The ring of modules with endo-permutation source*

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Abstract

In this paper we consider certain subalgebras of the Green algebra (representation algebra) of a finite group G. One such algebra is spanned by the isomorphism classes of all indecomposable modules whose source is an endo-permutation module. This algebra can be embedded into a finite direct product of Laurent polynomial rings in finitely many variables over a field. Another such algebra is spanned by the isomorphism classes of all irreducibly generated modules. When G is p-solvable then this algebra is finite-dimensional and split semisimple.

Introduction

Let p be a prime, let R be either an algebraically closed field of characteristic $p > 0$ or a suitable p-adic ring, and let G be a finite group. We denote by $a(RG)$ the Green ring and by $A(RG) = K \otimes_{\mathbb{Z}} a(RG)$ the Green algebra of the group ring RG; here K is a suitable field of characteristic 0. Then $A(RG)$ is a commutative K-algebra; a K-basis is given by the isomorphism classes $[M]$ of the indecomposable (finitely generated) RG-modules M. Here, and throughout this paper, the notion "RG-module" will always imply that the module is finitely generated and R-free. We do not consider any other modules over group algebras. The multiplication in $A(RG)$ is induced by the tensor product of RG-modules.

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It is known that, for "most" finite groups G, the Green algebra $A(RG)$ is infinite-dimensional and contains nilpotent elements. Thus one is interested in subalgebras of $A(RG)$ with a more transparent structure. An important well-known example of such a subalgebra is the trivial source algebra $A(RG;\mathrm{triv})$. It is spanned by the isomorphism classes of the indecomposable RG-modules whose sources are trivial modules (for their vertices). It is known that the K-algebra $A(RG;\mathrm{triv})$ is finite-dimensional and split semisimple. Moreover, the K-algebra homomorphisms (also called species) $A(RG;\mathrm{triv}) \to K$ can be determined explicitly, cf. for example [B95, Corollary 5.5.5].

In the first part of the paper we investigate certain subalgebras $A(RG, \mathcal{S})$ of the Green algebra $A(RG)$ containing $A(RG;\mathrm{triv})$. Our main example is $A(RG;\mathrm{ep})$; it is spanned by the isomorphism classes of all indecomposable RG-modules whose sources are endo-permutation modules.

We recall that a (finitely generated) RP-module M, for a finite p-group P, is called an endo-permutation module if $\mathrm{Hom}_R(M,M) \cong M \otimes_R M^*$ is a permutation RP-module; here $M^* = \mathrm{Hom}_R(M,R)$ denotes the dual of M. Endo-permutation modules were first investigated by Dade in [D78]. He introduced an important abelian group $\mathcal{D}(RP)$ which is now called the Dade group of RP. Its elements are the isomorphism classes of the indecomposable endo-permutation RP-modules with vertex P. The multiplication in $\mathcal{D}(RP)$ is induced by the tensor product.

By a theorem of Puig in [P90], $\mathcal{D}(RP)$ is always a finitely generated abelian group. Using this fact, we show that the K-algebra $A(RG;\mathrm{ep})$ is always noetherian and does not contain nilpotent elements; more precisely, it can be embedded into a finite direct product of Laurent polynomial rings over K in finitely many variables.

Since $\mathcal{D}(RP)$ is always a finitely generated abelian group, its torsion subgroup $\mathcal{D}_t(RP)$ is finite. The RP-modules M with $[M] \in \mathcal{D}_t(RP)$ are called torsion endo-permutation modules. The isomorphism classes of the indecomposable RG-modules whose sources are torsion endo-permutation modules span a finite-dimensional subalgebra $A(RG;\mathrm{tep})$ of $A(RG;\mathrm{ep})$ still containing $A(RG;\mathrm{triv})$. We show that $A(RG;\mathrm{tep})$ is a split semisimple K-algebra, and we classify the K-algebra homomorphisms $A(RG;\mathrm{tep}) \to K$.

In the last part of the paper, we suppose that $R = F$ is an algebraically closed field of characteristic $p > 0$ and that G is a p-solvable finite group. Following Alperin, an indecomposable FG-module is called irreducibly generated if it is isomorphic to a direct summand of a tensor product of simple FG-modules. By making use of results of Berger and Feit (depending on the classification of finite simple groups), cf. [F82, Theorem X.7.1], we show that the irreducibly generated FG-modules span a finite-dimensional subalgebra $A(FG;\mathrm{irr})$ of $A(FG;\mathrm{tep})$. It follows that $A(FG;\mathrm{irr})$ is a split semisimple K-algebra containing the isomorphism classes of the simple FG-modules.

1 The Green ring

1.1 Throughout, G denotes a finite group, p a prime number, and $\mathcal{O}$ a complete discrete valuation ring of characteristic 0, containing a root of unity of order $\exp(G)$, the exponent of G, with algebraically closed residue field F of characteristic p. By K we denote a field of characteristic 0 containing $\mathcal{O}$ and all roots of unity. We write $\nabla_p(G)$ for the set of p-subgroups of G. Many of the following considerations work for $\mathcal{O}$ as well as for F. For this reason we assume $R \in \{\mathcal{O}, F\}$. If M and N are modules over some ring we write $M \mid N$ if M is isomorphic to a direct summand of N.

1.2 For every subgroup H of G we denote by $\mathcal{I}(RH)$ the set of isomorphism classes $[M]$ of indecomposable RH-modules M (i.e., finitely generated R-free RH-modules, by our convention in the introduction). By $\mathcal{P}(RG)$ we denote the set of pairs $(H, [M])$ with $H \leqslant G$ and $[M] \in \mathcal{I}(RH)$. The group G acts on $\mathcal{P}(RG)$ by conjugation. Moreover, for $(H, [M]), (I, [N]) \in \mathcal{P}(RG)$ we define

$$(I, [N]) \leqslant (H, [M]) \quad \text{if and only if} \quad I \leqslant H \text{ and } N \mid \mathrm{Res}_I^H(M)\,.$$

This defines a partial order on $\mathcal{P}(RG)$ which is respected by G-conjugation: $\mathcal{P}(RG)$ is a G-poset. We will write $(I, [N]) \leqslant_G (H, [M])$ if $(I, [N]) \leqslant {}^g(H, [M])$ for some $g \in G$. The stabilizer of $(I, [N])$ in G will be denoted by $N_G(I, [N])$.

A *vertex-source pair* of $[M] \in \mathcal{I}(RG)$ is a pair $(P, [S]) \in \mathcal{P}(RG)$ which is maximal among all pairs $(Q, [T]) \in \mathcal{P}(RG)$ with the property that $Q \in \nabla_p(G)$, $[T]$ has vertex Q, and $(Q, [T]) \leqslant (G, [M])$. This is equivalent to P being a vertex of M and S being a P-source of M. A vertex-source pair of $[M]$ is uniquely determined up to G-conjugacy by $[M]$.

We will need the following general behaviour of vertex-source pairs with respect to restriction, induction and tensor products later.

