbf{(5,4,3,2,1)}$	3562	P_{15}	1375	-	
			$\mathbf{(9,4,1^2)}$	$\mathbf{(5,4^2,1^2)}$	15365	P_{15}		-	
			$\mathbf{(9,3^2)}$	$\mathbf{(8,4,3)}$	1716	P_{15}	1716	-	
			$\mathbf{(9,3,2,1)}$	$\mathbf{(6,4^2,1)}$	10582	P_{15}		-	
			$\mathbf{(8,5,2)}$	$\mathbf{(4^2,3,2,1^2)}$	19369	P_{15}		-	
			$\mathbf{(8,4,2,1)}$	$\mathbf{(6,3^2,2,1)}$	24114	P_{15}		-	
			$\mathbf{(8,3,2,1^2)}$	$\mathbf{(6,4,3,2)}$	29106			-	
			$\mathbf{(7,4,2^2)}$	$\mathbf{(7,3,2^2,1)}$	38661	P_{15}		-	
$(4,2)$	$(2^2,1^2)$	P_9	$(13,2)$	$(7,6,1^2)$	90	P_9	1	$(13,2)$	$(2^2,1^{11})$
			$(10,5)$	$(5^2,3,2)$	1548	P_9	28	-	-
			$(10,2^2,1)$	$(5^2,4,1)$	7722	P_3^3	1	$(10,2,1^3)$	$(5,2,1^8)$
			$\mathbf{(7,5,3)}$	$\mathbf{(4,3^2,2^2,1)}$	43497	E_9	6/3	-	-
			$(7,5,2,1)$	$(5,3^2,2^2)$	61425	P_3^3	1	$(7,5,1^3)$	$(5,2^4,1^2)$
			$(7,4,3,1)$	$(8,3^2,1)$	19305	P_3^3	1	$(7,2,1^6)$	$(8,2,1^5)$
			$(7,3^2,1^2)$	$(7,3^2,2)$	32670	P_3^3	4	-	-
			$(6,5,3,1)$	$(11,2,1^2)$	2925	P_9	1	$(4,2,1^9)$	$(11,2,1^2)$
			$(5^2,3,1^2)$	$(10,3,1^2)$	8163	P_9	7	-	-
			$(4^2,3,2^2)$	$(8,5,1^2)$	26937	P_9	41	-	-
$(5,3,1)$	$(3,2^2,1^2)$	P_3^2	$(11,3,1)$	$(6,5,2,1^2)$	2835	P_3^2	1	$(11,3,1)$	$(3,2^2,1^8)$
			$(8,6,1)$	$(4^2,3^2,1)$	11583	P_3^2	1	$(8,6,1)$	$(3,2^5,1^2)$
			$(8,3,2^2)$	$(6,4,3,1^2)$	44550	P_3^2	1	$(8,3,1^4)$	$(6,2^2,1^5)$
			$(5,4^2,2)$	$(9,2^2,1^2)$	24948	P_3^2	1	$(5,3,1^7)$	$(9,2^2,1^2)$
			$(5,3^2,2,1^2)$	$(7,4,2,1^2)$	159327	P_3^2	4	-	-
$(6,4,2)$	$(3^2,2^2,1^2)$	P_3	$(9,4,2)$	$(5,4,2^2,1^2)$	22113	P_3	1	$(9,4,2)$	$(3^2,2^2,1^5)$
			$(6,4,2^2,1)$	$(6,3,2^2,1^2)$	221130	P_3	1	$(6,4,2,1^3)$	$(6,3,2^2,1^2)$

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