Vertices, sources and Green correspondents of the simple modules for the large Mathieu groups

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Abstract

We investigate the simple modules for the sporadic simple Mathieu groups M_{22}, M_{23} and M_{24} as well as those of the automorphism group, the covering groups and the bicyclic extensions of M_{22} in characteristic 2 and 3. We determine the vertices and sources as well as the Green correspondents of these simple modules. We also find two 3-blocks with elementary abelian defect groups of order 9 in these groups which are Morita equivalent to their Brauer correspondents.

Subject classification: 20C20, 20C34, 20C40

Keywords: Alperin's Weight Conjecture, Broué's Abelian Defect Group Conjecture, covering group, Green correspondent, Mathieu group, Morita equivalence, simple module, source, vertex

1 Introduction

More than 20 years ago, G. Schneider [38] determined the vertices, sources and Green correspondents of the simple modules for the sporadic simple Mathieu groups M_{11} and M_{12} in characteristic 2. Later H. Gollan [20] settled the characteristic 3 case, with a few exceptions. Gollan's results were reproved by S. Koshitani and K. Waki [29] who also determined the vertices and Green correspondents of the modules left open in Gollan's thesis. Both Schneider's and Gollan's results build on computations with the computer algebra system CAYLEY, whereas Koshitani and Waki do without any computer calculations.

The purpose of this article is to determine the vertices of the simple modules for the large Mathieu groups M_{22}, M_{23} and M_{24} over a field F of characteristic 2 and 3, respectively. For each of these simple modules we also investigate its sources and its Green correspondent with respect to the normalizer of some vertex. Both the sources and the Green correspondents will be described in terms of their Loewy series. All of the simple modules for the Mathieu groups M_{22}, M_{23} and M_{24} in characteristics 2 and 3 have precisely the defect groups of the respective blocks as vertices, with two exceptions. The first of these exceptional cases occurs for M_{24} in characteristic 2 where FM_{24} has only one block. The simple FM_{24}-module D of dimension 1792 has the Sylow 2-subgroups of the commutator subgroups of the maximal subgroups of M_{24} with index 3795 as vertices. Moreover, each vertex of D is a maximal subgroup of some Sylow 2-subgroup of M_{24}. In characteristic 3 the principal block of FM_{24} contains a simple module of dimension 483 whose vertices are maximal subgroups of Sylow 3-subgroups of M_{24} and whose normalizers in M_{24} are isomorphic to the automorphism group of M_9 which is a split extension $M_9:\mathfrak{S}_3$.

The Mathieu groups M_{23} and M_{24} have no non-trivial outer automorphisms, whereas the automorphism group $M_{22}:2$ of M_{22} is a split extension. Moreover, while, by [9], both M_{23} and M_{24} have trivial Schur multipliers, M_{22} has a Schur multiplier which is cyclic of order 12. This has been shown by Mazet in [33]. In consequence, M_{22} admits covering groups of order

$n|M_{22}| = n\cdot 2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$ for $n \in \{2,3,4,6,12\}$, and each of these is unique up to isomorphism. We will also focus on these covering groups as well as on the automorphism group $M_{22}{:}2$ and the respective simple modules over a field F of characteristic 2 and 3, respectively. In analogy to the case of the Mathieu groups, we will determine the vertices of these simple modules, and give a description of their sources and Green correspondents in terms of Loewy series. We will see below that, for our purposes, only the covering groups $2.M_{22}$ and $4.M_{22}$ in characteristic 3, and the covering group $3.M_{22}$ in characteristic 2 need to be considered explicitly. Once having determined vertices, sources and Green correspondents for the simple modules in these cases, one obtains the respective results for the simple modules of the remaining covering groups of M_{22} as well.

Finally, we will also investigate the simple modules for the bicyclic extensions $n.M_{22}.2$ of M_{22} where $n \in \{2,3,4,6,12\}$. Here $n.M_{22}.2$ is an extension of the automorphism group $M_{22}{:}2$ of M_{22} by a cyclic group $C = \langle c \rangle$ of order n such that the outer automorphism of M_{22} lifts to an outer automorphism of $n.M_{22}$ mapping c to c^{-1}.

It turns out that all simple modules for the covering groups and the bicyclic extensions of M_{22} considered throughout this article have precisely the defect groups of their blocks as vertices, and in most cases this is a direct consequence of Knörr's Theorem [27]. Two of the blocks investigated here are particularly interesting, since all of their simple modules have trivial sources. We prove that both blocks are Morita equivalent to their Brauer correspondents. This confirms Broué's Abelian Defect Group Conjecture for these blocks, in a strong form.

Suppose that G is an arbitrary finite group and that D is a simple FG-module with vertex Q and trivial source. Then, by a theorem of Okuyama [36], the Green correspondent $f(D)$ of D in $N_G(Q)$ is again simple. Thus Q acts trivially on $f(D)$, and $f(D)$ can be viewed as a simple projective $F[N_G(Q)/Q]$-module. That is, the pair $(Q, [f(D)])$ is a weight in Alperin's sense [1]. In this way every simple FG-module with trivial source defines a unique conjugacy class of weights, and non-isomorphic simple FG-modules with trivial sources define different conjugacy classes of weights. Now suppose that B is a block of FG such that every simple FG-module belonging to B has trivial source. Then, in the way described above, we obtain an injection from the set of isomorphism classes of simple FG-modules belonging to B into the set of conjugacy classes of FG-weights belonging to B. According to Alperin's Weight Conjecture [1], this injection should actually be a bijection. It would be interesting to have a proof of this special case of Alperin's Weight Conjecture.

If B has abelian defect group P, and still every simple FG-module belonging to B has trivial source, then every simple FG-module belonging to B has vertex P, by Knörr's Theorem [27], and therefore the conjectured bijection of the previous paragraph should really be a one-to-one correspondence between the set of isomorphism classes of simple FG-modules belonging to B and the set of isomorphism classes of simple $FN_G(P)$-modules belonging to the Brauer correspondent b of B. In view of the examples in this paper and other examples we are wondering whether B and b have to be Morita equivalent in general.

We begin by introducing the notation used throughout and by summarizing some known facts which will be needed later in this article. Afterwards, in Section 3, we will present our results on the vertices, sources and Green correspondents of the simple modules under consideration.

In Section 4 we will then prove the existence of Morita equivalences between the blocks just mentioned above and their respective Brauer correspondents.

Most of our results are due to calculations with the computer algebra system MAGMA [4]. In this context we were faced with the task of constructing the simple modules for a finite group G over a finite field. MAGMA provides a function `AbsolutelyIrreducibleModulesBurnside` whose input are a finite permutation or matrix group `G` and a finite field `F`, and which constructs the absolutely simple modules of `G` over appropriate extension fields of `F`. According to the MAGMA instructions, the underlying algorithm is based on the Brauer-Burnside Theorem (cf. [5], [8]). The latter asserts that, given any faithful FG-module V, each indecomposable projective FG-module occurs as a direct summand of some tensor power of V. In particular, each simple FG-module then occurs as composition factor of some tensor power of V. Most of the simple modules investigated here have been constructed via the MAGMA function `AbsolutelyIrreducibleModulesBurnside`. The function considers modules for splitting only up to some dimension d. The bound d can be defined by the user, and is by default set to 2000. However, in order to get all simple modules of the given group, the chosen bound d might be too small so that MAGMA then does not return all simple modules. Moreover, for our purposes it is often not necessary to construct all simple modules of a group but only those in some particular block. Therefore, in some cases simple modules have been constructed somewhat more interactively. Details are given at the beginning of Section 3. In order to determine the Loewy series of the Green correspondents and the sources of the simple modules investigated, we repeatedly used the MAGMA function `JacobsonRadical` which computes the radical of a given module over a group algebra.

The programs for carrying out the actual vertex computation had been developed by R. Zimmermann in [43], and were later extended by R. Zimmermann and the authors in [16]. For details we refer the reader to [15], [16] and [43]. Moreover, the MAGMA source code of our algorithms as well as part of the results of this note are available on-line at
`http://users.minet.uni-jena.de/~susanned/`.
We have not yet developed actual code for computing the Green correspondent of a given indecomposable FG-module M. In most cases we just restricted M to the normalizer of some vertex P, and determined an indecomposable direct sum decomposition of this restriction applying the built-in MAGMA function `IndecomposableSummands`. Among these summands we then identified the unique one with vertex P. In the case of the simple FM_{24}-module of dimension 1243 in characteristic 3 we proceeded slightly differently; details are given in Section 3.8.

The information on blocks and decomposition numbers for the groups investigated has been taken from the GAP character table library [19] (see also
`http://www.math.rwth-aachen.de/homes/MOC/decomposition/`).

We would like to mention that sources and Green correspondents of simple modules for the Mathieu groups are also examined in [13] and [40] as has been pointed out by the referee.

2 Prerequisites

2.1 General notation

(1) In what follows, F will always denote a field of prime characteristic p, and all groups considered here are finite. Whenever A is an F-algebra then A is supposed to be finite-dimensional, associative and unitary, and any A-module is understood to be a finitely generated left A-module. Moreover, the endomorphism algebra $\operatorname{End}_A(V)$ is supposed to be acting on V from the left as well. By A° we denote the *opposite algebra* of A. Given A-modules V and W such that W is isomorphic to a direct summand of V, we write $W \mid V$.

(2) Suppose we are given groups G and H. For any FG-module V and any FH-module W, the outer tensor product $V \otimes_F W =: V \boxtimes W$ then becomes an $F[G \times H]$-module in the obvious way. Moreover, there is a canonical F-algebra isomorphism $F[G \times H] \longrightarrow FG \otimes_F FH$. Identifying $F[G \times H]$ and $FG \otimes_F FH$ via this isomorphism, each block of $F[G \times H]$ may be written as $B_1 \otimes B_2$, for some block B_1 of FG and some block B_2 of FH.

(3) When describing modules in terms of their Loewy series we will use the following notation: given any FG-module V of Loewy length $l \in \mathbb{N}$ such that

$$\operatorname{Rad}^{i-1}(V)/\operatorname{Rad}^i(V) = D_{i1} \oplus \cdots \oplus D_{ir_i},$$

for $i = 1, \ldots, l$, appropriate $r_i \in \mathbb{N}$, and simple FG-modules $D_{i1}, \ldots, D_{ir_i}$, we write

$$V \sim \begin{bmatrix} D_{11} \ldots D_{1r_1} \\ D_{21} \ldots D_{2r_2} \\ \vdots \\ D_{l1} \ldots D_{lr_l} \end{bmatrix}.$$

If G is a p-group then we will only record the dimensions of the layers of the Loewy series of V, since these then determine the Loewy structure of V.

Suppose we are given any groups G and H, an FG-module V and an FH-module W with the following properties:

- V and W have common Loewy length $l \in \mathbb{N}$ and
- there is some bijection φ between the set of isomorphism classes of composition factors of V and W such that $\operatorname{Rad}^{i-1}(V)/\operatorname{Rad}^i(V) \cong D_{i1} \oplus \cdots \oplus D_{ir_i}$ is equivalent to $\operatorname{Rad}^{i-1}(W)/\operatorname{Rad}^i(W) \cong \varphi(D_{i1}) \oplus \cdots \oplus \varphi(D_{ir_i})$, for $i = 1, \ldots, l$ and simple FG-modules $D_{i1}, \ldots, D_{ir_i}$.

Then we say that V and W have the *same Loewy structure*.

(4) For any integer $n \geq 1$ we denote the symmetric and alternating group, respectively, of degree n by $\mathfrak{S}_n$ and $\mathfrak{A}_n$, respectively. Moreover, by D^λ we understand the simple $F\mathfrak{S}_n$-module corresponding to the *p-regular partition* λ of n. For details concerning the representation theory of the symmetric groups we refer to [25].

2.2 Vertices, sources and the Green correspondence

Let G be a group, let V be an FG-module, and let H be a subgroup of G such that $V \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V))$. Then V is said to be relatively *H-projective*. In the case that V is indecomposable, a subgroup P of G which is minimal subject to the condition that V is relatively P-projective is called a *vertex* of V. The vertices of an indecomposable FG-module form a conjugacy class of p-subgroups of G. Moreover, if P is a vertex of an indecomposable FG-module V then there is an indecomposable FP-module W, unique up to isomorphism and conjugation with elements in $N_G(P)$, such that $V \mid \mathrm{Ind}_P^G(W)$. The module W is then called a *source* of V. For an introduction to the theory of vertices, sources and Green correspondents of indecomposable FG-modules we refer the reader to [35], Sections 4.3 and 4.4.

Suppose that D is a simple FG-module, and let P be a p-subgroup of G such that D is relatively P-projective. As far as the question of determining the vertices of D is concerned, by Knörr's Theorem [27], we may suppose further that D belongs to a block with non-abelian defect groups. In particular, D then cannot have cyclic vertices, by Erdmann's Theorem [17]. Clearly, $\mathrm{Res}_P^G(D)$ has an indecomposable direct summand whose vertices are also vertices of D and, as just mentioned, we may ignore all those indecomposable direct summands of $\mathrm{Res}_P^G(D)$ which have cyclic vertices. In [16] an algorithm is presented which detects and removes such summands from $\mathrm{Res}_P^G(D)$ without computing an explicit indecomposable direct sum decomposition first. We have applied this algorithm several times throughout our computations in order to make these less time and memory consuming. In the following, we will write $V = \mathrm{cyc}$ when V is an FG-module all of whose indecomposable direct summands have cyclic vertices. If V is projective then we write $V = \mathrm{proj}$.

Consider, for instance, the simple FM_{24}-module $D(483)_{24}$ of dimension 483 in characteristic 3 which occurs in Section 3.8. Let P be a Sylow 3-subgroup of M_{24}. Then $\mathrm{Res}_P^{M_{24}}(D(483)_{24}) = U_1 \oplus U_2 \oplus \mathrm{cyc}$. Here U_1 and U_2 are indecomposable of dimension 3 each, and both have elementary abelian vertices of order 9. Restricting $D := D(483)_{24}$ to P, splitting off all the indecomposable direct summands with cyclic vertices, and determining the vertices of U_1 and U_2 required 11MB of main memory and took roughly 10 seconds. Calling the function `VxStart(D,P)` (cf. [43]) directly required 2350MB of main memory and took 15 hours.

For another example, let again be $p = 3$, and consider the simple FM_{23}-module $D := D(770)_{23}$ of dimension 770 occuring in Section 3.7. Then D belongs to the principal block. If $P \in \mathrm{Syl}_3(M_{23})$ then P is elementary abelian of order 9. By Knörr's Theorem, D has thus vertex P. Moreover, $\mathrm{Res}_P^{M_{23}}(D) \cong F \oplus F \oplus \mathrm{cyc}$. So D has trivial source. Restricting D to P, and splitting off all indecomposable direct summands with cyclic vertices took 34 seconds and required 13MB of main memory. When trying to decompose $\mathrm{Res}_P^{M_{23}}(D)$ using the MAGMA function `IndecomposableSummands`, after 13 minutes MAGMA ran out of memory (16GB).

We mention that algorithms for splitting projective direct summands off a given module over some p-group can also be found in [11], [13] and [39].

Remark 2.1. Among the known criteria for testing a given FG-module V for relative projectivity with respect to subgroups of G, especially *Higman's criterion* which involves the relative trace map has proved to be a valuable tool (cf. [35], Thm. 4.2.2). Moreover, when H is a normal subgroup of G such that $|G : H| = p^s$, for some $s \in \mathbb{N}$, also the result below is often helpful for testing an FG-module for relative projectivity with respect to H. For this recall that the isomorphism classes of indecomposable direct summands of an FG-module V bijectively correspond to the simple $\mathrm{End}_{FG}(V)$-modules, via *Fitting correspondence.* More precisely, consider a fixed decomposition

$$V \cong a_1V_1 \oplus \cdots \oplus a_rV_r,$$

with mutually non-isomorphic indecomposable FG-modules $V_1, \ldots, V_r$, and corresponding multiplicities $a_1, \ldots, a_r \in \mathbb{N}$. Furthermore, set $E := \mathrm{End}_{FG}(V)$, and let

$${}_EE \cong b_1Ee_1 \oplus \cdots \oplus b_nEe_n$$

be a decomposition of the left E-module E, with pairwise orthogonal non-associate primitive idempotents $e_1, \ldots, e_n$ in E and multiplicities $b_1, \ldots, b_n \in \mathbb{N}$. Then $r = n$, and after appropriate re-numbering, we have $V_i \cong e_i(V) = e_iV$ and $a_i = b_i$, for $i = 1, \ldots, n$. This follows from [35], Thm. 1.5.4.

For each $f \in E$, denote its image under the natural epimorphism $E \longrightarrow \overline{E} := E/\mathbf{J}(E)$ by $\bar{f}$. Then, by [35], Thm. 1.4.5, $\{\overline{E}\bar{e}_1, \ldots, \overline{E}\bar{e}_n\}$ is a transversal for the isomorphism classes of simple E-modules, and each of these occurs as composition factor of the E-module V. With this notation, we obtain:

Proposition 2.2. *Let V be an FG-module such that*

$$V \cong a_1V_1 \oplus \cdots \oplus a_nV_n,$$

with pairwise non-isomorphic indecomposable FG-modules $V_1, \ldots, V_n$ and respective multiplicities $a_1, \ldots, a_n \in \mathbb{N}$. As above, let $\{\overline{E}\bar{e}_1, \ldots, \overline{E}\bar{e}_n\}$ be the corresponding transversal for the isomorphism classes of simple E-modules, and set $d_i := [V : \overline{E}\bar{e}_i]$, for $i = 1, \ldots, n$. Moreover, let $E_i := \mathrm{End}_E(\overline{E}\bar{e}_i)$, for $i = 1, \ldots, n$. Then $a_i \cdot \dim_F(E_i) = \dim_F(\overline{E}\bar{e}_i)$, and $d_i \cdot \dim_F(E_i) = \dim_F(V_i)$, for $i = 1, \ldots, n$.