1.3 Lemma *Let $[M], [M_1], [M_2] \in \mathcal{I}(RG)$ with vertex-source pairs $(P, [S]), (P_1, [S_1]), (P_2, [S_2])$, respectively. Moreover, let $H \leqslant G$ and let $[N] \in \mathcal{I}(RH)$ with vertex-source pair $(Q, [T])$.*

(a) *If $N \mid \mathrm{Res}_H^G(M)$, then $(Q, [T]) \leqslant_G (P, [S])$.*

(b) *If $M \mid \mathrm{Ind}_H^G(N)$, then $(P, [S]) \leqslant_G (Q, [T])$.*

(c) *If $M \mid M_1 \otimes_R M_2$, then there exist $[T_1], [T_2] \in \mathcal{I}(RP)$ with vertex P such that $S \mid T_1 \otimes_R T_2$ and $(P, [T_i]) \leqslant_G (P_i, [S_i])$ for $i = 1, 2$.*

Proof Parts (a) and (b) are easy consequences of the Mackey decomposition formula. Part (c) is a straightforward application of the Frobenius identity and the Mackey decomposition formula. □

1.4 For $H \leqslant G$, the *Green ring* $a(RH)$ of RH-modules is defined as the free abelian group on the set $\mathcal{I}(RH)$. If M is an arbitrary RH-module and $M = M_1 \oplus \cdots \oplus M_r$ is a decomposition into indecomposable RH-submodules we set

$[M] := [M_1] + \cdots + [M_r] \in a(RH)$. The abelian group $a(RH)$ is a commutative ring with a multiplication satisfying $[M_1] \cdot [M_2] = [M_1 \otimes_R M_2]$ for any RH-modules M_1 and M_2. Moreover, the rings $a(RH)$, $H \leqslant G$, form a Green functor on G under the usual conjugation, restriction and induction maps

$$\begin{aligned} c_{g,H} \colon a(RH) \to a(R\,{}^gH)\,, &\quad [M] \mapsto [{}^gM] = {}^g[M]\,,\\ \mathrm{res}^H_I \colon a(RH) \to a(RI)\,, &\quad [M] \mapsto [\mathrm{Res}^H_I(M)]\,,\\ \mathrm{ind}^H_I \colon a(RI) \to a(RH)\,, &\quad [N] \mapsto [\mathrm{Ind}^H_I(N)]\,, \end{aligned}$$

for $I \leqslant H \leqslant G$, $g \in G$, $[M] \in \mathcal{I}(RH)$, $[N] \in \mathcal{I}(RI)$, cf. for example [T88] for the complete definition of a Green functor. For $H \leqslant G$ we define $A(RH)$ as the K-vector space with basis $\mathcal{I}(RH)$ and we view $a(RH)$ as subset of $A(RH)$. The K-algebras $A(RH)$, $H \leqslant G$, then form a Green functor over K on G.

Recall that a finite group H is called *p-hypoelementary* if it has a normal (Sylow) p-subgroup P such that H/P is cyclic. We will denote by $\mathcal{H}_p(G)$ the set of p-hypoelementary subgroups of G. Recall also that subgroups and factor groups of p-hypoelementary groups are again p-hypoelementary. As usual we denote the largest normal p-subgroup of a finite group H by $\mathrm{O}_p(H)$.

1.5 Theorem (Conlon, [B95, Theorem 5.6.8]) *There exist elements $a_H \in \mathbb{Q}$, $H \in \mathcal{H}_p(G)$, satisfying*

$$[R] = \sum_{H \in \mathcal{H}_p(G)} a_H \cdot \mathrm{ind}^G_H([R]) \in A(RG)\,.$$

For $H \leqslant G$ and a group homomorphism $\varphi : H \to R^\times$ into the unit group of R, we write R_φ for the RH-module with underlying R-module R and H-action given by $h\alpha := \varphi(h)\alpha$ for $h \in H$ and $\alpha \in R$.

The following proposition is well-known. We include a proof for the reader's convenience.

1.6 Proposition *Let $H \in \mathcal{H}_p(G)$, set $P := \mathrm{O}_p(H)$ and let $[M] \in \mathcal{I}(RP)$ be H-stable.*

(a) *There exists an extension $[\tilde{M}] \in \mathcal{I}(RH)$ of $[M]$, i.e., $\mathrm{Res}^H_P(\tilde{M}) \cong M$.*

(b) *One has $\mathrm{Ind}^H_P(M) \cong \bigoplus_\varphi R_\varphi \otimes_R \tilde{M}$, where φ runs through all group homomorphisms $H \to R^\times$ with $P \leqslant \ker(\varphi)$. Moreover, the above summands $R_\varphi \otimes_R \tilde{M}$ are pairwise non-isomorphic. For every $[N] \in \mathcal{I}(RH)$ with $(P, [M]) \leqslant (H, [N])$, one has $N \cong R_\varphi \otimes_R \tilde{M}$ for some φ.*

Proof By Clifford theory, $E := \mathrm{End}_{RH}(\mathrm{Ind}^H_P(M))$ is an R-algebra and is graded by H/P. Since M is H-stable, E is in fact a crossed product algebra. Its 1-component E_1 is isomorphic to the local R-algebra $\mathrm{End}_{RP}(M)$. Thus $E_1/J(E_1)$ is isomorphic to F, and $EJ(E_1) = J(E_1)E$ is a graded ideal of E such that $E/EJ(E_1)$ is a twisted group algebra of H/P over F. Since H/P is cyclic, its Schur multiplier is trivial, so that $E/EJ(E_1)$ is isomorphic to the group algebra of H/P over F. Since H/P is a cyclic p'-group, $1_{E/EJ(E_1)}$ is the sum of $t = [H : P]$ pairwise orthogonal primitive idempotents in $E/EJ(E_1)$. Hence, by lifting theorems for idempotents, 1_E is the sum of t pairwise orthogonal primitive idempotents in E. Thus $\mathrm{Ind}_P^H(M)$ is the direct sum of t submodules $M_1, \ldots, M_t$. On the other hand, $\mathrm{Res}_P^H(M_i) \cong M$ for $i = 1, \ldots, t$, and (a) is proved.

The other parts of the result follow easily. □

Let $H \in \mathcal{H}_p(G)$ and $(I, [N]) \in \mathcal{P}(RH)$. We call $(I, [N])$ *basic* in $\mathcal{P}(RH)$, if $Q := \mathrm{O}_p(I)$ is a vertex of N, $S := \mathrm{Res}_Q^I(N)$ is a source of N, and I/Q is a Hall p'-subgroup of $N_H(Q, [S])/Q$. Moreover, we denote the set of all basic pairs in $\mathcal{P}(RH)$ by $\mathcal{B}(RH)$. Note that $\mathcal{B}(RH)$ is stable under the conjugation action of H on $\mathcal{P}(RH)$. The following result is related to [FK99, Proposition 3.3].

1.7 Proposition *Let $H \in \mathcal{H}_p(G)$. Then the map*

$$\mathcal{B}(RH) \to \mathcal{I}(RH)\,, \quad (I, [N]) \mapsto [\mathrm{Ind}_I^H(N)]\,,$$

defines a bijection $\mathcal{B}(RH)/H \to \mathcal{I}(RH)$. Moreover, if $[M] \in \mathcal{I}(RH)$ and $(I, [N]) \in \mathcal{B}(RH)$ with $M \cong \mathrm{Ind}_I^H(N)$, then the vertex-source pair of $[N]$ is also a vertex-source pair of $[M]$.

Proof (a) Let $[M] \in \mathcal{I}(RH)$. We will show in this part that $M \cong \mathrm{Ind}_I^H(N)$ for some $(I, [N]) \in \mathcal{B}(RH)$.

Set $P := \mathrm{O}_p(H)$, let $[M_0] \in \mathcal{I}(RP)$ be such that $M_0 \mid \mathrm{Res}_P^H(M)$ and set $H_0 := N_H(P, [M_0])$. By Clifford theory, there exists $[\tilde{M}_0] \in \mathcal{I}(RH_0)$ such that $M \cong \mathrm{Ind}_{H_0}^H(\tilde{M}_0)$ and $M_0 \mid \mathrm{Res}_P^{H_0}(\tilde{M}_0)$. By Proposition 1.6(b), we have $M_0 \cong \mathrm{Res}_P^{H_0}(\tilde{M}_0)$.