Proof. Let $i \in \{1, \ldots, n\}$. In the notation of Remark 2.1, $a_i = b_i$ which equals the multiplicity of the simple E-module $\overline{E}\bar{e}_i$ as composition factor of the E-module ${}_E\overline{E}$. By Wedderburn's Theorem, this multiplicity in turn equals $\dim_{E_i}(\overline{E}\bar{e}_i) = \dim_F(\overline{E}\bar{e}_i)/\dim_F(E_i)$. Therefore, $a_i \dim_F(E_i) = \dim_F(\overline{E}\bar{e}_i)$.

Furthermore, we have

$$E_i = \mathrm{End}_E(\overline{E}\bar{e}_i) \cong (\bar{e}_i\overline{E}\bar{e}_i)^\circ \cong ((e_iEe_i)/\mathbf{J}(e_iEe_i))^\circ,$$

so that the F-algebra e_iEe_i possesses exactly one simple module which has F-dimension $\dim_F(E_i)$. By [35], Thm. 1.8.11, the multiplicity $d_i = [V : \overline{E}\bar{e}_i]$ equals the composition length of the e_iEe_i-module e_iV. Consequently, $d_i \dim_F(E_i) = \dim_F(e_iV) = \dim_F(V_i)$. This proves the proposition. □

Corollary 2.3. *In the notation of Proposition 2.2, assume that $\overline{E}\bar{e}_i$ is absolutely simple, for some $i \in \{1, \ldots, n\}$. Then $a_i = \dim_F(\overline{E}\bar{e}_i)$, and $d_i = \dim_F(V_i)$. Moreover, the FG-module V_i is then absolutely indecomposable. In particular, if the E-module V has up to isomorphism precisely one composition factor and if this composition factor is one-dimensional then V is absolutely indecomposable.*

Proof. Consider any extension field F' of F. Then

$$\mathrm{End}_{F'G}(F' \otimes_F V) \cong F' \otimes_F \mathrm{End}_{FG}(V) \cong b_1(F' \otimes_F Ee_1) \oplus \cdots \oplus b_n(F' \otimes_F Ee_n),$$

by [35], Thm. 1.11.12. Hence the FG-module V_i is absolutely indecomposable if and only if the E-module Ee_i is absolutely indecomposable, or equivalently, if the E-module $\overline{E}\bar{e}_i$ is absolutely simple. The rest now follows from the previous proposition. ☐

Corollary 2.4. *Let H be a normal subgroup of G such that $|G : H| = p^s$, for some $s \in \mathbb{N}$. Moreover, let V be an indecomposable FG-module such that the $\mathrm{End}_{FH}(V)$-module $\mathrm{Res}^G_H(V)$ has an absolutely simple composition factor occuring with multiplicity different from $\dim(V)/p^s$. Then V is not relatively H-projective.*

Proof. If V were relatively H-projective then we would have $\mathrm{Res}^G_H(V) \cong k(W_1 \oplus \cdots \oplus W_m)$, for mutually non-isomorphic and in G conjugate indecomposable FH-modules $W_1, \ldots, W_m$. This follows from [24], Thm. VII.9.3. Moreover, $V \mid \mathrm{Ind}^G_H(W_i)$, for $i = 1, \ldots, m$. By our hypothesis and Corollary 2.3, $W_1, \ldots, W_m$ are absolutely indecomposable so that $V \cong \mathrm{Ind}^G_H(W_i)$ for $i = 1, \ldots, m$, by Green's Indecomposability Theorem [21]. This yields the contradiction $\dim(V) = p^s \dim(W_i) \neq \dim(V)$, and the assertion follows. ☐

Remark 2.5. (a) Corollary 2.4 is helpful for computational purposes, especially in the following situation: suppose that G is a p-group and that V is an indecomposable FG-module which we suspect to have vertex G. Let $\{H_1, \ldots, H_m\}$ be a transversal for the conjugacy classes of maximal subgroups of G. In order to verify that G is a vertex of V, it suffices to successively restrict V to each of the maximal subgroups H_i, compute the endomorphism algebras $E_i := \mathrm{End}_{FH_i}(\mathrm{Res}^G_{H_i}(V))$, and search for an absolutely simple composition factor of the FE_i-module $\mathrm{Res}^G_{H_i}(V)$ occuring with multiplicity different from $\dim_F(V)/p$. This enables us to avoid the computation of the relative trace maps $\mathrm{Tr}^G_{H_i} : \mathrm{End}_{FH_i}(\mathrm{Res}^G_{H_i}(V)) \longrightarrow \mathrm{End}_{FG}(V)$, for $i = 1, \ldots, m$, which is required when applying Higman's criterion.

(b) Let G be arbitrary again. Suppose that F' is any extension field of F, and let V be an indecomposable FG-module with vertex P. Then P is also a vertex of each indecomposable direct summand of the $F'G$-module $F' \otimes_F V$. For a proof see for instance [18], L. II.4.14. Each simple module D investigated in this paper has been constructed over a finite field $\mathbb{F}_q$, for some p-power q, such that:

- D is absolutely simple,
- the sources and the Green correspondent of D are absolutely indecomposable,
- the composition factors of the Green correspondent of D are absolutely simple.

In order to check whether a given indecomposable $\mathbb{F}_qG$-module is absolutely indecomposable, we applied the criterion given by Corollary 2.3. Hence, the $\overline{\mathbb{F}_q}G$-module $\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} D$ is also simple and has the same vertices as D. Moreover, if V and L are the Green correspondent and a source of D then both $\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} V$ and $\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} L$ are indecomposable, and are the Green correspondent and a source of the $\overline{\mathbb{F}_q}G$-module $\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} D$. Also the Loewy structures of the modules in question do not change under this field extension. Therefore, unless stated otherwise, for the remainder of this article we may assume the field F to be algebraically closed.

In the sequel, we will have to deal with FG-modules which are obtained from modules for factor groups of G via inflation. Therefore, recall that, given a normal subgroup N of G and an indecomposable $F[G/N]$-module $\overline{V}$, the inflation $V := \mathrm{Inf}_N^G(\overline{V})$ is an indecomposable FG-module. Moreover, if P is a vertex of V then PN/N is a vertex of $\overline{V}$. A proof for this can, for instance, be found in [31], Prop. 2.1. In particular, $P \cap N$ is then a Sylow p-subgroup of N, and P is a Sylow p-subgroup of PN which implies $N_{G/N}(PN/N) = N_G(P)N/N$. In analogy, also the Green correspondents and the sources of $\overline{V}$ and V are related as follows:

Proposition 2.6. *Let N be a normal subgroup of G, and let V be an indecomposable FG-module such that $V = \mathrm{Inf}_N^G(\overline{V})$, for an indecomposable $F[G/N]$-module $\overline{V}$. Suppose that V has vertex P so that $N_{G/N}(PN/N) = N_G(P)N/N$. Let V' be the Green correspondent of V with respect to $(G, P, N_G(P))$, and let $\overline{V'}$ be the Green correspondent of $\overline{V}$ with respect to $(G/N, PN/N, N_G(P)N/N)$. Then $V' \cong \mathrm{Res}_{N_G(P)}^{N_G(P)N}(\mathrm{Inf}_N^{N_G(P)N}(\overline{V'}))$. If L is a P-source of V then there is a PN/N-source $\overline{L}$ of $\overline{V}$ such that $L \cong \mathrm{Res}_P^{PN}(\mathrm{Inf}_N^{PN}(\overline{L}))$. In particular, the Loewy structures of V' and $\overline{V'}$, as well as those of L and $\overline{L}$ coincide.*

For a proof of Proposition 2.6 see [22].

Proposition 2.7. *Let N be a normal subgroup of G, and let V be a simple FG-module with vertex $Q \leq N$. Suppose that $|N_G(Q) : N_N(Q)| = |G : N|$, and denote the Green correspondence with respect to $(G, Q, N_G(Q))$ by f_1 and the Green correspondence with respect to $(N, Q, N_N(Q))$ by f_2. Consider an indecomposable direct sum decomposition $\mathrm{Res}_N^G(V) = V_1 \oplus \cdots \oplus V_n$. Then Q is a vertex of V_i for $i = 1, \ldots, n$, and*

$$\mathrm{Res}_{N_N(Q)}^{N_G(Q)}(f_1(V)) \cong f_2(V_1) \oplus \cdots \oplus f_2(V_n).$$

Proof. We set $H := N_G(Q)$ and $K := N_N(Q)$. Then $|G : H| = |N : K|$ so that the G-conjugates of Q are precisely the N-conjugates of Q. By Clifford's Theorem [35], Thm. 3.3.1, we have

$$\mathrm{Res}_N^G(V) \cong k(W_1 \oplus \cdots \oplus W_m),$$

for some $k \in \mathbb{N}$, and simple FN-modules $W_1, \ldots, W_m$ which are pairwise conjugate in G. We may assume that W_1 has vertex Q, and since the G-conjugates of Q are exactly the N-conjugates of Q, also $W_2, \ldots, W_m$ have vertex Q. Furthermore,

$$\begin{aligned}\mathrm{Res}_H^G(V) &= f_1(V) \oplus X_1 \oplus \cdots \oplus X_r,\\ \mathrm{Res}_K^N(W_i) &= f_2(W_i) \oplus Y_{i1} \oplus \cdots \oplus Y_{is_i}, \quad \text{for } i = 1, \ldots, m.\end{aligned}$$

Here, $X_1, \ldots, X_r$ and $Y_{i1}, \ldots, Y_{is_i}$ are indecomposable, and neither of these modules has vertex Q. Now consider

$$\begin{aligned}k(f_2(W_1) \oplus \cdots \oplus f_2(W_m) \oplus \bigoplus_{i=1}^{m}(Y_{i1} \oplus \cdots \oplus Y_{is_i})) &\cong \mathrm{Res}_K^G(V)\\ &= \mathrm{Res}_K^H(f_1(V)) \oplus \mathrm{Res}_K^H(X_1) \oplus \cdots \oplus \mathrm{Res}_K^H(X_r),\end{aligned}$$

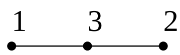
and assume that $k(f_2(W_1) \oplus \cdots \oplus f_2(W_m)) \nmid \mathrm{Res}_K^H(f_1(V))$. Then there are some $i \in \{1, \ldots, m\}$ and some $j \in \{1, \ldots, r\}$ such that $f_2(W_i) \mid \mathrm{Res}_K^H(X_j)$. In particular, Q is in H conjugate to a subgroup of some vertex of X_j, that is Q is itself a vertex of X_j, a contradiction. Consequently,

$k(f_2(W_1) \oplus \cdots \oplus f_2(W_m)) \mid \mathrm{Res}_K^H(f_1(V))$. Since $f_1(V)$ is relatively K-projective, $\mathrm{Res}_K^H(f_1(V))$ is a direct sum of indecomposable FK-modules which are pairwise conjugate in H. This follows from [24], Thm. VII. 9.3. In particular, all indecomposable direct summands of $\mathrm{Res}_K^H(f_1(V))$ have vertex Q, and this finally yields

$$f_2(V_1) \oplus \cdots \oplus f_2(V_n) \cong k(f_2(W_1) \oplus \cdots \oplus f_2(W_m)) \cong \mathrm{Res}_K^H(f_1(V)).$$

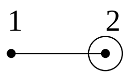
□

Remark 2.8. Suppose that B is a block of FG with cyclic defect group C. The representation theory of blocks with cyclic defect groups is well-understood. In particular, such a block B is a *Brauer tree algebra*. We refer to [23] for an introduction to the theory of Brauer trees. Given the Brauer tree of B and the Green correspondent of one of the simple modules in B, one can also determine the Green correspondents of the remaining simple modules in B. This is shown in [23], Sec. 4.4. In fact, if B is one of the blocks with cyclic defect groups investigated in this note then B occurs in characteristic 3, and has a defect group C of order 3. In this special case, B contains either one or two simple FG-modules, up to isomorphism. Moreover, by [23], Sections 6.4, 6.6 and 6.9, the Brauer tree of B is either of shape

(1) 

,

where the node 3 is of type $\circ$ and the nodes 1 and 2 are of type $\times$, or of shape

(2) 

,

where the *exceptional node* 2 has type $\circ$, and the node 1 has type $\times$. In either case, [23], L. 4.4.12 implies that all simple modules belonging to B have simple Green correspondents in $N_G(C)$. Hence, by [35], Thm. 4.7.8, these simple modules must have trivial sources.

We also mention a misprint in [23]. In the Brauer tree of the block B_8 of $F[12.M_{22}]$ in characteristic 3, the labelling of the nodes by $\times$ and $\circ$ should be interchanged. This block has a central defect group of order 3; its Brauer tree is of shape (2), and also here the exceptional node has type $\circ$, and the non-exceptional node has type $\times$.

2.3 Group extensions

In this subsection we briefly recall some facts concerning the covering groups of a finite group. For more details and proofs of the results quoted here, we refer to [3], Sec. 33. A *covering group* of a group G is a group $\widetilde{G}$ containing a central subgroup $\widetilde{H}$ such that $\widetilde{H} \leq \widetilde{G}'$ and $\widetilde{G}/\widetilde{H} \cong G$. Here $\widetilde{G}'$ denotes the commutator subgroup of $\widetilde{G}$. Provided that $\widetilde{H}$ has order 2, 3, etc, one also speaks of $\widetilde{G}$ as a *double cover*, *triple cover*, etc, of G.

Now suppose that G is a *perfect group*, i.e. G equals its commutator subgroup G'. Then there exists a covering group $\widetilde{G}$ of G such that each covering group of G is isomorphic to a factor group of $\widetilde{G}$. Furthermore, $\widetilde{G}$ is determined up to isomorphism, and is called the *Schur cover* or *Darstellungsgruppe* of G. Moreover, $\widetilde{G}$ is itself a perfect group. If $\widetilde{H} \leq Z(\widetilde{G})$ with $\widetilde{G}/\widetilde{H} \cong G$ then $\widetilde{H}$ is isomorphic to the *Schur multiplier* of G.

Remark 2.9. As mentioned in the introduction, the Schur multiplier of M_{22} is cyclic of order 12, and M_{22} therefore possesses covering groups of order $2|M_{22}|, 3|M_{22}|, 4|M_{22}|, 6|M_{22}|$ and $12|M_{22}|$. We will use the Atlas notation, and write $n.M_{22}$ to denote the covering group of M_{22} of order $n|M_{22}| = n \cdot 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, for $n \in \{2, 3, 4, 6, 12\}$.

Note that, as far as the determination of vertices of simple $F[n.M_{22}]$-modules is concerned, we only need to consider the cases where n is coprime to $p = \mathrm{char}(F)$. Namely, if $n = p^r q$, for some $q, r \in \mathbb{N}$ such that $p \nmid q$, then there is a cyclic subgroup $Z \leq Z(n.M_{22})$ of order p^r such that $n.M_{22}/Z \cong q.M_{22}$. Moreover, by [35], Thm. 4.7.8, every simple $F[n.M_{22}]$-module D corresponds to a simple $F[q.M_{22}]$-module D', via inflation. If $P \geq Z$ is a vertex of D then P/Z is a vertex of D', by [31], Prop. 2.1. Furthermore, in consequence of Proposition 2.6 the Loewy structures of the Green correspondents of D and D' coincide. The same applies to the sources of D and D'. Therefore, we only need to investigate the simple $F[3.M_{22}]$-modules for $p = 2$, and the simple $F[2.M_{22}]$-modules and the simple $F[4.M_{22}]$-modules for $p = 3$ explicitly.

Remark 2.10. There are five bicyclic extensions of M_{22}, namely $2.M_{22}.2$, $3.M_{22}{:}2$, $4.M_{22}.2$, $6.M_{22}.2$ and $12.M_{22}.2$. Let $C_{12} \cong C = \langle c \rangle$ be the centre of $12.M_{22}$. The outer automorphism of M_{22} lifts to an outer automorphism of $12.M_{22}$ which maps c to c^{-1}. In particular, $12.M_{22}.2$ has a centre of order 2. If $p = 2$ then we only need to consider the simple modules for $F[3.M_{22} : 2]$. For $n \in \{2, 4, 6, 12\}$ any simple $F[n.M_{22}.2]$-module D is the inflation of a simple module D' for $F[M_{22}{:}2]$ or $F[3.M_{22}{:}2]$, by [35], Thm. 4.7.8. Moreover, in these cases Proposition 2.6 implies that the vertices as well as the Loewy structures of the sources and the Green correspondents of D can be deduced from those of D'.

Analogously, in the case where $p = 3$, it suffices to focus on the simple modules for $F[2.M_{22}.2]$ and $F[4.M_{22}.2]$, respectively.

2.4 Equivalences between module categories

In Section 4 we will prove the existence of Morita equivalences between the faithful 3-block of $2.M_{22}$ of defect 2 and its Brauer correspondent, and between the faithful 3-block of $2.M_{22}.2$ of defect 2 and its Brauer correspondent. For this we now briefly recall some facts concerning equivalences between module categories. For more details on this subject we refer to [2].

Given any F-algebra A, we denote the category of finitely generated left A-modules by $A-\mathbf{mod}$. Furthermore, $A-\mathbf{stab}$ denotes the stable category of A. That is, the objects in $A-\mathbf{stab}$ are the same as the objects in $A-\mathbf{mod}$. But the morphisms between objects V and W in $A-\mathbf{stab}$ are equivalence classes of A-homomorphisms between V and W modulo A-homomorphisms which factor through projective A-modules.

Definition 2.11. Consider F-algebras A and B.

(i) The algebras A and B are called *Morita equivalent* if and only if there exist an A-B-bimodule V and a B-A-bimodule W such that $V \otimes_B W \cong A$ as A-A-bimodules and $W \otimes_A V \cong B$ as B-B-bimodules.

(ii) We say that there exists a *stable equivalence of Morita type* between A and B if and only if there exist an A-B-bimodule V and a B-A-bimodule W such that the following conditions are satisfied:

- V is projective both as left A-module and right B-module.
- W is projective both as left B-module and right A-module.
- $V \otimes_B W \cong A \oplus X$ as A-A-bimodules, where X is a projective A-A-bimodule.
- $W \otimes_A V \cong B \oplus Y$ as B-B-bimodules, where Y is a projective B-B-bimodule.

Remark 2.12. By Morita's Theorem, the F-algebras A and B are Morita equivalent if and only if the module categories $A-\mathbf{mod}$ and $B-\mathbf{mod}$ are equivalent. If there exists a stable equivalence of Morita type between A and B then the stable categories $A-\mathbf{stab}$ and $B-\mathbf{stab}$ are equivalent.

With this notation, the following result will be essential:

Theorem 2.13 (Linckelmann [32], Thm. 2.1)**.** *Let A and B be indecomposable non-simple symmetric F-algebras, and let W be a B-A-bimodule which is projective both as left B-module and right A-module. Suppose further that the functor $W \otimes_A -$ induces a stable equivalence between A and B. Then the following hold:*

(i) If W is indecomposable and if D is a simple A-module, then $W \otimes_A D$ is an indecomposable non-projective B-module.

(ii) If, for every simple A-module D, also $W \otimes_A D$ is a simple B-module then the functor $W \otimes_A -$ induces a Morita equivalence between A and B.

Remark 2.14. We consider subgroups $K \leq H \leq G$ and an FG-module V. Then V^K and V^H will denote the F-vector spaces of fixed points under the action of K and H, respectively, on V. Moreover, for $v \in V^K$, we define

$$\mathrm{Tr}_K^H(v) := \sum_{gK \in H/K} gv.$$

This yields an F-linear map $\mathrm{Tr}_K^H : V^K \longrightarrow V^H$ which does not depend on the choice of the transversal for H/K, and is called *relative trace.*

Definition 2.15. Let P be a p-subgroup of G, and let V be an FG-module. Then we set

$$V(P) := V^P / (\sum_{Q<P} \mathrm{Tr}_Q^P(V^Q)).$$

Remark 2.16. (a) In this way we obtain a functor

$$FG - \mathbf{mod} \longrightarrow FN_G(P) - \mathbf{mod},\ V \longmapsto V(P),$$

which is called the *Brauer functor* or the *Brauer construction with respect to P*. In the following we list some known facts concerning the Brauer functor. For details we refer to [6] and [41] §11, §27.

(b) The Brauer functor is additive. That is, if V_1 and V_2 are FG-modules then the canonical map $V_1(P) \oplus V_2(P) \longrightarrow (V_1 \oplus V_2)(P)$ is an isomorphism of $FN_G(P)$-modules. We identify $V_1(P) \oplus V_2(P)$ and $(V_1 \oplus V_2)(P)$ in this way. Also, we have a canonical homomorphism $V_1(P) \otimes_F V_2(P) \longrightarrow (V_1 \otimes_F V_2)(P)$.

(c) Suppose that A is a G-algebra over F. That is, A is both an F-algebra and an FG-module, and for $g \in G$ the map $A \longrightarrow A, a \longmapsto ga$, is an F-algebra automorphism of A. Then the canonical map

$$A(P) \otimes_F A(P) \longrightarrow (A \otimes_F A)(P) \longrightarrow A(P)$$

turns $A(P)$ into an $N_G(P)$-algebra over F.

(d) Let A be a G-algebra over F, and let V be an A-module. Suppose further that V is an FG-module and that the canonical map $A \otimes_F V \longrightarrow V$ is an FG-homomorphism. Then the induced map

$$A(P) \otimes_F V(P) \longrightarrow (A \otimes_F V)(P) \longrightarrow V(P)$$

gives an $A(P)$-module structure on $V(P)$ which is compatible with the action of $N_G(P)$.

(e) Suppose that V is a *p-permutation FG-module*. That is, for every p-subgroup P of G there is an F-basis of V stabilized by P. Let further P and Q be p-subgroups of G such that $Q \trianglelefteq P$. Then $V(P) \cong (V(Q))(P)$. Note that $V(Q)$ is a p-permutation $FN_G(Q)$-module. In particular, $V(P) \neq 0$ then also implies $V(Q) \neq 0$.

(f) Let V_1 and V_2 be FG-modules such that V_1 or V_2 is a p-permutation FG-module. Then the canonical map $V_1(P) \otimes_F V_2(P) \longrightarrow (V_1 \otimes_F V_2)(P)$ is an isomorphism.

(g) The group algebra FG is both a G-algebra over F and a p-permutation FG-module. Furthermore, its Brauer construction $(FG)(P)$ is canonically isomorphic to $FC_G(P)$, as an $N_G(P)$-algebra over F. We identify $(FG)(P)$ and $FC_G(P)$ in this way. If $FG = FGe_1 \oplus \cdots \oplus FGe_r$ is the block decomposition of FG then, via this identification and additivity, we get

$$(FG)(P) = (FGe_1)(P) \oplus \cdots \oplus (FGe_r)(P) = FC_G(P)\,\mathrm{Br}_P(e_1) \oplus \cdots \oplus FC_G(P)\,\mathrm{Br}_P(e_r),$$

where Br_P is the usual Brauer homomorphism. Thus the Brauer construction of a block is either 0 or a direct sum of blocks of $FC_G(P)$.

In the following, for each subgroup H of G, we denote the respective diagonal subgroup of $G \times G$ by $\Delta H := \{(h,h) \mid h \in H\}$. Then the next result will play a crucial role in Section 4.

Theorem 2.17 (Broué [7], Thm. 6.3)**.** *Let G and H be groups with a common Sylow p-subgroup P such that $N_G(P)/C_G(P) \cong N_H(P)/C_H(P)$. Moreover, let e and f be central idempotents in FG and FH, respectively, and set $A := FGe$ and $B := FHf$. We suppose that there are an A-B-bimodule V and a B-A-bimodule W such that*

(1) $V \mid \mathrm{Ind}_{\Delta P}^{G\times H}(X)$, for some permutation $F[\Delta P]$-module X.

(2) If Q is a non-trivial subgroup of P then $FC_G(Q)\,\mathrm{Br}_Q(e)$ and $FC_H(Q)\,\mathrm{Br}_Q(f)$ are Morita equivalent via the functors $W(\Delta Q) \otimes_{FC_G(Q)\,\mathrm{Br}_Q(e)} -$ and $V(\Delta Q) \otimes_{FC_H(Q)\,\mathrm{Br}_Q(f)} -$.

Then the functors $W \otimes_A -$ and $V \otimes_B -$ induce a stable equivalence of Morita type between A and B.

3 Results

3.1 Constructing simple modules

We now present our results most of which have been obtained with computer assistance.

In the following, we consider $M_{22} < M_{23} < M_{24} < \mathfrak{S}_{24}$ and $M_{22} \lhd M_{22} : 2 < \mathfrak{S}_{24}$. For $j \in \{22, 23, 24\}$, by $D(d_i)_j$ we understand the ith simple FM_j-module of dimension d. We omit the index i whenever FM_j has, up to isomorphism, exactly one simple module or two mutually dual simple modules of dimension d. Analogously, the simple $F[M_{22}:2]$-modules are denoted by $D(d_i)_{22:2}$, and for $n \in \{2, 3, 4, 6, 12\}$, the simple $F[n.M_{22}]$-modules and the simple $F[n.M_{22}.2]$-modules, respectively, are denoted by $D(d_i)_{n.22}$ and $D(d_i)_{n.22.2}$, respectively.

A permutation representation of $2.M_{22}$ on 660 points and two permutation representations of $4.M_{22}$ on 4928 points are given in [42]. In the case of $4.M_{22}$ we have worked with the first of those two permutation representations. The simple modules for $4.M_{22}$ in characteristic 3 investigated here all appear to be composition factors of the corresponding permutation module over $\mathbb{F}_9$. The simple modules for $2.M_{22}$ in characteristic 3 have been constructed via the MAGMA function `AbsolutelyIrreducibleModulesBurnside`. To obtain the simple modules for $2.M_{22}.2$ in characteristic 3, we proceeded as follows: we started with the 10-dimensional irreducible matrix representation of $2.M_{22}.2$ over $\mathbb{F}_3$ available at [42]. The GAP function `IsomorphismPermGroup` then enables us to construct a permutation representation on 9240 points of the respective matrix group. This also yields a permutation representation of $2.M_{22} \lhd 2.M_{22}.2$ on 9240 points. The simple $\mathbb{F}_9[2.M_{22}.2]$-modules investigated in Subsection 3.6 could then be obtained as composition factors of inductions of simple $\mathbb{F}_9[2.M_{22}]$-modules to $2.M_{22}.2$.

In order to construct the simple modules for $3.M_{22}$ in characteristic 2, we have taken the permutation representation for $3.M_{22}$ on 693 points from [42]. Then each of the non-projective faithful simple $\mathbb{F}_4[3.M_{22}]$-modules occurs as composition factor of the corresponding permutation module over $\mathbb{F}_4$.

The simple $\mathbb{F}_9 M_{24}$-modules $D(45)_{24}$ and $D(990)_{24}$ of dimension 45 and 990, respectively, have been obtained as composition factors of the induction of one of the 45-dimensional simple $\mathbb{F}_9 M_{23}$-modules. The remaining simple $\mathbb{F}_9 M_{24}$-modules have been constructed as composition factors of appropriate tensor powers of the simple $\mathbb{F}_9 M_{24}$-module $D(22)_{24}$ which is the unique non-trivial composition factor of the natural $\mathbb{F}_9 M_{24}$-permutation module of dimension 24.

All remaining simple modules considered below have actually been constructed via the MAGMA function `AbsolutelyIrreducibleModulesBurnside`. It turns out that all modules considered throughout, that is all simple modules as well as their sources and Green correspondents can be realized over $\mathbb{F}_{p^2}$, and that they are then also absolutely indecomposable. Absolute indecomposability has always been checked using the criterion given by Corollary 2.3.

3.2 Determining Brauer characters

For most of the simple modules occuring in this article we also determined their corresponding Brauer characters. In order to do this we proceeded as follows: suppose that $n \geq 1$, and let $q := p^n$. For our purposes n is at most 2. Then, following the MAGMA instructions, calling the function `GF(q)` the finite field $\mathbb{F}_q$ is constructed as $\mathbb{F}_p[x]/(C_n)$. Here C_n denotes the Conway polynomial of degree n. This is consistent with the ATLAS, see [26], Sec. 3. The image of x in $\mathbb{F}_p[x]/(C_n)$ is denoted by $\mathbb{F}_q.1$. If $n = 1$ then $\mathbb{F}_p.1 = 1$.

With this notation, $\mathbb{F}_q.1$ is a generator of the multiplicative group $\mathbb{F}_q^\times$. Let further $\zeta := \exp(2\pi i/(q-1))$. Then the lifting map

$$\mathbb{Z}[\zeta] \longrightarrow \mathbb{F}_q$$

used in the ATLAS is those mapping ζ to $\mathbb{F}_q.1$ (cf. [26], Sec. 2, 3).

In [42], standard presentations of the groups examined here are given. If in addition representatives for the p-regular conjugacy classes as words in the standard generators are given then one can identify the Brauer character of a given simple module.

Example 3.1. Let $p = 2$, and consider the Mathieu group M_{22}. Standard generators of M_{22} are a and b where a has order 2, b has order 4 and belongs to conjugacy class $4A$, and where both ab and $ababb$ have order 11. By [42], $ababababbbabb$ belongs to conjugacy class $7A$, and ab belongs to conjugacy class $11A$. The simple modules of dimension 10 occuring in Section 3.3 can be realized over $\mathbb{F}_2$, the ones of dimension 70 can be realized over $\mathbb{F}_4$. Using the standard permutation representation of M_{22} on 22 points from [42], and denoting Brauer characters as well as irrationalities in accordance with the ATLAS, we get:

module	conj. class	modular character value
$D(10)_{22}$	7A	$0 = \overline{b7}$
$D(70)_{22}$	11A	$\mathbb{F}_4.1^2 = \mathbb{F}_4.1 - 1 = \overline{-1 + b11}$

Hence, by [26], $D(10)_{22}$ has Brauer character φ_2, and $D(70)_{22}$ has Brauer character φ_5. This also determines the labels of the Brauer characters of the simple modules $D(10)_{22}^*$, $D(70)_{22}^*$, $D(34)_{22}$ and $D(98)_{22}$.

In characteristic 3, the double cover $2.M_{22}$ has two simple modules of dimension 154. The values of the corresponding Brauer characters coincide, except on the conjugacy classes of elements of order 8. There are two of the latter both of which are lying above the unique conjugacy class of elements of order 8 of M_{22}. Similarly, in characteristic 3, the covering group $4.M_{22}$ has two dual pairs of simple modules of dimension 56. Also here, in order to determine the labels of the Brauer characters afforded by these modules, one needs to know their values on the four conjugacy classes lying above the unique conjugacy class of elements of order 8 of M_{22}. There does not seem to be an obvious way to distinguish these conjugacy classes of elements of order 8. Moreover, in order to distinguish between the faithful blocks B_5 and $B_6 = B_5^*$ of $F[4.M_{22}]$ of defect 2, one has to make a choice for a generator for the central subgroup of $4.M_{22}$ of order 4. Therefore we do not determine the precise Brauer character labels of the 154-dimensional simple $F[2.M_{22}]$-modules belonging to the faithful block of defect 2, and we do not determine the precise Brauer character labels of the simple $F[4.M_{22}]$-modules belonging to the faithful blocks B_5 and B_6 just mentioned. For similar reasons we do not deduce the precise labelling of the Brauer characters of the simple $F[2.M_{22}.2]$-modules in characteristic 3 either.

In order to distinguish simple modules of equal dimension, in these three cases, we give elements in the respective groups, written as words in the standard generators, on which the Brauer character values of the modules in question differ. In addition, whenever we have simple modules of equal dimension for which representing matrices are available at [42], we attach to each of these its "ID" as given in [42].

Throughout we use the notation $D \leftrightarrow \varphi \leftrightarrow a$ to indicate that a simple module D has Brauer character φ, and ID a in [42]. The Brauer character afforded by the dual module D^* is denoted by $\overline{\varphi}$. Which conjugacy classes we actually used in order to determine the Brauer characters of the simple modules are listed at
`http://users.minet.uni-jena.de/~susanned/mathieu.html`.

In view of Remark 2.5 and the considerations in Subsection 3.1, from now on we may again suppose that F is algebraically closed. For the forthcoming three subsections let further $p = 2$.

3.3 M_{22} in characteristic 2

We begin by investigating the Mathieu group M_{22} and its extensions. For simplicity of notation, we set $M := M_{22}$, throughout this subsection.

Remark 3.2. (a) There is only one block of FM, i.e. the principal block of defect 7. Moreover, FM has the following seven simple modules:

$$\begin{array}{llll} D(1)_{22} = F \leftrightarrow \varphi_1, & D(10)_{22} \leftrightarrow \varphi_2 \leftrightarrow a, & D(10)^*_{22} \leftrightarrow \varphi_3 \leftrightarrow b, & D(34)_{22} \leftrightarrow \varphi_4, \\ D(70)_{22} \leftrightarrow \varphi_5 \leftrightarrow a, & D(70)^*_{22} \leftrightarrow \varphi_6 \leftrightarrow b, & D(98)_{22} \leftrightarrow \varphi_7. & \end{array}$$

(b) There is only one block of $F[M:2]$, i.e. the principal one, containing the following six simple modules:

$$\begin{array}{lll} D(1)_{22:2} = F \leftrightarrow \varphi_1, & D(10)_{22:2} \leftrightarrow \varphi_2 \leftrightarrow a, & D(10)^*_{22:2} \leftrightarrow \varphi_3 \leftrightarrow b, \\ D(34)_{22:2} \leftrightarrow \varphi_4, & D(98)_{22:2} \leftrightarrow \varphi_7, & D(140)_{22:2} \leftrightarrow \varphi_5. \end{array}$$

Here, $\mathrm{Ind}_M^{M:2}(D(70)_{22}) \cong D(140)_{22:2} \cong \mathrm{Ind}_M^{M:2}(D(70)^*_{22})$, and $\mathrm{Res}_M^{M:2}(D(d)_{22:2}) \cong D(d)_{22}$, for $d \in \{1, 10, 34, 98\}$. In particular, $D(140)_{22:2}$ is the only relatively M-projective simple $F[M:2]$-module.