First we will show that $\tilde{M}_0 \cong \mathrm{Ind}_I^{H_0}(N)$ for some $(I, [N]) \in \mathcal{B}(RH_0)$. Let $(Q, [S])$ be a vertex-source pair of $[M_0]$. Since $M_0 \cong \mathrm{Res}_P^{H_0}(\tilde{M}_0)$ and $\tilde{M}_0 \mid \mathrm{Ind}_P^{H_0}(M_0)$, $(Q, [S])$ is also a vertex-source pair of $[\tilde{M}_0]$. The Frattini argument implies that $H_0 = PN_{H_0}(Q, [S])$. Let I/Q be a Hall p'-subgroup of $N_{H_0}(Q, [S])/Q$. Then $H_0 = IP$ and $Q = I \cap P = \mathrm{O}_p(I)$. Since $\tilde{M}_0 \mid \mathrm{Ind}_Q^{H_0}(S) \cong \mathrm{Ind}_I^{H_0}(\mathrm{Ind}_Q^I(S))$, there exists $[N] \in \mathcal{I}(RI)$ such that $N \mid \mathrm{Ind}_Q^I(S)$ and $\tilde{M}_0 \mid \mathrm{Ind}_I^{H_0}(N)$. By Proposition 1.6(b), we have $\mathrm{Res}_Q^I(N) \cong S$. Thus, $(Q, [S])$ is a vertex-source pair of $[N]$ and $(I, [N]) \in \mathcal{B}(RH_0)$. Since

$$\mathrm{Res}_P^{H_0}\big(\mathrm{Ind}_I^{H_0}(N)\big) \cong \mathrm{Ind}_{P\cap I}^P\big(\mathrm{Res}_{P\cap I}^I(N)\big) \cong \mathrm{Ind}_Q^P(S)$$

is indecomposable by Green's theorem, $\mathrm{Ind}_I^{H_0}(N)$ is indecomposable. In particular, $\tilde{M}_0 \cong \mathrm{Ind}_I^{H_0}(N)$ and $M \cong \mathrm{Ind}_I^H(N)$.

We still need to show that $(I, [N]) \in \mathcal{B}(RH)$. It suffices to show that I/Q is a Hall p'-subgroup of $N_H(Q, [S])/Q$. Assume that there exists a p'-subgroup J/Q of $N_H(Q, [S])/Q$ strictly containing I/Q. Then Proposition 1.6 implies that $\mathrm{Ind}_I^J(N)$ is isomorphic to a direct sum of $[J : I]$ extensions of S. This contradicts the indecomposability of $M \cong \mathrm{Ind}_I^H(N)$.

(b) Next assume that $[M] \in \mathcal{I}(RH)$ and that $(I_1, [N_1]), (I_2, [N_2]) \in \mathcal{B}(RH)$ are such that $M \cong \mathrm{Ind}_{I_j}^H(N_j)$ for $j = 1, 2$. We will show in this part that $(I_1, [N_1])$ and $(I_2, [N_2])$ are H-conjugate.

Let $(Q, [S])$ be a vertex-source pair of $[M]$ and let $(Q_j, [T_j])$ be the vertex-source pair of N_j, for $j = 1, 2$. Since $M \mid \mathrm{Ind}_{I_j}^H(N_j)$ and $N_j \mid \mathrm{Res}_{I_j}^H(\mathrm{Ind}_{I_j}^H(N_j)) \cong \mathrm{Res}_{I_j}^H(M)$, Lemma 1.3 implies that $(Q, [S]) \leqslant_H (Q_j, [T_j])$ and $(Q_j, [T_j]) \leqslant_H (Q, [S])$ for $j = 1, 2$. This implies that $(Q, [S])$, $(Q_1, [T_1])$ and $(Q_2, [T_2])$ are H-conjugate (showing the last statement of the proposition) and we may assume from now on that they are equal. Note that $Q = \mathrm{O}_p(I_1) = \mathrm{O}_p(I_2)$ and that, as Hall p'-subgroups of $N_H(Q, [S])/Q$, I_1/Q and I_2/Q are conjugate in $N_H(Q, [S])/Q$. Thus, we may assume that $I_1 = I_2$ and set $I := I_1 = I_2$. Since $\mathrm{Ind}_I^H(N_j) \cong M$ is indecomposable, also $\mathrm{Ind}_I^{N_H(Q,[S])}(N_j)$ is indecomposable for $j = 1, 2$. But both of them are Green correspondents of M with respect to $(Q, [S])$, and therefore, $\mathrm{Ind}_I^{N_H(Q,[S])}(N_1) \cong \mathrm{Ind}_I^{N_H(Q,[S])}(N_2)$. Let $\tilde{Q}/Q$ be the Sylow p-subgroup of $N_H(Q, [S])/Q$ and set $\tilde{S} := \mathrm{Ind}_Q^{\tilde{Q}}(S)$. We will show that

$$\{[N] \in \mathcal{I}(RI) \mid \mathrm{Res}_P^I(N) \cong S\} \to \{[\tilde{N}] \in \mathcal{I}(RN_H(Q, [S])) \mid \mathrm{Res}_{\tilde{Q}}^{N_H(Q,[S])}(\tilde{N}) \cong \tilde{S}\}\,,$$
$$[N] \mapsto \left[\mathrm{Ind}_I^{N_H(Q,[S])}(N)\right],$$

is a bijection which then implies that $[N_1] = [N_2]$ and finishes the proof of part (b).

Note that $\tilde{S}$ is indecomposable by Green's theorem and also $N_H(Q, [S])$-stable. By Proposition 1.6, $\mathrm{Ind}_Q^{N_H(Q,[S])}(S) \cong \mathrm{Ind}_{\tilde{Q}}^{N_H(Q,[S])}(\tilde{S})$ is a direct sum of the $[N_H(Q, [S]) : \tilde{Q}]$ pairwise non-isomorphic extensions $\tilde{N}$ of $\tilde{S}$. On the other hand, for the same reason, $\mathrm{Ind}_Q^{N_H(Q,[S])}(S) \cong \mathrm{Ind}_I^{N_H(Q,[S])}(\mathrm{Ind}_Q^I(S))$ is isomorphic to the direct sum of the modules $\mathrm{Ind}_I^{N_H(Q,[S])}(N)$, where N runs through the $[I : Q] = [N_H(Q, [S]) : \tilde{Q}]$ non-isomorphic extensions of S to I. Moreover, the modules $\tilde{N}$ and $\mathrm{Ind}_I^{N_H(Q,[S])}(N)$ have the same R-rank. This implies that each $\mathrm{Ind}_I^{N_H(Q,[S])}(N)$ is indecomposable and that they are pairwise non-isomorphic. It follows that the above map is bijective.

(c) In order to finish the proof of the proposition it now suffices to show that for any $(I, [N]) \in \mathcal{B}(RH)$, the module $\mathrm{Ind}_I^H(N)$ is indecomposable. Let $(Q, [S])$ be the vertex-source pair of N. Then the arguments in part (b) of the proof show that $\tilde{N} := \mathrm{Ind}_I^{N_H(Q,[S])}(N)$ is indecomposable. Moreover, since $N \mid \mathrm{Res}_I^{N_H(Q,[S])}(\tilde{N})$, Lemma 1.3 implies that $(Q, [S])$ is a vertex-source pair of $\tilde{N}$. Let $[M] \in \mathcal{I}(RH)$ denote the Green correspondent of $\tilde{N}$ with respect to $(Q, [S])$. Then $M \mid \mathrm{Ind}_I^H(N)$. By part (a) there exists $(I_1, [N_1]) \in \mathcal{B}(RH)$ with $M \cong \mathrm{Ind}_{I_1}^H(N_1)$. By part (b) we know that the vertex-source pair $(Q_1, [S_1])$ of N_1 is also a vertex-source pair of M. Thus, after conjugating $(I_1, [N_1])$, if necessary, we may assume that $(Q, [S]) = (Q_1, [S_1])$. Now, I_1/Q and I/Q are Hall p'-subgroups of $N_H(Q, [S])/Q$ and therefore, $|I_1| = |I|$. Moreover, as extensions of S, N_1 and N have the same R-rank, so that $M \cong \mathrm{Ind}_{I_1}^H(N_1)$

and $\mathrm{Ind}_I^H(N)$ have the same R-rank. Since $M \mid \mathrm{Ind}_I^H(N)$, we obtain that $\mathrm{Ind}_I^H(N) \cong M$ is indecomposable. $\square$

2 Source systems

2.1 Definition (a) A *source system* $\mathcal{S} = (\mathcal{S}(P))_{P\in\nabla_p(G)}$ for G over R consists of a family of subsets $\mathcal{S}(P) \subseteq \mathcal{I}(RP)$, $P \in \nabla_p(G)$, satisfying the following conditions for all $P \in \nabla_p(G)$:

(i) If $[S] \in \mathcal{S}(P)$ then S has vertex P.

(ii) If $[S] \in \mathcal{S}(P)$ and $g \in G$, then $[{}^gS] \in \mathcal{S}({}^gP)$.