With this notation, we have the following:

Proposition 3.3. *All seven simple FM-modules have vertex $Q \in \mathrm{Syl}_2(M)$. Furthermore, the restriction of any simple FM-module D to Q is also a source of D. For $Q \in \mathrm{Syl}_2(M)$ we have $N_M(Q) = Q$, and thus the Green correspondents of the simple FM-modules in Q are also sources. They have the following Loewy series:*

module	$D(1)_{22}$	$D(10)_{22}$	$D(10)^*_{22}$	$D(34)_{22}$	$D(70)_{22}$
Green	1	10	10	34	70
layer dims.	1	$1,2,1,2,$ $1,1,1,1$	$1,1,1,2$ $2,1,1,1$	$2,3,3,5,4,$ $5,4,4,2,2$	$1,3,4,6,7,8,9,$ $8,7,6,5,3,2,1$

module	$D(70)^*_{22}$	$D(98)_{22}$
Green	70	98
layer dims.	$1,3,4,6,7,8,9,$ $8,7,6,5,3,2,1$	$2,5,6,9,10,12,$ $12,12,10,8,6,3,2,1$

Proof. These results have been obtained by computer calculations. □

Proposition 3.4. *The simple $F[M\!:\!2]$-module $D(140)_{22:2}$ has vertex $Q \in \mathrm{Syl}_2(M)$. Moreover, the restriction of $D(70)_{22}$ to Q is a source of $D(140)_{22:2}$. The remaining simple $F[M:2]$-modules have vertex $P \in \mathrm{Syl}_2(M\!:\!2)$, and restrict indecomposably to P. Choosing $Q \leq P$, we have $N_{M:2}(P) = P = N_{M:2}(Q)$, and the Green correspondents of the simple $F[M\!:\!2]$-modules in P have the following Loewy series:*

module	$D(1)_{22:2}$	$D(10)_{22:2}$	$D(10)^*_{22:2}$	$D(34)_{22:2}$	$D(98)_{22:2}$
Green	1	10	10	34	98
layer dims.	1	$1,2,1,2,$ $1,1,1,1$	$1,1,1,2,$ $2,1,1,1$	$2,3,3,5,4,$ $5,4,4,2,2$	$2,5,6,9,10,12,$ $12,12,10,8,6,3,2,1$

module	$D(140)_{22:2}$
Green	140
layer dims.	$1,4,7,10,13,15,17,$ $17,15,13,11,8,5,3,1$

Proof. As mentioned above, the only relatively M-projective simple $F[M:2]$-module is $D(140)_{22:2}$, and $\mathrm{Res}^{M:2}_M(D(140)_{22:2}) \cong D(70)_{22} \oplus D(70)^*_{22}$. Thus, by the previous proposition, $D(140)_{22:2}$ and $D(70)_{22}$ have common vertex Q and common source $\mathrm{Res}^M_Q(D(70)_{22})$. Moreover, we have $Q = N_M(Q) \lhd N_{M:2}(Q) = P$ and $|P:Q| = 2$. Consequently, $\mathrm{Res}^{M:2}_P(D(140)_{22:2})$ is indecomposable and therefore the Green correspondent of $D(140)_{22:2}$ in P. Its Loewy series has been determined computationally. The remaining simple $F[M\!:\!2]$-modules restrict irreducibly to M and are thus not relatively M-projective. Since $|(M\!:\!2):M| = 2$, Proposition 3.3 implies that all these modules have vertex P, and restrict indecomposably to P. The Loewy series of their Green correspondents have been computed to be as stated. □

Remark 3.5. We observe that the Loewy series of the Green correspondents of the not relatively M-projective simple $F[M:2]$-modules and those of the respective simple FM-modules coincide. However, we do not have a good explanation for this.

Remark 3.6. Consider the non-split central extension $3.M$ of M by a group of order 3. Besides the principal block whose simple modules are obtained from the simple FM-modules via inflation, the group algebra $F[3.M]$ has two faithful blocks B_2 and $B_3 = B_2^*$ of defect 7. The simple modules belonging to B_2 and B_3, respectively, are:

$$B_2 : D(6)_{3.22} \leftrightarrow \varphi_8 \leftrightarrow a,\ D(15)_{3.22} \leftrightarrow \varphi_9 \leftrightarrow a,\ D(45_1)_{3.22} \leftrightarrow \varphi_{11} \leftrightarrow b,$$
$$D(45_2)_{3.22} \leftrightarrow \varphi_{10} \leftrightarrow a,\ D(84)_{3.22} \leftrightarrow \varphi_{12} \leftrightarrow a;$$
$$B_3 : D(6)^*_{3.22} \leftrightarrow \overline{\varphi_8},\ D(15)^*_{3.22} \leftrightarrow \overline{\varphi_9},\ D(45_1)^*_{3.22} \leftrightarrow \overline{\varphi_{11}},\ D(45_2)^*_{3.22} \leftrightarrow \overline{\varphi_{10}},\ D(84)^*_{3.22} \leftrightarrow \overline{\varphi_{12}}.$$

Moreover, $F[3.M]$ has two blocks of defect 0. In view of [35], Thm. 4.7.8 and Proposition 2.6, it suffices to determine the vertices, sources and Green correspondents of the simple $F[3.M]$-modules belonging to the faithful blocks of positive defect.

Denoting a Sylow 2-subgroup of $3.M$ by P, we have $N_{3.M}(P) \cong Z(3.M) \times P \cong C_3 \times P$. Hence $F[N_{3.M}(P)]$ possesses three blocks each of which has defect 7 and contains one simple module. All of these simple modules are one-dimensional, and we denote them by $1_1 = F, 1_2, 1_3 = 1_2^*$ where 1_2 belongs to the Brauer correspondent of B_2, and 1_3 belongs to the Brauer correspondent of B_3.

Proposition 3.7. *All simple $F[3.M]$-modules belonging to the blocks B_2 and B_3, respectively, have vertex $P \in \mathrm{Syl}_2(3.M)$, and restrict indecomposably to P. Thus, with the above parametrization, 1_2 is the unique composition factor of the Green correspondents of the simple modules belonging to B_2, and 1_3 is the unique composition factor of the Green correspondents of the simple modules belonging to B_3. The Loewy series of these Green correspondents are as follows:*

module	$D(6)_{3.22}$	$D(6)^*_{3.22}$	$D(15)_{3.22}$	$D(15)^*_{3.22}$	$D(45_1)_{3.22}$	$D(45_1)^*_{3.22}$
Green	6	6	15	15	45	45
layer dims.	$1,1,1,$ $1,1,1$	$1,1,1,$ $1,1,1$	$1,2,2,2,$ $2,2,2,1,1$	$1,2,2,2,$ $2,2,2,1,1$	$2,3,4,6,5,6,$ $5,5,3,3,2,1$	$1,2,3,4,5,6,$ $6,6,5,4,2,1$

module	$D(45_2)_{3.22}$	$D(45_2)^*_{3.22}$	$D(84)_{3.22}$	$D(84)^*_{3.22}$
Green	45	45	84	84
layer dims.	$1,2,3,4,5,6,$ $6,6,5,4,2,1$	$2,3,4,6,5,6,$ $5,5,3,3,2,1$	$2,4,6,8,9,10,10,$ $10,8,6,5,3,2,1$	$2,4,6,8,9,10,10,$ $10,8,6,5,3,2,1$

Furthermore, for any simple $F[3.M]$-module belonging to B_2 or B_3, the Loewy series of its sources and those of its Green correspondent coincide.

Proof. Consider the modules $D(15)_{3.22}$, $D(45_1)^*_{3.22}$ and $D(45_2)_{3.22}$ first. Since their dimensions are not divisible by 2, they all have vertex P. Furthermore, we have determined the Loewy series of their restrictions to $N_{3.M}(P)$ with the computer. It turns out that each of these restricted modules has a one-dimensional head, and, in particular, is thus indecomposable. Therefore the Green correspondents of these three modules and their duals are precisely the indecomposable restrictions to $N_{3.M}(P)$. The concrete Loewy series have been determined computationally.

Finally we have computed that the simple modules $D(6)_{3.22}$ and $D(84)_{3.22}$ restrict indecomposably to P, and that neither is relatively projective with respect to any maximal subgroup of P. Thus both modules have vertex P, and their restrictions to P are also sources. The Loewy series of their Green correspondents in $N_{3.M}(P)$ have been determined by computer calculations to be as claimed.

The assertion concerning the Loewy series of the sources of the simple modules in question now follows from [24], Thm. VII.7.21 and Thm. VII.9.15. □

Remark 3.8. (a) Let $G := 3.M:2$. Then FG possesses three blocks: the principal block B_1 of defect 8, the faithful block B_2 of defect 7 and the block B_3 of defect 0. The simple modules belonging to B_1 are obtained from the simple $F[M:2]$-modules via inflation. The simple modules belonging to the block B_2 are the following:

$$D(12)_{3.22.2} \leftrightarrow \varphi_8,\ D(30)_{3.22.2} \leftrightarrow \varphi_9,\ D(90)_{3.22.2} \leftrightarrow \varphi_{11},$$
$$D(90)^*_{3.22.2} \leftrightarrow \varphi_{10},\ D(168)_{3.22.2} \leftrightarrow \varphi_{12}.$$

Moreover, the outer automorphism of $3.M$ interchanges the faithful blocks of defect 7 of $F[3.M]$. In consequence thereof, each of these faithful blocks of $F[3.M]$ is Morita equivalent to B_2 via the respective induction functors. This is Fong's first correspondence (cf. [30]). In particular, the simple modules in B_2 are precisely the inductions of the faithful simple $F[3.M]$-modules to G. We choose notation such that $\mathrm{Ind}_{3.M}^{G}(D(45_1)_{3.22}) \cong D(90)_{3.22.2} \cong \mathrm{Ind}_{3.M}^{G}(D(45_2)^*_{3.22})$.

(b) Let Q be a Sylow 2-subgroup of $3.M$, that is a defect group of B_2. Then $N_G(Q)$ is a split extension $N_{3.M}(Q):2 \cong (C_3 \times Q):2$. Moreover, $N_G(Q)/Q \cong \mathfrak{S}_3$ so that there are two simple $FN_G(Q)$-modules: the trivial module and the inflation of the projective simple $F\mathfrak{S}_3$-module $D^{(2,1)}$ of dimension 2; we denote the latter by 2. Here we obtain Morita equivalences between the block of $F[N_G(Q)]$ containing 2 and each of the two non-principal blocks of $F[N_{3.M}(Q)]$ both of which are nilpotent blocks. Again the Morita equivalences are obtained via the induction functors, and we have $\mathrm{Ind}_{N_{3.M}(Q)}^{N_G(Q)}(1_2) \cong 2 \cong \mathrm{Ind}_{N_{3.M}(Q)}^{N_G(Q)}(1_3)$ where 1_2 and 1_3 are as in Remark 3.6. The Green correspondence with respect to $(G, Q, N_G(Q))$ will be denoted by f_2.

Proposition 3.9. *Let $G := 3.M:2$. Let $Q \in \mathrm{Syl}_2(3.M)$ so that Q is a defect group of B_2. Then each simple FG-module belonging to B_2 has vertex Q, and the 2-dimensional simple $FN_G(Q)$-module is the unique composition factor of its Green correspondent. The Loewy series of the Green correspondents and those of the sources of the simple modules in B_2 are as follows:*

module	$D(12)_{3.22.2}$	$D(30)_{3.22.2}$	$D(90)_{3.22.2}$	$D(90)^*_{3.22.2}$	$D(168)_{3.22.2}$
Green	12	30	90	90	168
layer dims.	$2,2,2,$ $2,2,2$	$2,4,4,4,$ $4,4,4,2,2$	$4,6,8,12,10,12,$ $10,10,6,6,4,2$	$2,4,6,8,10,12,$ $12,12,10,8,4,2$	$4,8,12,16,18,20,20,$ $20,16,12,10,6,4,2$
sce.	6	15	45	45	84
layer dims.	$1,1,1,$ $1,1,1$	$1,2,2,2,$ $2,2,2,1,1$	$2,3,4,6,5,6,$ $5,5,3,3,2,1$	$1,2,3,4,5,6,$ $6,6,5,4,2,1$	$2,4,6,8,9,10,10,$ $10,8,6,5,3,2,1$

Proof. We consider a simple FG-module V belonging to B_2. Then there exists a simple $F[3.M]$-module U such that $V \cong \mathrm{Ind}_{3.M}^{G}(U)$ and $\mathrm{Res}_{3.M}^{G}(V) \cong U \oplus {}^{g}U$, for some $g \in G \setminus$

$3.M$. Both V and U have vertex $Q \in \mathrm{Syl}_2(3.M)$, by Proposition 3.7, and they also have common sources. Furthermore, we have $N_{3.M}(Q) \lhd N_G(Q) =: N$, and $|N : N_{3.M}(Q)| = 2$. By Proposition 3.7, the Green correspondents in $N_{3.M}(Q)$ of U and ${}^gU$ are isomorphic to $\mathrm{Res}^{3.M}_{N_{3.M}(Q)}(U)$ and $\mathrm{Res}^{3.M}_{N_{3.M}(Q)}({}^gU)$, respectively. From this, together with Proposition 2.7, we now deduce that $f_2(V) = \mathrm{Res}^G_N(V)$ and

$$\mathrm{Ind}^N_{N_{3.M}(Q)}(\mathrm{Res}^{3.M}_{N_{3.M}(Q)}(U)) \cong \mathrm{Res}^G_N(V) \cong \mathrm{Ind}^N_{N_{3.M}(Q)}(\mathrm{Res}^{3.M}_{N_{3.M}(Q)}({}^gU)).$$

As explained in Remark 3.8, the induction functors yield Morita equivalences between each of the two non-principal blocks of $F[N_{3.M}(Q)]$ and the Brauer correspondent of B_2. Thus, in particular, the Green correspondent of U in $N_{3.M}(Q)$ and $f_2(V)$ have the same Loewy structure, and the assertion now follows from Proposition 3.7. □

3.4 M_{23} in characteristic 2

Remark 3.10. There are three blocks of FM_{23}: the principal one and two dual blocks of defect 0. The principal block has defect 7, and contains the following nine simple modules:

$D(1)_{23} = F \leftrightarrow \varphi_1$, $D(11)_{23} \leftrightarrow \varphi_3 \leftrightarrow b$, $D(11)^*_{23} \leftrightarrow \varphi_2 \leftrightarrow a$, $D(44)_{23} \leftrightarrow \varphi_4 \leftrightarrow a$,
$D(44)^*_{23} \leftrightarrow \varphi_5 \leftrightarrow b$, $D(120)_{23} \leftrightarrow \varphi_6$, $D(220)_{23} \leftrightarrow \varphi_8 \leftrightarrow b$, $D(220)^*_{23} \leftrightarrow \varphi_7 \leftrightarrow a$,
$D(252)_{23} \leftrightarrow \varphi_9$.

Proposition 3.11. *All simple FM_{23}-modules belonging to the principal block have vertex $P \in \mathrm{Syl}_2(M_{23})$. Moreover, $D(120)_{23}$ has sources of dimension 56, and $D(252)_{23}$ has sources of dimension 28. The remaining modules in the principal block restrict indecomposably to P. We have $N_{M_{23}}(P) = P$, and the Green correspondents in P of the simple FM_{23}-modules in the principal block are thus also sources of these. Their Loewy series are as follows:*

module	$D(1)_{23}$	$D(11)_{23}$	$D(11)^*_{23}$	$D(44)_{23}$	$D(44)^*_{23}$	$D(120)_{23}$
Green	1	11	11	44	44	56
layer dims.	1	$2,2,1,2,$ $1,1,1,1$	$1,1,1,2,$ $2,2,1,1$	$3,4,4,6,5,$ $6,5,5,3,2,1$	$2,3,4,5,5,$ $6,6,6,3,3,1$	$2,4,5,7,8,8,$ $7,6,4,2,2,1$

module	$D(220)_{23}$	$D(220)^*_{23}$	$D(252)_{23}$
Green	220	220	28
layer dims.	$3,8,12,18,22,26,27,$ $27,24,20,16,9,6,2$	$5,9,13,20,23,26,$ $28,27,22,18,14,8,5,2$	$3,4,4,5,$ $3,3,3,2,1$

Proof. Consider the modules $D(120)_{23}$ and $D(252)_{23}$ first. Then $\mathrm{Res}^{M_{23}}_P(D(120)_{23}) = U \oplus \mathrm{cyc}$, and $\mathrm{Res}^{M_{23}}_P(D(252)_{23}) = V \oplus W \oplus \mathrm{cyc}$. Here U, V and W are indecomposable of dimension $56, 28$ and 32, respectively. As mentioned earlier, the indecomposable direct summands with cyclic vertices are easily detected. Both U and V have vertex P and are thus sources and Green correspondents of $D(120)_{23}$ and $D(252)_{23}$, respectively. All other simple FM_{23}-modules in the principal block restrict indecomposably to P, and are not relatively projective with respect to any maximal subgroup of P. The Loewy series of the Green correspondents have been computed to be as stated. □

3.5 M_{24} in characteristic 2

Remark 3.12. (a) There is only one block of FM_{24}, i.e. the principal one of defect 10. The 13 simple FM_{24}-modules are:

$D(1)_{24} = F \leftrightarrow \varphi_1$, $D(11)_{24} \leftrightarrow \varphi_2 \leftrightarrow a$, $D(11)^*_{24} \leftrightarrow \varphi_3 \leftrightarrow b$, $D(44)_{24} \leftrightarrow \varphi_5 \leftrightarrow b$, $D(44)^*_{24} \leftrightarrow \varphi_4 \leftrightarrow a$, $D(120)_{24} \leftrightarrow \varphi_6$, $D(220)_{24} \leftrightarrow \varphi_8 \leftrightarrow b$, $D(220)^*_{24} \leftrightarrow \varphi_7 \leftrightarrow a$, $D(252)_{24} \leftrightarrow \varphi_9$, $D(320)_{24} \leftrightarrow \varphi_{11} \leftrightarrow b$, $D(320)^*_{24} \leftrightarrow \varphi_{10} \leftrightarrow a$, $D(1242)_{24} \leftrightarrow \varphi_{12}$, $D(1792)_{24} \leftrightarrow \varphi_{13}$.