(iii) If $(Q, [T]) \leqslant (P, [S])$ are in $\mathcal{P}(RG)$ with $[S] \in \mathcal{S}(P)$ and if T has vertex Q, then $[T] \in \mathcal{S}(Q)$.

(b) A source system $\mathcal{S}$ is called *group-like*, if the following conditions hold for every $P \in \nabla_p(G)$:

(iv) $[R] \in \mathcal{S}(P)$.

(v) If $[S_1], [S_2] \in \mathcal{S}(P)$, then $[S_1]\cdot[S_2] = [S]+[T_1]+\cdots+[T_r]$ with $[S] \in \mathcal{S}(P)$ and with $[T_1], \ldots, [T_r] \in \mathcal{I}(RP)$ having vertices smaller than P.

(vi) The commutative monoid structure on $\mathcal{S}(P)$ defined by $[S_1]*[S_2] := [S]$ in the notation of (v) is a group structure (with identity element $[R]$).

(c) A group-like source system $\mathcal{S}$ is called *finitely generated* if, for every $P \in \nabla_p(G)$, the group $\mathcal{S}(P)$ is finitely generated.

(d) A group-like source system $\mathcal{S}$ is called *finite* if, for every $P \in \nabla_p(G)$, the group $\mathcal{S}(P)$ is finite.

2.2 Remark Note that by Definition 2.1(c) and (d), a finitely generated or finite source system is always group-like.

Let $\mathcal{S}$ be a group-like source system for G over R, and let $[S] \in \mathcal{S}(P)$. Then there exists $[S'] \in \mathcal{S}(P)$ such that $[S] * [S'] = [R]$. Hence there are $[T_1], \ldots, [T_r] \in \mathcal{I}(RP)$ with vertices properly contained in P such that

$$[S]\cdot[S'] = [R] + [T_1] + \cdots + [T_r]\,.$$

Then $R \mid S \otimes_R S'$, so a theorem of Benson (cf. [B95, Theorem 3.1.9], which also holds for $R = \mathcal{O}$, cf. the appendix below) implies that $S' \cong S^*$ so that, in the group $\mathcal{S}(P)$, inverse elements are given by R-duals. Since $p \mid \mathrm{rk}_R T_i$ for $i = 1, \ldots, r$ we also conclude that $(\mathrm{rk}_R S)^2 \equiv 1 \pmod p$, i.e. $\mathrm{rk}_R S \equiv \pm 1 \pmod p$.

2.3 Examples (a) Defining $\mathcal{S}(P) := \{[R]\}$ for $P \in \nabla_p(G)$ leads to a finite source system $\mathcal{S}$.

(b) If, for $P \in \nabla_p(G)$, we define $\mathcal{S}(P)$ to consist of all elements $[M] \in \mathcal{I}(RP)$, where M is an endo-permutation RP-module with vertex P, then $\mathcal{S}$ is a finitely generated source system. Similarly, if we require further that $[M]$ is a torsion element in the Dade group $\mathcal{D}(RP)$, we obtain a finite source system. This follows immediately from the results in [D78] and from the main result in [P90].

(c) The two versions of part (b) can be generalized to group-like source systems by replacing 'endo-permutation' by 'endo-monomial' throughout. This follows from the results in [H04].

(d) If for $P \in \nabla_p(G)$, we set $\mathcal{S}(P) := \{[\Omega^n(R)] \mid n \in \mathbb{Z}\}$, then we obtain a finitely generated source system for which $\mathcal{S}(P)$ is a cyclic group generated by $[\Omega(R)]$.

(e) Another example of a finitely generated source system arises from considering indecomposable endo-trivial RP-modules for $P \in \nabla_p(G)$. Recall that an RP-module M is called *endo-trivial* if $\mathrm{End}_R(M) \cong R \oplus Q$ as RP-modules, where R denotes the trivial RP-module and Q is a projective RP-module. This source system is finite if and only if G has cyclic or generalized quaternion Sylow p-subgroups, cf. [D78, Proposition 9.16].

2.4 Assume that $\mathcal{S}$ is a source system for G over R. For $H \leqslant G$ we define $\tilde{\mathcal{S}}(H) \subseteq \mathcal{I}(RH)$ as the subset of all elements $[M] \in \mathcal{I}(RH)$ which have a vertex-source pair $(P, [S]) \in \mathcal{P}(RH)$ with $[S] \in \mathcal{S}(P)$. We define $a(RH, \mathcal{S})$ as the free abelian group with basis $\tilde{\mathcal{S}}(H)$ and $A(RH, \mathcal{S})$ as $K \otimes_{\mathbb{Z}} a(RH, \mathcal{S})$. Then we can view $a(RH, \mathcal{S})$ and $A(RH, \mathcal{S})$, $H \leqslant G$, in a natural way as subsets of $a(RH)$ and $A(RH)$, respectively. We also view $a(RH, \mathcal{S})$ as a subset of $A(RH, \mathcal{S})$.

For the definition of the notions of a Mackey subfunctor and a Green subfunctor occurring in the following proposition we refer to [T88].

2.5 Proposition *Assume that $\mathcal{S}$ is a source system for G over R.*

(a) *The subgroups $a(RH, \mathcal{S})$, $H \leqslant G$, form a Mackey subfunctor of the family $a(RH)$, $H \leqslant G$.*

(b) *If $\mathcal{S}$ is group-like, then $a(RH, \mathcal{S})$ is a unitary subring of $a(RH)$ for every $H \leqslant G$, and the family $a(RH, \mathcal{S})$, $H \leqslant G$, is a Green subfunctor of the family $a(RH)$, $H \leqslant G$.*

Proof (a) This follows immediately from the definition of a source system and from Lemma 1.3(a) and (b).

(b) This follows immediately from the definition of a group-like source system and Lemma 1.3(c). □

2.6 Examples The examples of group-like source systems from 2.3 lead to Green subfunctors

$$a(RH; \mathrm{triv}) \subseteq a(RH; \mathrm{tep}) \subseteq a(RH; \mathrm{ep}) \subseteq a(RH; \mathrm{em})$$

of $a(RH)$, $H \leqslant G$, which are generated as abelian groups by the elements $[M] \in \mathcal{I}(RH)$ with the following respective properties: M has a source which is a trivial module, a torsion endo-permutation module, an endo-permutation module, and an endo-monomial module, respectively. After tensoring with K we obtain Green subfunctors of $A(RH)$ which we denote as $A(RH, \mathrm{triv})$, etc.

3 Representation rings for group-like source systems

For a source system $\mathcal{S}$ for G over R and for $H \leqslant G$ we define

$$\bar{A}(RH, \mathcal{S}) := A(RH, \mathcal{S}) \Big/ \sum_{I<H} \operatorname{ind}_I^H(A(RI, \mathcal{S}))\,.$$

If $\mathcal{S}$ is group-like, then the above sum is an ideal of $A(RH, \mathcal{S})$ and $\bar{A}(RH, \mathcal{S})$ is a K-algebra.

The following proposition goes back to work of Conlon and Thévenaz.

3.1 Proposition *Let $\mathcal{S}$ be a source system for G over R. Then the map*

$$A(RG, \mathcal{S}) \to \Big(\prod_{H \in \mathcal{H}_p(G)} \bar{A}(RH, \mathcal{S}) \Big)^G$$

induced by the restriction maps res_H^G, $H \in \mathcal{H}_p(G)$, is an isomorphism of K-vector spaces onto the G-fixed points of the above product under the conjugation action of G. If $\mathcal{S}$ is group-like, then the above map is an isomorphism of K-algebras.

Proof This is an easy consequence of [T88, Corollary 4.4] together with Conlon's theorem 1.5. ∎

3.2 Let $\mathcal{S}$ be a source system for G over R, let $H \in \mathcal{H}_p(G)$ and set $P := \mathrm{O}_p(H)$. Moreover let I be a subgroup of H containing P.

For any subset $\mathcal{T} \subseteq \bigcup_{Q \leqslant P} \mathcal{S}(Q)$ we define the set

$$\tilde{\mathcal{S}}(I, \mathcal{T}) := \{[M] \in \tilde{\mathcal{S}}(I) \mid M \text{ has a source contained in } \mathcal{T}\}$$

and the K-vector space

$$A(RI, \mathcal{T}) := \langle \tilde{\mathcal{S}}(I, \mathcal{T}) \rangle_K \subseteq A(RI, \mathcal{S})\,.$$

If we define $\mathcal{S}_{<P} := \bigcup_{Q<P} \mathcal{S}(Q)$, and $\mathcal{S}(P)^H$ as the set of H-stable elements of $\mathcal{S}(P)$ under the conjugation action, then we have a decomposition

$$\tilde{\mathcal{S}}(I) = \tilde{\mathcal{S}}(I, \mathcal{S}_{<P}) \;\cup\; \tilde{\mathcal{S}}(I, \mathcal{S}(P) \smallsetminus \mathcal{S}(P)^H) \;\cup\; \tilde{\mathcal{S}}(I, \mathcal{S}(P)^H) \qquad (3.2.a)$$

into three disjoint subsets and a resulting decomposition

$$A(RI, \mathcal{S}) = A(RI, \mathcal{S}_{<P}) \oplus A(RI, \mathcal{S}(P) \smallsetminus \mathcal{S}(P)^H) \oplus A(RI, \mathcal{S}(P)^H) \qquad (3.2.b)$$

into K-subspaces.

Note that, by Proposition 1.7, for an element $[M] \in \tilde{\mathcal{S}}(I)$, each of the following conditions is equivalent to $[M] \in \tilde{\mathcal{S}}(I, \mathcal{S}(P)^I)$:

(i) $(P, \operatorname{res}_P^I([M]))$ is a vertex-source pair of $[M]$.

(ii) $[M]$ has vertex P and $\operatorname{Res}_P^I(M)$ is indecomposable.

(iii) $[M]$ has a vertex-source pair of the form $(P, [S])$ with I-stable source $[S]$.

3.3 Proposition *Let $\mathcal{S}$ be a source system for G over R, let $H \in \mathcal{H}_p(G)$ and set $P := \mathrm{O}_p(H)$. Then the inclusion $A(RH, \mathcal{S}(P)^H) \to A(RH, \mathcal{S})$ induces an isomorphism*

$$A(RH, \mathcal{S}(P)^H) \Big/ \sum_{P \leqslant I < H} \mathrm{ind}_I^H(A(RI, \mathcal{S}(P)^H)) \to \bar{A}(RH, \mathcal{S})$$

of K-vector spaces. If $\mathcal{S}$ is group-like this is an isomorphism of K-algebras.

Proof Equation (3.2.b) with $I = H$ shows that

$$A(RH, \mathcal{S}) = A(RH, \mathcal{S}_{<P}) \oplus A(RH, \mathcal{S}(P) \smallsetminus \mathcal{S}(P)^H) \oplus A(RH, \mathcal{S}(P)^H). \quad (3.3.a)$$

Moreover, Proposition 1.7 implies:

$$A(RH, \mathcal{S}_{<P}) = \sum_{P \not\leqslant I \leqslant H} \mathrm{ind}_I^H(A(RI, \mathcal{S})), \quad (3.3.b)$$

$$A(RH, \mathcal{S}(P) \smallsetminus \mathcal{S}(P)^H) = \sum_{P \leqslant I < H} \mathrm{ind}_I^H(A(RI, \mathcal{S}(P) \smallsetminus \mathcal{S}(P)^H)), \quad (3.3.c)$$

$$A(RH, \mathcal{S}(P)^H) \supseteq \sum_{P \leqslant I < H} \mathrm{ind}_I^H(A(RI, \mathcal{S}(P)^H)). \quad (3.3.d)$$

Since $\sum_{I<H} \mathrm{ind}_I^H(A(RI, \mathcal{S}))$ is the sum of the right hand sides of Equations (3.3.b), (3.3.c) and (3.3.d), the statement of the proposition follows. □

3.4 Now let $\mathcal{S}$ be a group-like source system for G over R, let $H \in \mathcal{H}_p(G)$, and set $P := \mathrm{O}_p(H)$. We first show that the K-subspace $A(RH, \mathcal{S}_{<P}) \oplus A(RH, \mathcal{S}(P)^H)$ of $A(RH, \mathcal{S})$ is even a K-subalgebra. Since $A(RH, \mathcal{S}_{<P})$ is an ideal in $A(RH, \mathcal{S})$ and since $[R] \in \mathcal{S}(P)^H$ by Definition 2.1(iv), it suffices to show that $[M_1] \cdot [M_2] \in A(RH, \mathcal{S}_{<P}) \oplus A(RH, \mathcal{S}(P)^H)$ for any $[M_1], [M_2] \in \tilde{\mathcal{S}}(H, \mathcal{S}(P)^H)$. However, $[S_i] := \mathrm{res}_P^H([M_i])$ is a source of M_i for $i = 1, 2$, and by Definition 2.1(v) we have $[S_1] \cdot [S_2] = [S] + [T_1] + \cdots + [T_r]$ with $[S] \in \mathcal{S}(P)$ and $[T_i] \in \tilde{\mathcal{S}}(P, \mathcal{S}_{<P})$ for $i = 1, \ldots, r$. Applying conjugation by elements in H to the last equation shows that with $[S_1]$ and $[S_2]$ also $[S]$ lies in $\mathcal{S}(P)^H$. By Proposition 1.7 and the equivalent reformulations (i)–(iii) in 3.2 it follows that there exists an element $[M] \in \tilde{\mathcal{S}}(H, \mathcal{S}(P)^H)$ and elements $[N_1], \ldots, [N_s] \in \tilde{\mathcal{S}}(H, \mathcal{S}_{<P})$ such that

$$[M_1] \cdot [M_2] = [M] + [N_1] + \cdots + [N_s] \quad (3.4.a)$$

and

$$\mathrm{res}_P^H([M]) = [S] \quad \text{and} \quad \mathrm{res}_P^H([N_1] + \cdots + [N_s]) = [T_1] + \cdots + [T_r]\,.$$

This proves that $A(RH, \mathcal{S}_{<P}) \oplus A(RH, \mathcal{S}(P)^H)$ is a subalgebra of $A(RH, \mathcal{S})$.

Since $A(RH, \mathcal{S}_{<P})$ is an ideal in $A(RH, \mathcal{S}_{<P}) \oplus A(RH, \mathcal{S}(P)^H)$ we can endow $A(RH, \mathcal{S}(P)^H)$ with a K-algebra structure via the canonical isomorphism

$$A(RH, \mathcal{S}(P)^H) \to A(RH, \mathcal{S}_{<P}) \oplus A(RH, \mathcal{S}(P)^H) \Big/ A(RH, \mathcal{S}_{<P})$$

onto the latter factor algebra. More precisely, if $[M_1], [M_2] \in \tilde{\mathcal{S}}(H, \mathcal{S}(P)^H)$ are as above, then using Equation (3.4.a) we have

$$[M_1] * [M_2] = [M]\,. \tag{3.4.b}$$

If in the sequel we view $A(RH, \mathcal{S}(P)^H)$ as a K-algebra, we implicitly assume the multiplication $*$ defined above. The last equation also shows that the set $\tilde{\mathcal{S}}(H, \mathcal{S}(P)^H)$ is a monoid under the multiplication $*$ and that $A(RH, \mathcal{S}(P)^H)$ is a monoid algebra.

In the sequel we will use the abbreviation

$$\mathcal{S}(H) := \tilde{\mathcal{S}}(H, \mathcal{S}(P)^H)\,.$$

Note, that by Proposition 1.7 we have a short exact sequence

$$1 \to \mathrm{Hom}(H/P, R^\times) \to \mathcal{S}(H) \to \mathcal{S}(P)^H \to 1\,, \tag{3.4.c}$$

where the first map sends an element φ to R_φ with φ regarded as an element of $\mathrm{Hom}(H, R^\times)$ via inflation, and where the second map is just the restriction map. It is easy to verify that, since the two outer components are groups, so is $\mathcal{S}(H)$. The identity element of $\mathcal{S}(H)$ is the isomorphism class $[R]$ of the trivial RH-module R, and the inverse of $[M] \in \mathcal{S}(H)$ is the isomorphism class $[M^*]$ of the dual module M^*, since $R \mid M \otimes_R M^*$. The latter follows from Benson's Theorem [B95, Theorem 3.1.9] (which also holds for $R = \mathcal{O}$; cf. the appendix below). We conclude that the K-algebra $A(RH, \mathcal{S}(P)^H)$ is a group algebra of the abelian group $\mathcal{S}(H)$ over K.