(b) By [12], M_{24} has three conjugacy classes of maximal subgroups with odd index in M_{24}. These are $2^4{:}\mathfrak{A}_8$ of index 759, $2^6{:}3.\mathfrak{S}_6$ of index 1771, and $2^6{:}(L_3(2) \times \mathfrak{S}_3)$ of index 3795.

Proposition 3.13. *All simple FM_{24}-modules, except $D(1792)_{24}$, have vertex $P \in \mathrm{Syl}_2(M_{24})$. Furthermore, $N_{M_{24}}(P) = P$ so that the Green correspondents of these modules are also sources. Their Loewy series are as follows:*

module	$D(1)_{24}$	$D(11)_{24}$	$D(11)^*_{24}$	$D(44)_{24}$	$D(44)^*_{24}$
Green	1	11	11	44	44
layer dims.	1	$1,1,2,2,2,1,1,1$	$1,1,1,2,2,2,1,1$	$1,2,4,4,6,6,6,5,5,3,1,1$	$2,2,3,4,4,6,6,6,4,4,2,1$

module	$D(120)_{24}$	$D(220)_{24}$	$D(220)^*_{24}$	$D(252)_{24}$
Green	120	220	220	252
layer dims.	$2,4,6,9,10,14,14,14,13,12,9,6,4,2,1$	$1,3,6,10,14,20,23,26,26,25,21,18,12,8,4,2,1$	$2,4,7,11,14,18,22,24,24,23,21,17,14,9,6,3,1$	$3,5,7,13,17,22,25,29,28,27,23,20,14,9,6,3,1$

module	$D(320)_{24}$	$D(320)^*_{24}$	$D(1242)_{24}$
Green	320	320	218
layer dims.	$1,3,6,11,17,23,29,34,36,36,34,29,23,17,11,6,3,1$	$1,3,6,11,17,23,29,34,36,36,34,29,23,17,11,6,3,1$	$3,6,9,14,17,21,23,25,23,22,18,14,10,7,3,2,1$

The vertices of $D(1792)_{24}$ have order 512, and are the M_{24}-conjugates of the Sylow 2-subgroups of the commutator subgroup of $2^6{:}(L_3(2)\times\mathfrak{S}_3) \leq M_{24}$. Furthermore, $N_{M_{24}}(Q)$ has order 3072, and $N_{M_{24}}(Q)/Q \cong \mathfrak{S}_3$. Denoting the trivial $F[N_{M_{24}}(Q)]$-module by 1, and the inflation of the two-dimensional projective simple $F\mathfrak{S}_3$-module by 2, the Green correspondent of $D(1792)_{24}$ has dimension 256 and the following Loewy series:

layer	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
multiplicity of 1	0	1	4	8	11	11	8	5	5	8	11	11	8	4	1	0
multiplicity of 2	1	2	2	3	4	6	10	13	13	10	6	3	2	2	2	1

Moreover, $D(1792)_{24}$ has sources of dimension 128, and the dimensions of their Loewy layers are $1,3,5,9,13,15,18,18,15,13,9,5,3,1$.

Proof. All simple FM_{24}-modules, except $D(1792)_{24}$ and $D(1242)_{24}$, remain indecomposable when restricted to P, and neither is relatively projective with respect to any maximal subgroup of P. Moreover, $\mathrm{Res}_P^{M_{24}}(D(1242)_{24}) = U \oplus \mathrm{proj}$, where U is indecomposable of dimension 218 and has vertex P. Finally, consider $D(1792)_{24}$. Our computations show that $\mathrm{Res}_P^{M_{24}}(D(1792)_{24}) = V \oplus W \oplus \mathrm{proj}$, where V and W are indecomposable of dimension 256 and 512, respectively. It turns out that W has vertices of order 128. Furthermore, there is a maximal subgroup Q of P such that $\mathrm{Res}_Q^P(V) = V_1 \oplus V_2$ and $\mathrm{Ind}_Q^P(V_1) \cong V \cong \mathrm{Ind}_Q^P(V_2)$, for indecomposable FQ-modules V_1 and V_2 of dimension 128 both of which have vertex Q. Therefore also $D(1792)_{24}$ has vertex Q, and V_1 and V_2 are sources of $D(1792)_{24}$. Each maximal subgroup of M_{24} of odd index in M_{24} clearly contains an M_{24}-conjugate of Q. In fact, Q is conjugate to a Sylow 2-subgroup of $(2^6 : (L_3(2) \times \mathfrak{S}_3))'$. The Loewy structures of the Green correspondents and the sources of the simple FM_{24}-modules have then also been determined with the computer to be as claimed. □

3.6 M_{22} in characteristic 3

For the remainder of this article, let $p = 3$. We investigate the Mathieu group M_{22}, its automorphism group $M_{22} : 2$ and their extensions $2.M_{22}$, $4.M_{22}$, $2.M_{22}.2$, $4.M_{22}.2$ first. For this we again set $M := M_{22}$.

Remark 3.14. There are five blocks of FM: the principal block B_1 of defect 2 whose defect groups are elementary abelian, the block B_2 of defect 1, and the blocks B_3, $B_4 = B_3^*$, and B_5 of defect 0. Thus, in particular, all simple FM-modules have the defect groups of their blocks as vertices, by Knörr's Theorem [27]. For the sake of completeness, we will determine the sources and Green correspondents of the simple modules belonging to the blocks of positive defect.

(a) The principal block B_1 contains the following simple modules:

$$D(1)_{22} = F \leftrightarrow \varphi_1,\ D(49)_{22} \leftrightarrow \varphi_5 \leftrightarrow a,\ D(49)_{22}^* \leftrightarrow \varphi_6 \leftrightarrow b,\ D(55)_{22} \leftrightarrow \varphi_7,\ D(231)_{22} \leftrightarrow \varphi_{10}.$$

Let P be a Sylow 3-subgroup of M. The normalizer $N_M(P)$ has order $2^3 \cdot 3^2 = 72$, and is isomorphic to M_9. Therefore $F[N_M(P)]$ has five simple modules: four of dimension 1, and one of dimension 2. We denote these by $1_1 = F, 1_2, 1_3, 1_4, 2$. There are outer automorphisms φ and ψ of M_9 mapping 1_2 to 1_3 and 1_4, respectively. Moreover, let f_1 be the Green correspondence with respect to $(M, P, N_M(P))$.

(b) Now consider the block B_2 with defect group $C \cong C_3$ which contains the simple modules $D(21)_{22} \leftrightarrow \varphi_2$ and $D(210)_{22} \leftrightarrow \varphi_9$. The normalizer $N_M(C)$ has order $2^3 \cdot 3^2 = 72$; its Sylow 3-subgroups are elementary abelian of order 9, and its Sylow 2-subgroups are isomorphic to the dihedral group D_8 of order 8. Furthermore, $F[N_M(C)]$ possesses four simple modules: two of dimension 1, and two of dimension 3. We denote them by $1_1 = F, 1_2, 3_1, 3_2$. In fact, C acts trivially on these simple modules, by [35], Thm. 4.7.8. Moreover, we have $N_M(C)/C \cong \mathfrak{S}_4$. We choose notation such that 3_1 is isomorphic to $D^{(3,1)}$, and 3_2 is isomorphic to $D^{(2,1^2)}$, when regarded as $F\mathfrak{S}_4$-modules. Considered as $F\mathfrak{S}_4$-module, the one-dimensional module 1_2 is isomorphic to the alternating $F\mathfrak{S}_4$-module $D^{(2^2)}$. Denoting the Green correspondence with respect to $(M, C, N_M(C))$ by f_2, we obtain the following:

Proposition 3.15. *All simple FM-modules have the defect groups of their blocks as vertices. Furthermore, the Loewy series of the Green correspondents and the sources of the simple modules in B_1 and B_2 are as follows:*

block	B_1					B_2	
module	$D(1)_{22}$	$D(49)_{22}$	$D(49^*)_{22}$	$D(55)_{22}$	$D(231)_{22}$	$D(21)_{22}$	$D(210)_{22}$
Green	F	$\begin{bmatrix} 1_3 \\ 2 \\ 1_2 \end{bmatrix}$	$\begin{bmatrix} 1_2 \\ 2 \\ 1_3 \end{bmatrix}$	1_4	$\begin{bmatrix} 2 \\ 1_4\ 1_1 \\ 2 \end{bmatrix}$	3_1	3_2
source	F	$\begin{bmatrix} F \\ F\ F \\ F \end{bmatrix}$	$\begin{bmatrix} F \\ F\ F \\ F \end{bmatrix}$	F	$\begin{bmatrix} F \\ F \\ F \end{bmatrix}$	F	F

Proof. The assertion concerning the vertices is clear. The Loewy series of the Green correspondents of the simple modules belonging to B_1 are given in the proof of [28], Prop. 4.2, and the Loewy series of the sources of the simple modules in B_1 have been determined with the computer to be as claimed. As mentioned in Remark 2.8, by [23], Sec. 6.4 and L. 4.4.12, the simple modules belonging to B_2 have simple Green correspondents and thus trivial sources, by [35], Thm. 4.7.8. We have explicitly checked that $3_1 \cong f_2(D(21)_{22})$ which then implies $f_2(D(210)_{22}) \cong 3_2$. □

Remark 3.16. Now, consider the automorphism group $M{:}2$ of M. There are nine blocks of $F[M{:}2]$. The ones of positive defect are the principal block B_1 of defect 2, and the blocks B_2 and B_3 of defect 1.

(a) The principal block of $F[M{:}2]$ contains the following simple modules:

$$D(1_1)_{22:2} = F \leftrightarrow \varphi_{1,0},\ D(1_2)_{22:2} \leftrightarrow \varphi_{1,1},\qquad D(55_1)_{22:2} \leftrightarrow \varphi_{7,0} \leftrightarrow a,\ D(55_2)_{22:2} \leftrightarrow \varphi_{7,1},$$
$$D(98)_{22:2} \leftrightarrow \varphi_5,\qquad D(231_1)_{22:2} \leftrightarrow \varphi_{10,0} \leftrightarrow a,\ D(231_2)_{22:2} \leftrightarrow \varphi_{10,1}\quad .$$

Here, we have $\mathrm{Ind}_M^{M:2}(D(49)_{22}) \cong D(98)_{22:2} \cong \mathrm{Ind}_M^{M:2}(D(49)^*_{22})$ and $\mathrm{Res}_M^{M:2}(D(98)_{22:2}) \cong D(49)_{22} \oplus D(49)^*_{22}$. The remaining simple $F[M{:}2]$-modules are the extensions of the simple FM-modules $F = D(1)_{22}$, $D(55)_{22}$ and $D(231)_{22}$, respectively. The Brauer characters $\varphi_{i,0}$ are those whose values are listed in the printed ATLAS [26]. Let P be a Sylow 3-subgroup of M. By Proposition 3.15, all simple FM-modules belonging to the principal block and thus also all simple $F[M{:}2]$-modules belonging to the principal block have vertex P. Furthermore, given a simple $F[M:2]$-module D belonging to B_1 and a simple FM-module D' such that $D' \mid \mathrm{Res}_M^{M:2}(D)$ then D and D' have common sources.

Moreover, $N_{M:2}(P)$ is isomorphic to a split extension $N_M(P) : 2 \cong M_9 : 2$, and $F[M_9 : 2]$ has seven simple modules: four of dimension 1, and three of dimension 2. We denote these by $1_1 = F, 1_2, 1_3, 1_4, 2_1, 2_2, 2_3 = 2_2^*$. In the notation of Remark 3.14 and Proposition 3.15, the restrictions of 2_2 and 2_3 to M_9 are isomorphic to the 2-dimensional simple FM_9-module, the restriction of 2_1 to M_9 splits into the direct sum of the simple FM_9-modules 1_2 and 1_3. Furthermore, the restrictions of the $F[M_9\!:\!2]$-modules 1_2 and 1_4 to M_9 are isomorphic to the FM_9-module 1_4, and the restriction of 1_3 to M_9 is trivial. We identify $N_M(P)$ with M_9 and $N_{M:2}(P)$ with $M_9\!:\!2$. The Green correspondence with respect to $(M\!:\!2, P, N_M(P)\!:\!2)$ will be denoted by f_1.

(b) The blocks of defect 1 contain the simple modules:

$$\begin{aligned} B_2 &: D(21_1)_{22:2} \leftrightarrow \varphi_{2,0} \leftrightarrow a,\ D(210_1)_{22:2} \leftrightarrow \varphi_{9,1}, \\ B_3 &: D(21_2)_{22:2} \leftrightarrow \varphi_{2,1}, \qquad D(210_2)_{22:2} \leftrightarrow \varphi_{9,0} \leftrightarrow a, \end{aligned}$$

where

$$\begin{aligned} \mathrm{Res}_M^{M:2}(D(21_1)_{22:2}) &\cong D(21)_{22} \cong \mathrm{Res}_M^{M:2}(D(21_2)_{22:2}), \\ \mathrm{Res}_M^{M:2}(D(210_1)_{22:2}) &\cong D(210)_{22} \cong \mathrm{Res}_M^{M:2}(D(210_2)_{22:2}). \end{aligned}$$

Again, the Brauer characters $\varphi_{i,0}$ are those whose values occur in [26]. The simple modules in B_2 and B_3 have trivial sources and simple Green correspondents, by [23], Sec. 6.4 and L. 4.4.12, and [35], Thm. 4.7.8. If $C \le M$ is a defect group of both B_2 and B_3 then $N_{M:2}(C)$ has order 144. Moreover, $N_{M:2}(C)/C \cong \mathfrak{S}_4 \times C_2$. In particular, the simple $F[N_{M:2}(C)]$-modules are precisely the inflations of the simple $F[\mathfrak{S}_4 \times C_2]$-modules.

Consider the natural epimorphism

$$\nu : \mathfrak{S}_4 \longrightarrow \mathfrak{S}_4/\mathfrak{A}_4 \xrightarrow{\cong} C_2.$$

Then $\mathfrak{S}_4 \times C_2$ has two subgroups isomorphic to $\mathfrak{S}_4$, namely $H_1 := \{(x,1) | x \in \mathfrak{S}_4\}$ and $H_2 := \{(x,\nu(x)) | x \in \mathfrak{S}_4\}$. Both of these have C_2 as a complement. In the following, we set $N := N_{M:2}(C)$, we identify N/C with $\mathfrak{S}_4 \times C_2$, and consider N/C as the inner direct product of H_1 and C_2. Then there are eight simple $F[N/C]$-modules, namely

$$\begin{array}{llll} F \boxtimes F, & D^{(2^2)} \boxtimes F, & F \boxtimes \mathbf{sgn}, & D^{(2^2)} \boxtimes \mathbf{sgn}, \\ D^{(3,1)} \boxtimes F, & D^{(2,1^2)} \boxtimes F, & D^{(3,1)} \boxtimes \mathbf{sgn}, & D^{(2,1^2)} \boxtimes \mathbf{sgn}. \end{array}$$

Here **sgn** denotes the alternating FC_2-module. Note that this labelling depends on the identification of N/C with $\mathfrak{S}_4 \times C_2$. If we choose an essentially different identification, that is if we replace H_1 by H_2, then, for any simple $F\mathfrak{S}_4$-module D, the labelling of the module $D \boxtimes F$ remains the same, but the module $D \boxtimes \mathbf{sgn}$ is replaced by $(D \otimes D^{(2^2)}) \boxtimes \mathbf{sgn}$.