The short exact sequence (3.4.c) does not split, in general. An example is given by the symmetric group $H = S_3$, and a field F of characteristic 3. In this case $\mathcal{S}(P)^H$ is cyclic of order 2 and generated by the isomorphism class of the Heller translate $\Omega_P(F)$ of the trivial module. On the other hand, $\mathcal{S}(H)$ is easily seen to be cyclic of order 4; for the Heller translate $M := \Omega_H(F)$ is uniserial with two non-isomorphic composition factors, so M is not isomorphic to its dual M^*. Thus $[M]$ has order 4 in $\mathcal{S}(H)$.

The following proposition follows immediately from Proposition 3.3 and Subsection 3.4.

3.5 Proposition *Let $\mathcal{S}$ be a group-like source system for G over R, let $H \in \mathcal{H}_p(G)$ and set $P := \mathrm{O}_p(H)$. Then, for any $[M_1], [M_2] \in \mathcal{S}(H)$, there exist $[N] \in \mathcal{S}(H)$ and an RH-module L with $[L] \in A(RH, \mathcal{S}_{<P})$ such that*

$$[M_1] \cdot [M_2] = [N] + [L]$$

in $A(RH,\mathcal{S})$. Setting $[M_1]*[M_2] := [N]$ defines a group structure on $\mathcal{S}(H)$ such that, with $A(RH,\mathcal{S}(P)^H)$ viewed as the group algebra, the map

$$A(RH,\mathcal{S}(P)^H) \to \bar{A}(RH,\mathcal{S})$$

induced by the inclusion is a surjective K-algebra homomorphism with kernel

$$\sum_{P\leqslant I<H} \mathrm{ind}_I^H(A(RI,\mathcal{S}(P)^H))\,.$$

In Proposition 3.1 the problem of determining the K-algebra structure of $A(RG,\mathcal{S})$ was reduced to the problem of determining the K-algebra structure of $\bar{A}(RH,\mathcal{S})$, for $H \in \mathcal{H}_p(G)$, together with the conjugation action of $N_G(H)$ on $\bar{A}(RH,\mathcal{S})$. In the following Propositions 3.6 and 3.9 and in Corollary 3.11 we investigate the K-algebra structure of $\bar{A}(RH,\mathcal{S})$ in the situations where the group-like source system $\mathcal{S}$ is arbitrary, finitely generated, and finite.

3.6 Proposition *Let $\mathcal{S}$ be a group-like source system for G over R, let $H \in \mathcal{H}_p(G)$ and let $P := \mathrm{O}_p(H)$. Then there exists an isomorphism of K-algebras*

$$\bar{A}(RH,\mathcal{S}) \cong \prod_{\substack{hP\in H/P \\ \langle hP\rangle = H/P}} K_*[\mathcal{S}(P)^H],$$

where the hP-component $K_*[\mathcal{S}(P)^H]$ is a twisted group algebra of $\mathcal{S}(P)^H$ over K (with a cocycle possibly depending on hP).

Proof By 3.4 we can identify the K-algebra $A(RH,\mathcal{S}(P)^H)$ with the group algebra $K[\mathcal{S}(H)]$. The short exact sequence (3.4.c) turns this group algebra into a K-algebra which is graded by the group $\mathcal{S}(P)^H$; more precisely, this $\mathcal{S}(P)^H$-graded K-algebra is a crossed product algebra with 1-component

$$K[\mathrm{Hom}(H/P,R^\times)] = A(R[H/P]) = \bigoplus_{hP\in H/P} K\cdot e_{hP},$$

where

$$e_{hP} = \frac{1}{[H:P]}\sum_{\varphi} \varphi(h^{-1}P)\cdot\varphi$$

is the primitive idempotent which, interpreted as a (class) function on H/P, is the characteristic function of the element hP.

The ideal $\sum_{P\leqslant I<H}\mathrm{ind}_I^H(A(RI,\mathcal{S}(P)^H))$ of $A(RH,\mathcal{S}(P)^H)$ is graded. Thus it is generated (as a left ideal and as a right ideal) by its 1-component which (under the above identification) equals

$$\sum_{P\leqslant I<H}\mathrm{ind}_{I/P}^{H/P}(A(R[I/P])) = \bigoplus_{\substack{hP\in H/P \\ \langle hP\rangle < H/P}} K\cdot e_{hP}.$$

Hence the K-algebra

$$\bar{A}(RH,\mathcal{S}) \cong A(RH,\mathcal{S}(P)^H)/\sum_{P\leqslant I<H} \mathrm{ind}_I^H(A(RI,\mathcal{S}(P)^H)) \\ \cong \bigoplus_{\substack{hP\in H/P\\ \langle hP\rangle = H/P}} A(RH,\mathcal{S}(P)^H)\cdot e_{hP} \tag{3.6.a}$$

is graded by the group $\mathcal{S}(P)^H$ and in fact is a crossed product algebra, with 1-component

$$A(R[H/P])/\sum_{P\leqslant I<H} \mathrm{ind}_{I/P}^{H/P}(A(R[I/P])) \cong \bigoplus_{\substack{hP\in H/P\\ \langle hP\rangle = H/P}} K\cdot e_{hP}.$$

Now it is clear that each K-algebra summand $A(RH,\mathcal{S}(P)^H)\cdot e_{hP}$ of $\bar{A}(RH,\mathcal{S})$ is a twisted group algebra over K with identity element e_{hP}. □

3.7 Remark As apparent from the proof of Proposition 3.6 the group $N_G(H)$ permutes the summands $A(RH,\mathcal{S}(P)^H)\cdot e_{hP}$ in the decomposition (3.6.a) according to the conjugation action of $N_G(H)$ on H/P. We denote the stabilizer of the hP-component by $N_G(H,hP)$ and observe that it acts on $A(RH,\mathcal{S}(P)^H)\cdot e_{hP}$ by conjugation.

3.8 Taking torsion subgroups in the short exact sequence (3.4.c) yields the short exact sequence

$$1\to \mathrm{Hom}(H/P,R^\times)\to \mathcal{S}(H)_t\to \mathcal{S}(P)_t^H\to 1. \tag{3.8.a}$$

Applying the contravariant functor $\mathrm{Hom}(-,K^\times)$ we obtain a sequence

$$1\to \mathrm{Hom}(\mathcal{S}(P)_t^H,K^\times)\to \mathrm{Hom}(\mathcal{S}(H)_t,K^\times)\to H/P\to 1 \tag{3.8.b}$$

after identifying $\mathrm{Hom}(\mathrm{Hom}(H/P,R^\times),K^\times)$ with H/P in the natural way. This sequence is exact since, on the sequence (3.8.a), the functor $\mathrm{Hom}(-,K^\times)$ coincides with the functor $\mathrm{Hom}(-,\mu_K)$, where μ_K denotes the group of roots of unity in K. Note that the group μ_K is divisible since K contains all roots of unity.

3.9 Proposition *In the situation of Proposition 3.6, suppose that $\mathcal{S}$ is finitely generated. Then there exists a K-algebra isomorphism*

$$\bar{A}(RH,\mathcal{S}) \cong \prod_{\lambda\in\Lambda} K[\mathcal{S}(P)^H/\mathcal{S}(P)_t^H],$$

where Λ is the set of all elements in $\mathrm{Hom}(\mathcal{S}(H)_t,K^\times)$ which are mapped to generators of H/P in the exact sequence (3.8.b).

Furthermore, the group algebra $K[\mathcal{S}(P)^H/\mathcal{S}(P)_t^H]$ of the finitely generated free abelian group $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$ over K is isomorphic to the ring $K[x_1,\ldots,x_r,x_1^{-1},\ldots,x_r^{-1}]$ of Laurent polynomials in variables $x_1,\ldots,x_r$, where r denotes the rank of $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$.

Proof Let B be the direct summand $K_*[\mathcal{S}(P)^H]$ corresponding to a fixed generator hP of H/P in the decomposition of $\bar{A}(RH,\mathcal{S})$ in Proposition 3.6. Then B can be viewed as a K-algebra graded by the free abelian group $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$ whose 1-component B_1 is a twisted group algebra of the finite abelian group $\mathcal{S}(P)_t^H$. In fact, B is a crossed product algebra. The 1-component B_1 decomposes as a K-algebra into

$$B_1 = \bigoplus_{\lambda\in\Lambda_{hP}} K\cdot e_\lambda.$$

Here, Λ_{hP} denotes the set of all elements in $\mathrm{Hom}(\mathcal{S}(H)_t,K^\times)$ which are mapped to the element hP in the short exact sequence (3.8.b), as follows by inspecting the proof of Proposition 3.6. Also, e_λ denotes the primitive idempotent of B_1 corresponding to λ. Therefore, we obtain a resulting decomposition

$$B = \bigoplus_{\lambda\in\Lambda_{hP}} B\cdot e_\lambda,$$

where each $B\cdot e_\lambda$ is a twisted group algebra of $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$ over K. In order to see that this twisted group algebra is in fact a group algebra of $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$ over K, consider the short exact sequence

$$1\to K^\times\to U_{gr}(B\cdot e_\lambda)\to \mathcal{S}(P)^H/\mathcal{S}(P)_t^H\to 1, \tag{3.9.a}$$

where $U_{gr}(B\cdot e_\lambda)$ denotes the group of graded units of $B\cdot e_\lambda$. This sequence splits since $U_{gr}(B\cdot e_\lambda)$ is abelian and $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$ is a free abelian group. Now it is clear that, mapping a $\mathbb{Z}$-basis of $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$ bijectively onto a set $\{x_1,\ldots,x_r\}$ of indeterminates, induces a K-algebra isomorphism

$$K[\mathcal{S}(P)^H/\mathcal{S}(P)_t^H]\to K[x_1,\ldots,x_r,x_1^{-1},\ldots,x_r^{-1}]\,,$$

and the proposition follows. □

3.10 Remark The proof of Proposition 3.9 shows that the natural action of $N_G(H)$ on $\bar{A}(RH,\mathcal{S})$ induces an action on Λ and a resulting permutation action on the components $K[\mathcal{S}(P)^H/\mathcal{S}(P)_t^H]$ which is compatible with the action on the set of generators of H/P. The stabilizer $N_G(H,\lambda)$ of each λ in $N_G(H)$ acts on the λ-component. However, it is not clear whether the action of $N_G(H,\lambda)$ on the group algebra $K[\mathcal{S}(P)^H/\mathcal{S}(P)_t^H]$ is simply induced by the action of $N_G(H,\lambda)$ on the group $\mathcal{S}(P)^H/\mathcal{S}(P)_t^H$; the splitting of the exact sequence (3.9.a) may or may not be invariant under the action of $N_G(H,\lambda)$.

The following corollary is a special case of Proposition 3.9.

3.11 Corollary *In the situation of Proposition 3.6, now suppose that $\mathcal{S}$ is a finite source system. Then there exists a K-algebra isomorphism*

$$\bar{A}(RH, \mathcal{S}) \cong \prod_{\lambda \in \Lambda} K$$

where Λ is the set of all elements in $\mathrm{Hom}(\mathcal{S}(H), K^\times)$ *which are mapped to generators of H/P in the exact sequence* (3.8.b).

3.12 For a finitely generated source system $\mathcal{S}$ of G over R we denote by $\mathcal{T}_\mathcal{S}$ the set of all triples

$$(P, h, \lambda)$$

with the following property: $P \in \nabla_p(G)$, h is a p'-element of $N_G(P)$ and, after setting $H := \langle P, h\rangle$, λ is an element in $\mathrm{Hom}(\mathcal{S}(H)_t, K^\times)$ which is mapped to hP in the short exact sequence (3.8.b). Clearly, G acts by conjugation on $\mathcal{T}_\mathcal{S}$. The stabilizer of $(P, h, \lambda) \in \mathcal{T}_\mathcal{S}$ is denoted by $N_G(P, h, \lambda)$.

In order to make the connection between $\mathcal{T}_\mathcal{S}$ and our previous results clearer we associate to $(P, h, \lambda) \in \mathcal{T}_\mathcal{S}$ the triple (H, hP, λ) where $H := \langle P, h\rangle$. This induces a bijection between $\mathcal{T}_\mathcal{S}/G$ and the set of G-conjugacy classes of all triples (H, hP, λ), where $H \in \mathcal{H}_p(G)$, $P = \mathrm{O}_p(H)$, $\langle hP\rangle = H/P$, and λ is an element in $\mathrm{Hom}(\mathcal{S}(H)_t, K^\times)$ which is mapped to hP in the short exact sequence (3.8.b). Note that the conjugacy classes of p'-elements of $N_G(P)$ are in natural bijection with the conjugacy classes of p'-elements of $N_G(P)/P$. We are now in a position to formulate the main result of this paper.

3.13 Theorem *Let $\mathcal{S}$ be a finitely generated source system for G over R. Then there exists a K-algebra isomorphism*

$$A(RG, \mathcal{S}) \cong \prod_{(P,h,\lambda)\in\mathcal{T}_\mathcal{S}/G} K[\mathcal{S}(P)^{\langle h\rangle}/\mathcal{S}(P)_t^{\langle h\rangle}]^{N_G(P,h,\lambda)},$$

where each factor $K[\mathcal{S}(P)^{\langle h\rangle}/\mathcal{S}(P)_t^{\langle h\rangle}]^{N_G(P,h,\lambda)}$ is a noetherian integral domain.

Proof The isomorphism follows from Proposition 3.1, Proposition 3.9, Remark 3.10, and the last part of 3.12. The $N_G(P, h, \lambda)$-fixed points of the Laurent polynomial ring $K[\mathcal{S}(P)^{\langle h\rangle}/\mathcal{S}(P)_t^{\langle h\rangle}]$ in finitely many variables are noetherian by [MS81, Theorem 1]. □

When $\mathcal{S}$ is finite we can be more precise.

3.14 Proposition *Let $\mathcal{S}$ be a finite source system for G over R. Then there exists a K-algebra isomorphism*

$$A(RG, \mathcal{S}) \to \prod_{(P,h,\lambda)\in\mathcal{T}_\mathcal{S}/G} K.$$

The projection $s_{(P,h,\lambda)} : A(RG, \mathcal{S}) \to K$ corresponding to the triple (P, h, λ) is given by

$$A(RG, \mathcal{S}) \xrightarrow{\mathrm{res}^G_H} A(RH, \mathcal{S}) \to \bar{A}(RH, \mathcal{S}) \xrightarrow{\sim} \bar{A}(RH, \mathcal{S}(P)^H) \xrightarrow{\tilde{\lambda}} K^\times,$$

where $H = \langle P, h \rangle$, $\bar{A}(RH, \mathcal{S}(P)^H)$ is the factor algebra of $A(RH, \mathcal{S}(P)^H)$ in Proposition 3.3, the second map is the canonical epimorphism, the third map is the isomorphism from Proposition 3.3, and $\tilde{\lambda}$ is induced by the K-linear extension of λ to the group algebra $K[\mathcal{S}(H)] = A(RH, \mathcal{S}(P)^H)$.