With this notation we obtain:

Proposition 3.17. *The simple $F[M\!:\!2]$-modules belonging to the blocks of positive defect have the defect groups of their blocks as vertices. Their Green correspondents have the following Loewy series:*

block	B_1						
mod.	$D(1_1)_{22:2}$	$D(1_2)_{22:2}$	$D(55_1)_{22:2}$	$D(55_2)_{22:2}$	$D(98)_{22:2}$	$D(231_1)_{22:2}$	$D(231_2)_{22:2}$
Green	F	1_3	1_2	1_4	$\begin{bmatrix} 2_1 \\ 2_2\ 2_3 \\ 2_1 \end{bmatrix}$	$\begin{bmatrix} 2_2 \\ 1_1\ 1_4 \\ 2_3 \end{bmatrix}$	$\begin{bmatrix} 2_3 \\ 1_2\ 1_3 \\ 2_2 \end{bmatrix}$

block	B_2		B_3	
module	$D(21_1)_{22:2}$	$D(210_1)_{22:2}$	$D(21_2)_{22:2}$	$D(210_2)_{22:2}$
Green	$\mathrm{Inf}_C^N(D^{(3,1)} \boxtimes F)$	$\mathrm{Inf}_C^N(D^{(2,1^2)} \boxtimes \mathbf{sgn})$	$\mathrm{Inf}_C^N(D^{(3,1)} \boxtimes \mathbf{sgn})$	$\mathrm{Inf}_C^N(D^{(2,1^2)} \boxtimes F)$

Proof. We have explicitly determined the isomorphism types of the Green correspondents of the simple modules belonging to B_2. These then also determine the isomorphism types of the Green correspondents of the simple modules belonging to B_3. We also obtain that $f_1(D(1_1)_{22:2}) = F$, $f_1(D(1_2)_{22:2}) \cong 1_3$, $f_1(D(55_1)_{22:2}) \cong 1_2$ and $f_1(D(55_2)_{22:2}) \cong 1_4$. From Proposition 3.15, Proposition 2.7 and [24], Thm. VII.7.21 we further deduce that

$$f_1(D(98)_{22:2}) \sim \begin{bmatrix} 2_1 \\ X_1\ X_2 \\ 2_1 \end{bmatrix},\ f_1(D(231_1)_{22:2}) \sim \begin{bmatrix} Y_1 \\ Z_1\ Z_2 \\ Y_2 \end{bmatrix},\ f_1(D(231_2)_{22:2}) \sim \begin{bmatrix} Y_3 \\ Z_3\ Z_4 \\ Y_4 \end{bmatrix},$$

where $X_1,\ X_2 \in \{2_2, 2_3\}$, $Y_1, \ldots, Y_4 \in \{2_2, 2_3\}$, $Z_1, Z_3 \in \{1_1, 1_3\}$ and $Z_2, Z_4 \in \{1_2, 1_4\}$. The actual isomorphism types of these simple modules can be read off from [28], proof of L. 4.5. □

Remark 3.18. The group algebra $F[2.M]$ possesses nine blocks. In view of [35], Thm. 4.7.8 and Proposition 2.6, it suffices to consider the simple $F[2.M]$-modules belonging to the faithful blocks of $F[2.M]$. These are the block B_3 of defect 2 whose defect groups are elementary abelian, and the block B_4 of defect 1.

(a) The simple modules belonging to B_3 are

$$D(10)_{2.22} \leftrightarrow \varphi_{11} \leftrightarrow a,\ D(10)^*_{2.22} \leftrightarrow \varphi_{12} \leftrightarrow b,\ D(56)_{2.22} \leftrightarrow \varphi_{13},\ D(154)_{2.22},\ D(154)^*_{2.22}.$$

The Brauer character afforded by $D(154)_{2.22}$ is either φ_{17} or φ_{18}, and representing matrices for $D(154)_{2.22}$ are available at [42]. Let A and B be standard generators of $2.M$, let $\zeta := \exp(2\pi i/8)$, and consider the lifting map $^- : \mathbb{Z}[\zeta] \longrightarrow \mathbb{F}_9$ as in Section 3.2. Then $g := ABABABBBABB$ has order 8. For the Brauer character φ afforded by $D(154)_{2.22}$, we have $\overline{\varphi(g)} = \mathbb{F}_9.1 + 1 = \overline{-2i}$, and hence $\overline{\overline{\varphi}(g)} = \overline{2i}$.

Let $P \in \mathrm{Syl}_3(2.M)$, i.e. a defect group of B_3. Then $N_{2.M}(P)$ has order 144, and if $Q \in \mathrm{Syl}_2(N_{2.M}(P))$ then $Q/Z(2.M)$ is isomorphic to the quaternion group Q_8 of order 8. Furthermore, $F[N_{2.M}(P)]$ has ten simple modules which are obtained from the simple $F[N_{2.M}(P)/P]$-modules via inflation. These will be denoted by $F = 1_1, 1_2, \ldots, 1_8, 2_1, 2_2$.

Moreover, we may choose notation such that the simple modules belonging to the Brauer correspondent b_3 of B_3 are

$$1_5, \quad 1_6 = 1_5^*, \quad 2_1, \quad 1_7, \quad 1_8 = 1_7^*.$$

(b) The block B_4 of defect 1 contains the simple modules $D(120)_{2.22} \leftrightarrow \varphi_{14}$ and $D(210_1)_{2.22} \leftrightarrow \varphi_{19}$. Let $C \cong C_3$ be a defect group of B_4. Then $N_{2.M}(C)$ has order 144. Moreover, each simple $F[N_{2.M}(C)]$-module can be regarded as a simple $F[N_{2.M}(C)/C]$-module. Actually, $N_{2.M}(C)/C \cong \mathfrak{S}_4 \times C_2$ where C_2 denotes a cyclic group of order 2. In analogy to Remark 3.16 above, we now identify $N_{2.M}(C)/C$ with $\mathfrak{S}_4 \times C_2$, and regard $N_{2.M}(C)/C$ as the inner direct product of the subgroups $H_1 := \{(x, 1) \mid x \in \mathfrak{S}_4\}$ and C_2. The eight simple $F[N_{2.M}(C)/C]$-modules are also denoted as in Remark 3.16.

For a suitable labelling we get:

Proposition 3.19. *All simple $F[2.M]$-modules belonging to B_3 and B_4, respectively, have the defect groups of their blocks as vertices. Moreover, all of these have trivial sources, and the following Green correspondents:*

block	B_3				
module	$D(10)_{2.22}$	$D(10)^*_{2.22}$	$D(56)_{2.22}$	$D(154)_{2.22}$	$D(154)^*_{2.22}$
Green	1_5	$1_6 = 1_5^*$	2_1	1_7	$1_8 = 1_7^*$

block	B_4	
module	$D(120)_{2.22}$	$D(210_1)_{2.22}$
Green	$\mathrm{Inf}_C^{N_{2.M_{22}}(C)}(D^{(3,1)} \boxtimes \mathbf{sgn})$	$\mathrm{Inf}_C^{N_{2.M_{22}}(C)}(D^{(2,1^2)} \boxtimes \mathbf{sgn})$

Proof. The assertion concerning the vertices is clear, by Knörr's Theorem [27]. The sources and Green correspondents have been determined computationally to be as claimed. □

Remark 3.20. Next we turn to the group $4.M$. Again it suffices to focus on the faithful blocks of $F[4.M]$. These are the blocks B_5 and $B_6 = B_5^*$ of defect 2 containing the following simple modules:

block	module	ID	Brauer
B_5	$D(56_1)_{4.22}$	a	$\varphi_{20}, \varphi_{21}, \overline{\varphi_{20}}, \overline{\varphi_{21}}$
	$D(56_2)_{4.22}$	b	$\varphi_{20}, \varphi_{21}, \overline{\varphi_{20}}, \overline{\varphi_{21}}$
	$D(64)_{4.22}$	a	$\varphi_{22}, \overline{\varphi_{22}}$
	$D(160_1)_{4.22}$	a	$\varphi_{25}, \overline{\varphi_{26}}$
	$D(160_2)_{4.22}$	b	$\varphi_{26}, \overline{\varphi_{25}}$
B_6	$D(56_1)^*_{4.22}$		$\varphi_{20}, \varphi_{21}, \overline{\varphi_{20}}, \overline{\varphi_{21}}$
	$D(56_2)^*_{4.22}$		$\varphi_{20}, \varphi_{21}, \overline{\varphi_{20}}, \overline{\varphi_{21}}$
	$D(64)^*_{4.22}$		$\varphi_{22}, \overline{\varphi_{22}}$
	$D(160_1)^*_{4.22}$		$\varphi_{26}, \overline{\varphi_{25}}$
	$D(160_2)^*_{4.22}$		$\varphi_{25}, \overline{\varphi_{26}}$

Column "Brauer" displays the possible Brauer characters of the simple modules in B_5 and B_6, respectively. Again, we also record the ID's of the modules appearing in [42]. Let A and B be standard generators of $4.M$ as in [42], let φ be the Brauer character afforded by $D(56_1)_{4.22}$, and let ψ be the Brauer character afforded by $D(56_2)_{4.22}$. Let further $\zeta := \exp(2\pi i/8)$, and let $\bar{\ } : \mathbb{Z}[\zeta] \longrightarrow \mathbb{F}_9$ be the lifting map as in Section 3.2. Then $g := ABABABBBBABB$ has order 8, $\overline{\varphi(g)} = 2\mathbb{F}_9.1 = \overline{2z_8}$, and $\overline{\psi(g)} = \mathbb{F}_9.1 = \overline{-2z_8}$. The element $z := (ABABABABAB^2ABAB^2AB^2)^{63} \in Z(4.M_{22})$ of order 4 acts on the simple modules in B_5 via multiplication with $\mathbb{F}_9.1^2$, and on the simple modules in B_6 via multiplication with $\mathbb{F}_9.1^6$.

Consider further $P \in \mathrm{Syl}_3(4.M)$. Then P is elementary abelian of order 9 and a defect group of B_5 and B_6. Its normalizer $N_{4.M}(P)$ has order 288, and is isomorphic to a split extension $P:(C_4:C_8)$. We denote the Green correspondence with respect to $(4.M, P, N_{4.M}(P))$ by f. There are precisely 20 simple $F[N_{4.M}(P)]$-modules, namely the inflations of the simple $F[C_4:C_8]$-modules. These will be denoted by $F = 1_1, \ldots, 1_{16}, 2_1, \ldots, 2_4$.

In [34] J. Müller and M. Schaps have proved that the blocks B_5 and B_6 are derived equivalent to their Brauer correspondents b_5 and $b_6 = b_5^*$, respectively, thereby proving Broué's conjecture for B_5 and B_6. We choose notation such that the simple $N_{4.M}(P)$-modules belonging to the block b_5 are 1_2, 1_3, 1_4, 1_5, 2_1, and the simple $N_{4.M}(P)$-modules belonging to the block b_6 are $1_6 = 1_2^*$, $1_7 = 1_3^*$, $1_8 = 1_4^*$, $1_9 = 1_5^*$, $2_2 = 2_1^*$. For a suitable labelling of the simple $FN_{4.M}(P)$-modules, we obtain the following:

Proposition 3.21. *The simple $F[4.M]$-modules belonging to B_5 and B_6, respectively, have vertex P. Moreover, the Loewy series of their Green correspondents are as follows:*

block	B_5				
module	$D(56_1)_{4.22}$	$D(56_2)_{4.22}$	$D(64)_{4.22}$	$D(160_1)_{4.22}$	$D(160_2)_{4.22}$
Green	$\begin{bmatrix} 1_2\ 2_1 \\ 1_3\ 1_4\ 1_5 \\ 2_1\ 2_1 \\ 1_2 \end{bmatrix}$	$\begin{bmatrix} 1_3\ 2_1 \\ 1_3\ 2_1 \\ 1_2\ 1_4\ 1_5 \\ 2_1 \end{bmatrix}$	$\begin{bmatrix} 1_2\ 1_4\ 1_5 \\ 2_1\ 2_1 \\ 1_3\ 1_4\ 1_5 \end{bmatrix}$	$\begin{bmatrix} 1_4\ 2_1 \\ 1_2\ 1_3\ 1_5\ 2_1 \\ 1_2\ 1_3\ 1_5\ 2_1 \\ 1_4\ 2_1 \end{bmatrix}$	$\begin{bmatrix} 1_5\ 2_1 \\ 1_2\ 1_3\ 1_4\ 2_1 \\ 1_2\ 1_3\ 1_4\ 2_1 \\ 1_5\ 2_1 \end{bmatrix}$

block	B_6				
module	$D(56_1)^*_{4.22}$	$D(56_2)^*_{4.22}$	$D(64)^*_{4.22}$	$D(160_1)^*_{4.22}$	$D(160_2)^*_{4.22}$
Green	$\begin{bmatrix} 1_2^*\ 2_1^* \\ 1_2^*\ 2_1^* \\ 1_3^*\ 1_4^*\ 1_5^* \\ 2_1^* \end{bmatrix}$	$\begin{bmatrix} 1_3^*\ 2_1^* \\ 1_2^*\ 1_4^*\ 1_5^* \\ 2_1^*\ 2_1^* \\ 1_3^* \end{bmatrix}$	$\begin{bmatrix} 1_3^*\ 1_4^*\ 1_5^* \\ 2_1^*\ 2_1^* \\ 1_2^*\ 1_4^*\ 1_5^* \end{bmatrix}$	$\begin{bmatrix} 1_4^*\ 2_1^* \\ 1_2^*\ 1_3^*\ 1_5^*\ 2_1^* \\ 1_2^*\ 1_3^*\ 1_5^*\ 2_1^* \\ 1_4^*\ 2_1^* \end{bmatrix}$	$\begin{bmatrix} 1_5^*\ 2_1^* \\ 1_2^*\ 1_3^*\ 1_4^*\ 2_1^* \\ 1_2^*\ 1_3^*\ 1_4^*\ 2_1^* \\ 1_5^*\ 2_1^* \end{bmatrix}$

If D is any of the simple modules belonging to B_5 or B_6 then the restriction of its Green correspondent $f(D)$ to P is also a source of D. Moreover, $f(D)$ and $\mathrm{Res}_P^{N_{4.M}(P)}(f(D))$ have the same Loewy lengths, and the dimensions of the respective Loewy layers coincide.

Proof. The assertion concerning the vertices is obvious. The dimensions of the Green correspondents of the simple modules belonging to B_5 and B_6, respectively, have already been determined in [34]. Moreover, the Green correspondents restrict indecomposably to P, by [34]. We have further calculated the explicit Loewy structures of all Green correspondents with the computer, as stated. The assertion regarding the sources follows from [24], Thm. VII.7.21. □

To close this subsection, we investigate the simple modules for the bicyclic extensions $2.M.2$ and $4.M.2$ of M.

Remark 3.22. (a) The group algebra $F[2.M.2]$ has 13 blocks, and due to our previous considerations it suffices to investigate only the faithful ones of positive defect explicitly. These are the block B_4 of defect 2 and the blocks B_5 and B_6 of defect 1. In fact, the blocks B_5 and B_6 only differ by a linear character, and are thus isomorphic.

(b) Consider the blocks B_5 and B_6 of defect 1 first. Here the block B_5 contains the simple modules $D(120_1)_{2.22.2}$ and $D(210_4)_{2.22.2}$, and the block B_6 contains $D(120_2)_{2.22.2}$ and $D(210_3)_{2.22.2}$ where

$$\mathrm{Res}_{2.M}^{2.M.2}(D(120_1)_{2.22.2}) \cong D(120)_{2.22} \cong \mathrm{Res}_{2.M}^{2.M.2}(D(120_2)_{2.22.2}),$$
$$\mathrm{Res}_{2.M}^{2.M.2}(D(210_3)_{2.22.2}) \cong D(210_1)_{2.22} \cong \mathrm{Res}_{2.M}^{2.M.2}(D(210_4)_{2.22.2}).$$

Let c and d be standard generators of $2.M.2$ as in [42]. Then $g := (ccd)^3$ is an element of order 2 belonging to one of the conjugacy classes of $2.M.2$ lying above the conjugacy class $2B$ of $M:2$. Let $^{-} : \mathbb{Z} \longrightarrow \mathbb{F}_3$ be the residue map. If φ denotes the Brauer character afforded by $D(120_2)_{2.22.2}$ and if ψ denotes the Brauer character afforded by $D(210_4)_{2.22.2}$ then $\overline{\varphi(g)} = 2 = \overline{8}$, and $\overline{\psi(g)} = 1 = \overline{28}$.

If $C \leq 2.M$ is a defect group of both B_5 and B_6 then $N_{2.M.2}(C) \cong N_{2.M}(C).2$ has order 288, and each simple $FN_{2.M}(C)$-module extends to two simple $FN_{2.M.2}(C)$-modules. Thus there are 16 simple $FN_{2.M.2}(C)$-modules eight of which have dimension 1, and the remaining eight have dimension 3. We may choose notation such that the Brauer correspondent b_5 of B_5 contains the simple modules 3_3 and 3_4, and the Brauer correspondent b_6 of B_6 contains the simple modules 3_5 and 3_6 where

$$\mathrm{Res}_{N_{2.M}(C)}^{N_{2.M.2}(C)}(3_3) \cong \mathrm{Inf}_C^{N_{2.M}(C)}(D^{(2,1^2)} \boxtimes \mathbf{sgn}) \cong \mathrm{Res}_{N_{2.M}(C)}^{N_{2.M.2}(C)}(3_5)$$

and

$$\mathrm{Res}_{N_{2.M}(C)}^{N_{2.M.2}(C)}(3_4) \cong \mathrm{Inf}_C^{N_{2.M}(C)}(D^{(3,1)} \boxtimes \mathbf{sgn}) \cong \mathrm{Res}_{N_{2.M}(C)}^{N_{2.M.2}(C)}(3_6),$$

in the notation of Remark 3.16 (b).

Let f_5 be the Green correspondence with respect to $(2.M.2, C, N_{2.M.2}(C))$. In consequence of Proposition 3.19 and Proposition 2.7, the simple modules belonging to B_5 and B_6, respectively, have simple Green correspondents and trivial sources. More precisely, we have:

$$f_5(D(120_1)_{2.22.2}) \cong 3_4,\ f_5(D(120_2)_{2.22.2}) \cong 3_6,\ f_5(D(210_3)_{2.22.2}) \cong 3_3,\ f_5(D(210_4)_{2.22.2}) \cong 3_5.$$

(c) The faithful block B_4 contains the simple modules

$$D(10_1)_{2.22.2},\ D(10_2)_{2.22.2},\ D(10_1)^*_{2.22.2},\ D(10_2)^*_{2.22.2},$$
$$D(56_1)_{2.22.2},\ D(56_2)_{2.22.2},\ D(308)_{2.22.2},$$

where notation is chosen such that

$$\begin{aligned}
\mathrm{Ind}_{2.M}^{2.M.2}(D(10)_{2.22}) &\cong D(10_1)_{2.22.2} \oplus D(10_2)_{2.22.2},\\
\mathrm{Ind}_{2.M}^{2.M.2}(D(10)^*_{2.22}) &\cong D(10_1)^*_{2.22.2} \oplus D(10_2)^*_{2.22.2},\\
\mathrm{Ind}_{2.M}^{2.M.2}(D(56)_{2.22}) &\cong D(56_1)_{2.22.2} \oplus D(56_2)_{2.22.2},\\
\mathrm{Ind}_{2.M}^{2.M.2}(D(154)_{2.22}) &\cong D(308)_{2.22.2} \cong \mathrm{Ind}_{2.M}^{2.M.2}(D(154)^*_{2.22}).
\end{aligned}$$

In particular, all these simple modules have trivial sources, by Proposition 3.19, and thus simple Green correspondents, by Okuyama's Theorem [36]. Let $P \leq 2.M$ be a Sylow 3-subgroup of $2.M.2$, i.e. a defect group of B_4. Then $N_{2.M.2}(P) \cong N_{2.M}(P).2$, and there are 14 simple $F[N_{2.M.2}(P)]$-modules seven of which belong to the Brauer correspondent b_4 of B_4. Among these are four of dimension 1 and three of dimension 2. Denoting the simple $FN_{2.M}(P)$-modules as in Remark 3.18, we may label the simple $FN_{2.M.2}(P)$-modules belonging to b_4 as $1_5, 1_6, 1_7 = 1_6^*, 1_8 = 1_5^*, 2_3, 2_5, 2_6$ such that:

$2.M.2$	Brauer	$N_{2.M.2}(P)$	$2.M$	$N_{2.M}(P)$
$D(10_1)_{2.22.2}$	$\varphi_{11,0}, \varphi_{11,1}$	1_5	$D(10)_{2.22}$	1_5
$D(10_2)_{2.22.2}$	$\varphi_{11,0}, \varphi_{11,1}$	1_6	$D(10)_{2.22}$	1_5
$D(10_1)^*_{2.22.2}$	$\varphi_{12,0}, \varphi_{12,1}$	1_5^*	$D(10)^*_{2.22}$	1_5^*
$D(10_2)^*_{2.22.2}$	$\varphi_{12,0}, \varphi_{12,1}$	1_6^*	$D(10)^*_{2.22}$	1_5^*
$D(56_1)_{2.22.2}$	$\varphi_{13,0}, \varphi_{13,1}$	2_5	$D(56)_{2.22}$	2_1
$D(56_2)_{2.22.2}$	$\varphi_{13,0}, \varphi_{13,1}$	2_6	$D(56)_{2.22}$	2_1
$D(308)_{2.22.2}$	φ_{17}	2_3	$D(154)_{2.22} \oplus D(154)^*_{2.22}$	$1_7 \oplus 1_8$

The entries of this table should be read as follows: the first column displays the simple modules in B_4, the fourth column their restrictions to $2.M$, and the third column their Green correspondents in $N_{2.M.2}(P)$. The last column contains the restrictions of these Green correspondents to $N_{2.M}(P)$. Column "Brauer" lists for each simple module in B_4 the corresponding possible Brauer characters. Again the $\varphi_{i,0}$ denote the Brauer characters whose values are printed in [26]. Let c and d be standard generators of $2.M.2$ as in [42], let $\zeta := \exp(2\pi i/8)$, and let $^- : \mathbb{Z}[\zeta] \longrightarrow \mathbb{F}_9$ be the lifting map as in Section 3.2. Then $g := ccdcdcdcddcdd$ and $h := g^3$ are non-conjugate elements of order 14. Moreover, $x := cdcdd$ has order 10. The modular character values of the above simple modules on these elements are as follows:

module	g	h	x
$D(10_1)_{2.22.2}$	$\mathbb{F}_9.1+2=\overline{b7**}$	$2\mathbb{F}_9.1=\overline{b7}$	0
$D(10_2)_{2.22.2}$	$\mathbb{F}_9.1^3=\overline{-b7**}$	$\mathbb{F}_9.1=\overline{-b7}$	0
$D(10_1)^*_{2.22.2}$	$\overline{b7}$	$\overline{b7**}$	0
$D(10_1)^*_{2.22.2}$	$\overline{-b7}$	$\overline{-b7**}$	0
$D(56_1)_{2.22.2}$	0	0	$-(\mathbb{F}_9.1+1)=\overline{-r5}$
$D(56_2)_{2.22.2}$	0	0	$\mathbb{F}_9.1+1=\overline{r5}$

Remark 3.23. Finally consider $4.M.2$. There are 16 blocks of $F[4.M.2]$, and in view of our previous results we only need to investigate the faithful block B_7 of defect 2 containing five simple modules. These are precisely the inductions of the simple $F[4.M]$-modules belonging to the faithful blocks B_5 and B_6 of $F[4.M]$. Here we actually have Morita equivalences between B_7 and each of the blocks B_5 and B_6 of $F[4.M]$, induced by the respective induction functors. Again this is Fong's first correspondence, see [30]. Similarly, via the induction functors, we also have Morita equivalences between the Brauer correspondent b_7 of B_7 and each of the Brauer correspondents b_5 and b_6 of the blocks B_5 and B_6, respectively. Consequently, any simple $F[4.M.2]$-module in B_7 and the corresponding simple $F[4.M]$-modules belonging to B_5 and B_6, respectively, have common vertices and common sources. Furthermore, the Loewy structures of their Green correspondents coincide.

3.7 M_{23} in characteristic 3

Remark 3.24. There are seven blocks of FM_{23}, namely the principal block B_1 of defect 2 whose defect groups are elementary abelian, the block B_4 of defect 1, and the blocks B_2, $B_3 = B_2^*$, B_5, $B_6 = B_5^*$ and B_7 of defect 0. Thus also all simple FM_{23}-modules have the defect groups of their blocks as vertices, by Knörr's Theorem [27].

(a) The simple modules belonging to the principal block B_1 are:

$$D(1)_{23} = F \leftrightarrow \varphi_1,\quad D(22)_{23} \leftrightarrow \varphi_2,\quad D(104)_{23} \leftrightarrow \varphi_6 \leftrightarrow b,\quad D(104)^*_{23} \leftrightarrow \varphi_5 \leftrightarrow a,$$
$$D(253)_{23} \leftrightarrow \varphi_8,\quad D(770)_{23} \leftrightarrow \varphi_{10},\quad D(770)^*_{23} \leftrightarrow \varphi_9 \leftrightarrow a.$$

The normalizer $N_{M_{23}}(P)$ of a Sylow 3-subgroup P of M_{23} has order $2^4 \cdot 3^2 = 144$. Actually, $N_{M_{23}}(P) \cong M_9{:}2$, and we denote the seven simple $F[M_9{:}2]$-modules as in Remark 3.16. The Green correspondence with respect to $(M_{23}, P, N_{M_{23}}(P))$ is denoted by f_1.

(b) The block B_4 contains precisely one simple FM_{23}-module, namely $D(231)_{23} \leftrightarrow \varphi_7$. Let $C \cong C_3$ be a defect group of B_4. The normalizer $N_{M_{23}}(C)$ has order $5 \cdot 3^2 \cdot 2^3 = 360$, and it is isomorphic to a split extension $(\mathfrak{A}_5 \times C_3){:}2$. Furthermore, $N_{M_{23}}(C)/C \cong \mathfrak{S}_5$, and $F[N_{M_{23}}(C)]$ has five simple modules. These are the inflations of the simple $F\mathfrak{S}_5$-modules. Precisely one of them is projective, namely the inflation of the simple $F\mathfrak{S}_5$-module $D^{(3,1^2)}$ of dimension 6. Denoting the Green correspondence with respect to $(M_{23}, C, N_{M_{23}}(C))$ by f_4, we then have:

Proposition 3.25. *All simple FM_{23}-modules have the defect groups of their blocks as vertices. The Loewy series of the sources and Green correspondents of the simple modules in B_1*

and B_4 are as follows:

block	B_1						
module	$D(1)_{23}$	$D(22)_{23}$	$D(104)_{23}$	$D(104)^*_{23}$	$D(253)_{23}$	$D(770)_{23}$	$D(770)^*_{23}$
Green	F	1_3	$\begin{bmatrix} 1_4 \\ 2_3 \\ 2_1 \end{bmatrix}$	$\begin{bmatrix} 2_1 \\ 2_2 \\ 1_4 \end{bmatrix}$	1_2	2_2	2_3
source	F	F	$\begin{bmatrix} F \\ F\ F \\ F\ F \end{bmatrix}$	$\begin{bmatrix} F\ F \\ F\ F \\ F \end{bmatrix}$	F	F	F

block	B_4
module	$D(231)_{23}$
Green	$\mathrm{Inf}_C^{N_{M_{23}}(C)}(D^{(3,1^2)})$
source	F

Proof. Again the assertion about the vertices is obvious. The simple module $D(22)_{23}$ is a composition factor, and hence a direct summand, of the natural permutation FM_{23}-module on 23 points. Therefore, $D(22)_{23}$ has trivial source. Furthermore, $\mathrm{Res}_P^{M_{23}}(D(104)_{23}) = V \oplus \mathrm{cyc}$, $\mathrm{Res}_P^{M_{23}}(D(253)_{23}) \cong F \oplus \mathrm{cyc}$ and $\mathrm{Res}_P^{M_{23}}(D(770)_{23}) \cong F \oplus F \oplus \mathrm{cyc}$. Here V is indecomposable of dimension 5 and therefore a source of $D(104)_{23}$. By Okuyama's Theorem [36], $D(22)_{23}$, $D(770)_{23}$, $D(770)^*_{23}$ and $D(253)_{23}$ have simple Green correspondents. Their actual isomorphism types, the Loewy series of the Green correspondents of the remaining simple modules in B_1 and the sources of $D(104)_{23}$ and $D(104)^*_{23}$ have been determined computationally.

Finally, consider $D(231)_{23}$. By [23], Sec. 6.6 and L. 4.4.12, it has a simple Green correspondent and thus, by [35], Thm. 4.7.8, a trivial source. In view of Remark 3.24, when regarded as $F\mathfrak{S}_5$-module, $f_4(D(231)_{23})$ has to be the unique projective simple $F\mathfrak{S}_5$-module $D^{(3,1^2)}$. This proves the proposition. □

3.8 M_{24} in characteristic 3

Remark 3.26. There are six blocks of FM_{24}. These are the principal block B_1 of defect 3 whose defect groups are extraspecial of exponent 3, the blocks B_2, $B_3 = B_2^*$, B_4, B_5 of defect 1, and B_6 of defect 0. The simple modules belonging to the blocks $B_2, \dots, B_6$ clearly have the respective defect groups as their vertices.

(a) The simple FM_{24}-modules belonging to B_1 are:

$$D(1)_{24} = F \leftrightarrow \varphi_1, \quad D(22)_{24} \leftrightarrow \varphi_2, \quad D(231)_{24} \leftrightarrow \varphi_5, \quad D(483)_{24} \leftrightarrow \varphi_7,$$
$$D(770)_{24} \leftrightarrow \varphi_9 \leftrightarrow b, \quad D(770)^*_{24} \leftrightarrow \varphi_8 \leftrightarrow a, \quad D(1243)_{24} \leftrightarrow \varphi_{13}.$$

The normalizer $N_{M_{24}}(P)$ of some Sylow 3-subgroup P of M_{24} has order 216, and $N_{M_{24}}(P)/P$ is isomorphic to the dihedral group D_8 of order 8. Consequently, the simple $F[N_{M_{24}}(P)]$-modules are precisely the inflations of the simple FD_8-modules, and are denoted by $F = 1_1, 1_2, 1_3, 1_4, 2$. The modules 1_2 and 1_4 are interchanged by an outer automorphism of $N_{M_{24}}(P)$. The Green correspondence with respect to $(M_{24}, P, N_{M_{24}}(P))$ will be denoted by f_1.

(b) By [37], M_{24} has two conjugacy classes of subgroups of order 3. If $C_{3,1}$ and $C_{3,2}$ are representatives for these, then $N_{M_{24}}(C_{3,1}) \cong L_2(7) \times \mathfrak{S}_3$ and $N_{M_{24}}(C_{3,2})$ is isomorphic to an extension $3.\mathfrak{S}_6$. We will from now on identify $N_{M_{24}}(C_{3,1})$ with $L_2(7) \times \mathfrak{S}_3$ and $N_{M_{24}}(C_{3,2})$ with $3.\mathfrak{S}_6$ via these isomorphisms. As subgroups of $\mathfrak{S}_{24}$ the group $C_{3,1}$ is generated by a product of eight disjoint 3-cycles, and $C_{3,2}$ is generated by a product of six disjoint 3-cycles. The defect groups of B_2, B_3 and B_5 are conjugate to $C_{3,1}$, and the defect groups of B_4 are conjugate to $C_{3,2}$. By [26], $FL_2(7)$ has five simple modules: $F = 1_1, 3_1, 3_2 = 3_1^*, 6_1, 7_1$. Hence $F[L_2(7) \times \mathfrak{S}_3]$ has ten simple modules: $F = 1_1 \boxtimes F$, $F \boxtimes \mathbf{sgn}$, $3_1 \boxtimes F$, $3_2 \boxtimes F$, $3_1 \boxtimes \mathbf{sgn}$, $3_2 \boxtimes \mathbf{sgn}$, $6_1 \boxtimes F$, $6_1 \boxtimes \mathbf{sgn}$, $7_1 \boxtimes F$, $7_1 \boxtimes \mathbf{sgn}$. Here, $\mathbf{sgn}$ denotes the alternating $F\mathfrak{S}_3$-module $D^{(2,1)}$. Note that $L_2(7) \times \mathfrak{S}_3$ possesses exactly one normal subgroup isomorphic to $\mathfrak{S}_3$, namely $H := \{(1, x) | x \in \mathfrak{S}_3\}$. For $i = 2, 3, 5$ we denote the Brauer correspondent of B_i by b_i.

The group algebra $F[3.\mathfrak{S}_6]$ has seven simple modules. Namely, by [35], Thm. 4.7.8, each simple $F[3.\mathfrak{S}_6]$-module is also a simple $F\mathfrak{S}_6$-module, and there are seven of those. There are two projective simple $F\mathfrak{S}_6$-modules: $D^{(4,2)}$ and $D^{(2^2,1^2)} = D^{(4,2)} \otimes \mathbf{sgn}$ which have dimension 9. We denote the Brauer correspondent of B_5 by b_5. For $i = 2, 3, 4, 5$ and a suitable labelling, the blocks B_i and b_i, respectively, contain the following simple modules:

B_2	B_3	B_5	B_4
$D(45)_{24} \leftrightarrow \varphi_3 \leftrightarrow a$	$D(45)^*_{24} \leftrightarrow \varphi_4 \leftrightarrow b$	$D(1035)_{24} \leftrightarrow \varphi_{12}$	$D(252)_{24} \leftrightarrow \varphi_6$
$D(990)_{24} \leftrightarrow \varphi_{10}$	$D(990)^*_{24} \leftrightarrow \varphi_{11} \leftrightarrow b$	$D(2277)_{24} \leftrightarrow \varphi_{14}$	$D(5544)_{24} \leftrightarrow \varphi_{15}$
b_2	b_3	b_5	b_4
$3_1 \boxtimes F$	$3_2 \boxtimes F$	$6_1 \boxtimes F$	$\mathrm{Inf}_{C_{3,2}}^{N_{M_{24}}(C_{3,2})}(D^{(4,2)})$
$3_1 \boxtimes \mathbf{sgn}$	$3_2 \boxtimes \mathbf{sgn}$	$6_1 \boxtimes \mathbf{sgn}$	$\mathrm{Inf}_{C_{3,2}}^{N_{M_{24}}(C_{3,2})}(D^{(2^2,1^2)})$

With the above notation, the following holds:

Proposition 3.27. *Apart from the module $D(483)_{24}$, all simple FM_{24}-modules belonging to the principal block have vertex $P \in \mathrm{Syl}_3(M_{24})$. The vertices of $D(483)_{24}$ are the M_{24}-conjugates of a subgroup of M_{24} of order 9 whose normalizer in M_{24} is isomorphic to the automorphism group of M_9 which is a split extension $M_9 : \mathfrak{S}_3$. Furthermore, the Loewy series of the sources are as follows:*

module	$D(1)_{24}$	$D(22)_{24}$	$D(231)_{24}$	$D(483)_{24}$	$D(770)_{24}$	$D(770)^*_{24}$	$D(1243)_{24}$
sce.	1	4	3	1	5	5	19
layer dims.	1	$1,2,1$	$1,1,1$	1	$1,2,2$	$2,2,1$	$2,3,4,2,4,2,2$

Proof. The modules $D(1)_{24}$, $D(22)_{24}$, $D(770)_{24}$, $D(770)^*_{24}$ and $D(1243)_{24}$ obviously have vertex P, since their dimensions are not divisible by 3. Moreover, $\mathrm{Res}_P^{M_{24}}(D(231)_{24}) = V_1 \oplus V_2 \oplus \mathrm{cyc}$, where V_1 and V_2 are indecomposable of dimension 3 with vertex P and therefore sources of $D(231)_{24}$. For $D(483)_{24}$ we have $\mathrm{Res}_P^{M_{24}}(D(483)_{24}) = U_1 \oplus U_2 \oplus \mathrm{cyc}$, where U_1 and U_2 are indecomposable of dimension 3 and have vertex Q of order 9. Actually, $N_{M_{24}}(Q) \cong \mathrm{Aut}(M_9) \cong M_9 : \mathfrak{S}_3$ which determines $N_{M_{24}}(Q)$ up to M_{24}-conjugacy, by [37]. The Loewy series of the sources of the non-trivial simple FM_{24}-modules belonging to B_1 have been obtained by computer calculations. □

Remark 3.28. Let Q be a vertex of $D(483)_{24}$. By the above proposition, its normalizer $N_{M_{24}}(Q)$ is isomorphic to $M_9 : \mathfrak{S}_3$. The factor group $N_{M_{24}}(Q)/Q$ is isomorphic to $\mathrm{GL}(2,3)$ which is one of the double covers of $\mathfrak{S}_4$. So each simple $F[N_{M_{24}}(Q)]$-module can be regarded as simple $F[\mathrm{GL}(2,3)]$-module. Furthermore, $F[\mathrm{GL}(2,3)]$ has six simple modules two of which are projective. The latter are precisely the inflations of the projective simple $F\mathfrak{S}_4$-modules $D^{(3,1)}$ and $D^{(2,1^2)}$. We denote the simple $FN_{M_{24}}(Q)$-module obtained from $D^{(3,1)}$ via inflation as 3_1, and the one obtained from $D^{(2,1^2)}$ via inflation by 3_2. For the following, let further f be the Green correspondence with respect to $(M_{24}, Q, N_{M_{24}}(Q))$.