Proof The isomorphism is a special case of Proposition 3.13. That $s_{(P,h,\lambda)}$ is of the given form follows from combining Proposition 3.1, Proposition 3.3, and the determination of the projections $\bar{A}(RH, \mathcal{S}(P)^H) \to K$ in the proof of Proposition 3.9. In fact, such a projection is given by multiplication with e_λ which coincides with applying $\tilde{\lambda}$ after identifying $K \cdot e_\lambda$ and K. □

4 Irreducibly generated modules for p-solvable groups

4.1 Following Alperin, an indecomposable FG-module M is called *irreducibly generated* if there exist simple FG-modules $L_1, \ldots, L_n$ such that

$$M \mid L_1 \otimes_F \cdots \otimes_F L_n$$

(cf. [F82, Section II.5]). Of course, every simple FG-module is irreducibly generated. In the following, we will denote the subgroup of $a(FG)$ generated by all isomorphism classes of irreducibly generated FG-modules by $a(FG; \mathrm{irr})$. It is immediate that $a(FG; \mathrm{irr})$ is a subring of the Green ring $a(FG)$. In the following, we will be interested in the K-algebra $A(FG; \mathrm{irr}) := K \otimes_{\mathbb{Z}} a(FG; \mathrm{irr})$.

4.2 An FG-module M is called *algebraic* if its isomorphism class $[M]$ is a root of a non-zero polynomial with integer coefficients in $a(FG)$. In this case the tensor powers $F,\ M,\ M \otimes_F M, M \otimes_F M \otimes_F M, \ldots$ of M contain only finitely many indecomposable direct summands, up to isomorphism, by [F82, Lemma I.5.1]. In fact, algebraic modules are characterized by this property. Moreover, direct summands, direct sums, tensor products, restrictions and inductions of algebraic modules are again algebraic.

4.3 Now suppose that G is p-solvable. Then, by a result of T. Berger and W. Feit, every simple FG-module is algebraic (cf. [F82, Theorem X.7.1]); the proof of this fact depends on the classification of finite simple groups. Hence every irreducibly generated FG-module M is also algebraic, and the sources of irreducibly generated FG-modules are again algebraic.

On the other hand, it is known that the sources of the simple FG-modules are endo-permutation modules (cf. [T95, Theorem (30.5)]). We prove:

4.4 Proposition *Let G be p-solvable, and let M be an irreducibly generated FG-module with vertex P and P-source S. Then $[S]$ is a torsion element in the Dade group $\mathcal{D}(FP)$.*

Proof By definition, there are simple FG-modules $L_1, \ldots, L_n$ such that $M \mid L_1 \otimes_F \cdots \otimes_F L_n$. For $i = 1, \ldots, n$, we denote by P_i a vertex and by S_i a P_i-source of L_i. Then $L_i \mid \mathrm{Ind}_{P_i}^G(S_i)$ and

$$M \mid L_1 \otimes_F \cdots \otimes_F L_n \mid \mathrm{Ind}_{P_1}^G(S_1) \otimes_F \cdots \otimes_F \mathrm{Ind}_{P_n}^G(S_n)\,.$$

Now Mackey's Tensor Product Theorem implies that there are $g_1, \ldots, g_n \in G$ such that $M \mid \mathrm{Ind}_D^G(N)$ where $D := g_1 P_1 g_1^{-1} \cap \cdots \cap g_n P_n g_n^{-1}$ and N is an indecomposable direct summand of

$$\mathrm{Res}_D^{g_1 P_1 g_1^{-1}}(\,{}^{g_1}S_1) \otimes_F \cdots \otimes_F \mathrm{Res}_D^{g_n P_n g_n^{-1}}(\,{}^{g_n}S_n)\,.$$

Let $(Q, [T])$ be a vertex-source pair of N. Then $(P, [S]) \leqslant_G (Q, [T])$. Since $S_1, \ldots, S_n$ are endo-permutation modules, so are N, T, and S. We assume that $[S]$ has infinite order in $\mathcal{D}(FP)$. Then the powers

$$[S_0] = [F],\ [S_1] = [S],\ [S_2] = [S] * [S],\ [S_3] = [S] * [S] * [S], \ldots$$

of $[S]$ in $\mathcal{D}(FP)$ yield an infinite set of pairwise non-isomorphic indecomposable FP-modules $S_0,\ S_1,\ S_2,\ S_3, \ldots$ with vertex P which occur as direct summands of the tensor powers $F,\ S,\ S \otimes_F S,\ S \otimes_F S \otimes_F S, \ldots$. Thus S cannot be algebraic, and we have reached a contradiction. □

A combination of Proposition 4.4 with the results of Section 3 now implies the following.

4.5 Theorem *Let G be p-solvable. Then $A(FG; \mathrm{irr})$ is a finite-dimensional split semisimple K-algebra.*

Proof By Proposition 4.4, the K-algebra $A(FG; \mathrm{irr})$ is a subalgebra of $A(FG, \mathrm{tep})$. By Puig's theorem in [P90] and Theorem 3.14, $A(FG, \mathrm{tep})$ is a finite-dimensional split semisimple K-algebra, and so is its subalgebra $A(FG; \mathrm{irr})$. □

5 Appendix

The following result is due to Benson, cf. [B95, Theorem 3.1.9], in the case $R = F$. It is easy to see that it also holds for $R = \mathcal{O}$. We provide a proof for the convenience of the reader (cf. also [AC86]).

5.1 Theorem *Let M and N be indecomposable RG-modules. If $R \mid M \otimes_R N$ then $M \cong N^*$ and $p \nmid \mathrm{rk}_R\, M$. Conversely, if $p \nmid \mathrm{rk}_R\, M$ then $R \mid M \otimes_R M^*$.*

Proof Suppose first that $R \mid M \otimes_R N$. Then $R \mid (M \otimes_R N)^* \cong \mathrm{Hom}_R(M, N^*)$ and there exist $f_1 \in \mathrm{Hom}_{RG}(M, N^*)$ and an RG-submodule X of $\mathrm{Hom}_R(M, N^*)$ such that $\mathrm{Hom}_R(M, N^*) = Rf_1 \oplus X$. Let $f_2, \ldots, f_n$ be an R-basis of X. The map

$$\mathrm{Hom}_R(M, N^*) \times \mathrm{Hom}_R(N^*, M) \to R, \quad (\phi, \psi) \mapsto \mathrm{tr}_M(\psi \circ \phi),$$

is a non-degenerate G-equivariant bilinear form. Hence there exist $g_1, \ldots, g_n \in \mathrm{Hom}_R(N^*, M)$ such that $\mathrm{tr}_M(g_i \circ f_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. Then we have $g_1 \in \mathrm{Hom}_{RG}(N^*, M)$, and $Y := Rg_2 + \cdots + Rg_n$ is an RG-submodule of $\mathrm{Hom}_R(N^*, M)$ such that $\mathrm{Hom}_R(N^*, M) = Rg_1 \oplus Y$.

We claim that $\mathrm{tr}_M(h) \in J(R)$ for $h \in J(\mathrm{End}_{RG}(M))$. In fact, if $h \in J(\mathrm{End}_{RG}(M))$ then h is nilpotent modulo $J(R) \cdot \mathrm{End}_{RG}(M)$. Thus its trace $\mathrm{tr}_M(h)$ is zero modulo $J(R)$, and our claim is proved.

Now we have $g_1 \circ f_1 \in \mathrm{End}_{RG}(M) = R \cdot \mathrm{id}_M + J(\mathrm{End}_{RG}(M))$ and $\mathrm{tr}_M(g_1 \circ f_1) = 1$. Thus $g_1 \circ f_1 \notin J(\mathrm{End}_{RG}(M))$ which implies that $g_1 \circ f_1$ is an RG-isomorphism. Since N^* is indecomposable we conclude that f_1 is an RG-isomorphism; in particular, we have $M \cong N^*$.

Also, $\mathrm{tr}_M(g_1 \circ f_1) = 1$ and $\mathrm{End}_{RG}(M) = R \cdot \mathrm{id}_M + J(\mathrm{End}_{RG}(M))$ imply that $\mathrm{tr}_M(\mathrm{id}_M) \notin J(R)$. But $\mathrm{tr}_M(\mathrm{id}_M) = (\mathrm{rk}_R\, M)1_R$, so p does not divide the R-rank of M, and we have proved the first statement of the theorem.

Now suppose conversely that $p \nmid \mathrm{rk}_R\, M$. Then

$$M \otimes_R M^* \cong \mathrm{End}_R(M) = R \cdot \mathrm{id}_M \oplus \ker(\mathrm{tr}_M)$$

is a decomposition into RG-submodules. Since G acts trivially on $R \cdot \mathrm{id}_M$, the result follows. □

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