Proposition 3.29. *With the notation of Remark 3.26 and Remark 3.28, the Green correspondents of the simple FM_{24}-modules belonging to the principal block have the following Loewy series:*

module	$D(1)_{24}$	$D(22)_{24}$	$D(231)_{24}$	$D(770)_{24}$	$D(770)^*_{24}$	$D(1243)_{24}$	$D(483)$
Green	F	$\begin{bmatrix} 1_2 \\ 2 \\ 1_2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1_1\ 1_3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1_4 \\ 2 \\ 1_3\ 1_4 \end{bmatrix}$	$\begin{bmatrix} 1_3\ 1_4 \\ 2 \\ 1_4 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1_1\ 1_2\ 1_3 \\ 2\ 2 \\ 1_3\ 1_4 \\ 2\ 2 \\ 1_1\ 1_2 \\ 2 \end{bmatrix}$	3_1

Proof. Since $D(483)_{24}$ has trivial source, its Green correspondent $f(D(483)_{24})$ is simple, by Okuyama's Theorem [36]. Furthermore, $f(D(483)_{24})$ can also be regarded as a simple projective $F[N_{M_{24}}(Q)/Q]$-module. By the previous remark, $f(D(483)_{24})$ has to be the inflation of one of the projective simple $F\mathfrak{S}_4$-modules. We have checked that it is actually isomorphic to that of $D^{(3,1)}$.

The remaining Green correspondents and their Loewy series have been determined with the computer. In order to determine $f_1(D(1243)_{24})$, let W be a source of $D(1243)_{24}$, and set $N := N_{M_{24}}(P)$. By Proposition 3.27, $\dim(W) = 19$. Furthermore, our computations show that $\mathrm{Ind}_P^N(W) = U_1 \oplus \cdots \oplus U_6$ where $U_1, \ldots, U_6$ are pairwise non-isomorphic indecomposable modules such that $\dim(U_i) = 19$ and $\dim(U_j) = 38$, for $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6\}$. Precisely one of these summands is isomorphic to $f_1(D(1243)_{24})$. Now $\mathrm{Res}_P^{M_{24}}(D(1243)_{24}) = W \oplus 2W_1 \oplus 2W_2 \oplus \mathrm{proj}$ where W_1 and W_2 are non-isomorphic indecomposable modules of dimension 9. Our computations also show that $\mathrm{Res}_N^{M_{24}}(D(1243)_{24})$ has a submodule U isomorphic to one of the 19-dimensional modules $U_1, \ldots, U_4$ such that U has vertex P, and

$$\mathrm{Res}_P^N(U) \oplus \mathrm{Res}_P^N(\mathrm{Res}_N^{M_{24}}(D(1243)_{24})/U) \cong W \oplus 2W_1 \oplus 2W_2 \oplus \mathrm{proj}\,.$$

Hence $\mathrm{Res}_P^N(U)$ is a direct summand of $\mathrm{Res}_P^{M_{24}}(D(1243)_{24})$, by [10], Thm. 1.5.8. Since U is relatively P-projective, [14], Thm. 19.2 implies that U is a direct summand of $\mathrm{Res}_N^{M_{24}}(D(1243)_{24})$. Consequently, $f_1(D(1243)_{24}) \cong U$, and its Loewy series has been computed to be as stated. □

Proposition 3.30. *The simple FM_{24}-modules belonging to the blocks B_2, B_3, B_4 and B_5 have the defect groups of their blocks as vertices. Moreover, all these modules have trivial sources, and the following Green correspondents:*

block	B_2		B_3		B_5	
module	$D(45)_{24}$	$D(990)_{24}$	$D(45)^*_{24}$	$D(990)^*_{24}$	$D(1035)_{24}$	$D(2277)_{24}$
Green	$3_1 \boxtimes \mathbf{sgn}$	$3_1 \boxtimes F$	$3_2 \boxtimes \mathbf{sgn}$	$3_2 \boxtimes F$	$6_1 \boxtimes F$	$6_1 \boxtimes \mathbf{sgn}$

block	B_4	
module	$D(252)_{24}$	$D(5544)_{24}$
Green	$\mathrm{Inf}_{C_{3,2}}^{N_{M_{24}}(C_{3,2})}(D^{(4,2)})$	$\mathrm{Inf}_{C_{3,2}}^{N_{M_{24}}(C_{3,2})}(D^{(2^2,1^2)})$

Proof. The assertion about the vertices is obvious, and, by [23], Sec. 6.9 and L. 4.4.12, we also know that all of these simple modules have simple Green correspondents and therefore trivial sources, by [35], Thm. 4.7.8. Let f_2 be the Green correspondence with respect to $(M_{24}, C_{3,1}, N_{M_{24}}(C_{3,1}))$, and let f_4 be the Green correspondence with respect to $(M_{24}, C_{3,2}, N_{M_{24}}(C_{3,2}))$. We have constructed $f_2(D(45)_{24})$, $f_2(D(1035)_{24})$ and $f_4(D(252)_{24})$. The restriction of $f_2(D(45)_{24})$ to the normal subgroup $H \cong \mathfrak{S}_3$ of $N_{M_{24}}(C_{3,1})$ decomposes into the direct sum of three copies of the alternating module, and the restriction of $f_2(D(1035)_{24})$ to H decomposes into the direct sum of six copies of the trivial module. This determines the Green correspondents of the simple modules in B_2, B_3 and B_5. Moreover, we have computed that $f_4(D(252)_{24})$ is isomorphic to $D^{(4,2)}$ when regarded as $F\mathfrak{S}_6$-module. This then also yields the claimed assertion on the Green correspondents of the simple modules in B_4. □

4 Morita equivalent blocks

In the following, let $p = 3$, $M := M_{22}$, $G = 2.M$ and $P \in \mathrm{Syl}_3(G)$. As we have seen in Proposition 3.19, all simple FG-modules belonging to the faithful block $B := B_3$ of FG with defect group P have vertex P, trivial sources and simple Green correspondents. The aim of this section is to show that B is Morita equivalent to its Brauer correspondent b in $N_G(P) =: H$.

The block B is an indecomposable $F[G \times G]$-module with vertex $\Delta P = \{(g, g) \mid g \in P\}$ and trivial sources, by [35], Thm. 5.10.8. In particular, $G \times H$ and $H \times G$ both contain $N_{G \times G}(\Delta P)$.

We write

$$V := \mathrm{Res}^{G\times G}_{G\times H}(B) = V_0 \oplus V_1 \quad \text{and} \quad W := \mathrm{Res}^{G\times G}_{H\times G}(B) = W_0 \oplus W_1,$$

with submodules V_0, V_1, W_0, W_1 where V_0 and W_0 are the Green correspondents of B in $G \times H$ and $H \times G$, respectively. Thus V_0 is an indecomposable $F[G \times H]$-module with vertex ΔP belonging to the block $B \otimes b$, and W_0 is an indecomposable $F[H \times G]$-module with vertex ΔP belonging to the block $b \otimes B$. We will show that the functor $W_0 \otimes_B -$ gives a Morita equivalence between B and b.

Remark 4.1. (a) We recall that $P \cong C_3 \times C_3$, $|H| = 144$ and $C_G(P) = C_H(P) \cong C_3 \times C_6$. Moreover, if Q is a subgroup of P of order 3 then $C_G(Q) \cong \mathfrak{A}_4 \times C_6$ where $\mathfrak{A}_4$ denotes the alternating group of degree 4, and $C_H(Q) = C_H(P) \cong C_3 \times C_6$.

(b) We denote the block idempotents of B and b by e and f, respectively. Then f is the unique non-principal block idempotent of $FC_G(P)$, by (a), and $f = z - 1$ where $Z(G) = \langle z \rangle$. Thus $ef = e$.

By making use of Theorem 2.13, we now show:

Proposition 4.2. *The functors $V_0 \otimes_b -$ and $W_0 \otimes_B -$ induce a stable equivalence of Morita type between B and b.*

Proof. Since V_0 has vertex ΔP and trivial source, it suffices to show that condition (2) of Theorem 2.17 is satisfied, with V_0 and W_0 in place of V and W. We show first that the functors $V_0(\Delta P) \otimes_{FC_H(P)\,\mathrm{Br}_P(f)} -$ and $W_0(\Delta P) \otimes_{FC_G(P)\,\mathrm{Br}_P(e)} -$ induce Morita equivalences between $FC_G(P)\,\mathrm{Br}_P(e)$ and $FC_H(P)\,\mathrm{Br}_P(f)$. In fact, by Remark 4.1, we have $C_G(P) = C_H(P)$ and $\mathrm{Br}_P(e) = \mathrm{Br}_P(f) = f = z - 1$. Similarly,

$$V_0(\Delta P) \oplus V_1(\Delta P) = V(\Delta P) = B(\Delta P) = FC_G(P)\,\mathrm{Br}_P(e) = FC_G(P)f$$

is an indecomposable $FC_G(P)f$-$FC_G(P)f$-bimodule. Since $V_0(\Delta P) \neq 0$, this implies $V_1(\Delta P) = 0$. In the same way, $W_0(\Delta P) = FC_G(P)f$ is an indecomposable $FC_G(P)f$-$FC_G(P)f$-bimodule. Trivially the functor $FC_G(P)f \otimes_{FC_G(P)f} -$ induces a Morita equivalence between $FC_G(P)f$ and $FC_G(P)f$.

Now let Q be a subgroup of P of order 3. It remains to show that the functors

$$V_0(\Delta Q) \otimes_{FC_H(Q)\,\mathrm{Br}_Q(f)} - \quad \text{and} \quad W_0(\Delta Q) \otimes_{FC_G(Q)\,\mathrm{Br}_Q(e)} -$$

induce Morita equivalences between $FC_G(Q)\,\mathrm{Br}_Q(e)$ and $FC_H(Q)\,\mathrm{Br}_Q(f)$. By Remark 4.1, we have $C_H(Q) \cong C_3 \times C_6$ and $\mathrm{Br}_Q(f) = f = z - 1$. Also, we have $C_G(Q) \cong \mathfrak{A}_4 \times C_6$. Note that $\mathrm{Br}_Q(e)f = \mathrm{Br}_Q(e)\,\mathrm{Br}_Q(f) = \mathrm{Br}_Q(ef) = \mathrm{Br}_Q(e)$. Since $\mathrm{Br}_Q(e)$ is a sum of block idempotents with defect group P, this means that $\mathrm{Br}_Q(e) = e_1 f$ where e_1 is the principal block idempotent of $F\mathfrak{A}_4$. Since $F\mathfrak{A}_4 e_1 \cong FC_3$, this implies that $FC_G(Q)\,\mathrm{Br}_Q(e) \cong F[C_3 \times C_3] \cong FC_H(Q)\,\mathrm{Br}_Q(f)$. Arguing as above, we also obtain

$$V_0(\Delta Q) = FC_G(Q)\,\mathrm{Br}_Q(e) \cong FC_H(Q)\,\mathrm{Br}_Q(f)$$

and $V_1(\Delta Q) = 0$. Hence the functors $V_0(\Delta Q) \otimes_{FC_H(Q)\,\mathrm{Br}_Q(f)} -$ and $W_0(\Delta Q) \otimes_{FC_G(Q)\,\mathrm{Br}_Q(e)} -$ trivially induce Morita equivalences between the blocks $FC_G(Q)\,\mathrm{Br}_Q(e)$ and $FC_H(Q)\,\mathrm{Br}_Q(f)$. □

We now prove the main result of this section:

Theorem 4.3. *The functor $W_0 \otimes_B -$ induces a Morita equivalence between B and b.*

Proof. By Proposition 4.2 and Theorem 2.13 (ii), it suffices to show that $W_0 \otimes_B D$ is a simple b-module, for every simple B-module D. But

$$W_0 \otimes_B D \,\big|\, W \otimes_B D = \mathrm{Res}^{G\times G}_{H\times G}(B) \otimes_B D = \mathrm{Res}^G_H(D),$$

and $W_0 \otimes_B D$ is an indecomposable non-projective FH-module, by Proposition 4.2 and Theorem 2.13 (i). Since computer calculations show that $\mathrm{Res}^G_H(D)$ has a unique non-projective indecomposable direct summand, $W_0 \otimes_B D$ is the Green correspondent of D in any case, and Proposition 3.19 (i) shows that $W_0 \otimes_B D$ is in fact a simple b-module. This proves the theorem. □

To close, we now consider the bicyclic extension $\widetilde{G} := 2.M.2$ of M and the faithful block B_4 of $F\widetilde{G}$ which has defect group P. As we have seen in Remark 3.22, also all simple modules belonging to B_4 have vertex P, trivial sources and simple Green correspondents. In analogy to the previous theorem, we will show that B_4 is Morita equivalent to its Brauer correspondent b_4 in $N_{\widetilde{G}}(P) =: \widetilde{H}$. For this we set

$$\widetilde{V} := \mathrm{Res}^{\widetilde{G}\times\widetilde{G}}_{\widetilde{G}\times\widetilde{H}}(B_4) = \widetilde{V}_0 \oplus \widetilde{V}_1 \quad \text{and} \quad \widetilde{W} := \mathrm{Res}^{\widetilde{G}\times\widetilde{G}}_{\widetilde{H}\times\widetilde{G}}(B_4) = \widetilde{W}_0 \oplus \widetilde{W}_1,$$

where $\widetilde{V}_0$ and $\widetilde{W}_0$ denote the Green correspondents of B_4 in $\widetilde{G} \times \widetilde{H}$ and $\widetilde{H} \times \widetilde{G}$, respectively. With this we now obtain:

Theorem 4.4. *The functor $\widetilde{W}_0 \otimes_{B_4} -$ induces a Morita equivalence between the blocks B_4 and b_4.*

Proof. The proof is similar to that of Theorem 4.3, so we omit the details here. Notice that

$$C_{\widetilde{H}}(P) = C_{\widetilde{G}}(P) = C_G(P) \cong C_3 \times C_6,$$

and if Q is any subgroup of P of order 3 then

$$C_{\widetilde{G}}(Q) \cong \mathfrak{S}_4 \times C_6 \quad \text{and} \quad C_{\widetilde{H}}(Q) \cong \mathfrak{S}_3 \times C_6.$$

Arguing as in the proof of Proposition 4.2 and applying Theorem 2.17, we deduce that the functors $\widetilde{V}_0 \otimes_{b_4} -$ and $\widetilde{W}_0 \otimes_{B_4} -$ induce a stable equivalence of Morita type between the blocks B_4 and b_4. Moreover, our computations show that, given any simple B_4-module D, the $F\widetilde{H}$-module $\mathrm{Res}^{\widetilde{G}}_{\widetilde{H}}(D)$ has a unique non-projective indecomposable direct summand, namely the simple Green correspondent of D in $\widetilde{H}$. Since, by Theorem 2.13 (i),

$$\widetilde{W}_0 \otimes_{B_4} D \,\big|\, \mathrm{Res}^{\widetilde{G}}_{\widetilde{H}}(D)$$

is indecomposable and non-projective, $\widetilde{W}_0 \otimes_{B_4} D$ is the Green correspondent of D and thus a simple b_4-module. The assertion of the theorem now follows from Theorem 2.13 (ii). □

Acknowledgements. The authors are grateful to the Mathematical Sciences Research Institute in Berkeley for its hospitality and financial support during their stay in the framework of the program 'Representation Theory of Finite Groups and Related Topics'. The first author's research was also supported by the Deutsche Forschungsgemeinschaft through grant # DA 1115/1-1. Moreover, both authors wish to thank R. Kessar, S. Koshitani, M. Linckelmann, K. Lux and J. Müller for helpful discussions on the topic of this note. They also gratefully acknowledge the comments of referees on earlier versions of the manuscript.

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