On the Depth 2 Condition for Group Algebra and Hopf Algebra Extensions*

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Abstract

Let $R \neq 0$ be a commutative ring, and let H be a subgroup of finite index in a group G. We prove that the group ring RG is a ring extension of the group ring RH of depth two if and only if H is a normal subgroup of G. We also show that, under suitable additional hypotheses, an analogous result holds for extensions of Hopf algebras over R.

Introduction

A *ring extension* consists of (associative unitary) rings A, B and a (unitary) ring homomorphism $f\colon A \to B$. Often, A is a (unitary) subring of B and f is the corresponding inclusion map. In general, B becomes an (A,A)-bimodule via $a_1ba_2 := f(a_1)bf(a_2)$ for $a_1, a_2 \in A$ and $b \in B$. General background on ring extensions can be found in [K99].

In the following, we denote by ${}_A\mathsf{Mod}$ the category of all left A-modules, by Mod_A the category of all right A-modules, and by ${}_A\mathsf{Mod}_B$ the category of all (A,B)-bimodules, for an arbitrary ring B. Whenever A and B are algebras over a commutative ring R we will tacitly assume that, for an (A,B)-bimodule M, the induced actions of R on the left and on the right of M coincide. For objects M, N in an abelian category C, we write $M \mid N$ if M is isomorphic to a direct summand of N. Equivalently there exist morphisms $i\colon M \to N$ and $p\colon N \to M$ in C such that $p \circ i = \mathrm{id}_M$.

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A ring extension $A \to B$ is said to have *left depth two* (resp. *right depth two*) if there exists a positive integer k such that

$$B \otimes_A B \mid B^k \quad \text{in} \quad {}_A\mathsf{Mod}_B \quad (\text{resp. } {}_B\mathsf{Mod}_A).$$

It is said to have *depth two* if it has both left depth two and right depth two. These notions of depth were introduced in [KL03]. They were motivated by and extend concepts in [GHJ89] and [KN01].

For a category C, an abelian category A, and functors $\mathcal{F}, \mathcal{G} \colon \mathsf{C} \to \mathsf{A}$ we define $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F}^k$ for a positive integer k in the obvious way. They are again functors from C to A. Also, if I is a finite set we write $\mathcal{F}^I$ for the direct sum of $|I|$ copies of $\mathcal{F}$, each copy indexed by an element in I. Finally, we write $\mathcal{F} \mid \mathcal{G}$ if there exist natural transformations $\iota \colon \mathcal{F} \to \mathcal{G}$ and $\pi \colon \mathcal{G} \to \mathcal{F}$ with $\pi \circ \iota = \mathrm{id}_{\mathcal{F}}$.

The depth 2 conditions can also be interpreted through restriction and induction functors: The ring extension $A \to B$ has left depth two if and only if there exists a positive integer n such that

$$\mathrm{Res}_A^B \mathrm{Ind}_A^B \mathrm{Res}_A^B \mid (\mathrm{Res}_A^B)^n \tag{$*$}$$

as functors from ${}_B\mathsf{Mod}$ to ${}_A\mathsf{Mod}$; or equivalently if there exists a positive integer n such that

$$\mathrm{Ind}_A^B \mathrm{Res}_A^B \mathrm{Ind}_A^B \mid (\mathrm{Ind}_A^B)^n \tag{$**$}$$

as functors from Mod_A to Mod_B. A similar statement holds for the right depth 2 condition. We will show (under mild assumptions) that in the case of group algebra extensions (see Section 1) and Hopf algebra extensions (see Section 2) there already exist natural transformations between the functors in $(*)$ and $(**)$ which govern the depth 2 conditions. These natural transformations are split if and only if the conditions in $(*)$ and $(**)$ hold. A similar phenomenon occurs for the notion of *relativ projectivity* in the case of group algebras, as is pointed out in Section 3.

A ring extension $A \to B$ is called a *Frobenius extension* if there exist an (A, A)-bimodule homomorphism $E \colon B \to A$ and elements $x_1, y_1, \ldots, x_n, y_n \in B$ such that

$$\sum_{i=1}^n x_i E(y_i b) = b = \sum_{i=1}^n E(b x_i) y_i \quad \text{for} \quad b \in B.$$

Other characterizations of Frobenius extensions can be found in [K99]. By Proposition 6.4 in [KL03], a Frobenius extension $A \to B$ has left depth two if and only if it has right depth two.

Suppose that $R \neq 0$ is a commutative ring and that H is a subgroup of finite index in a group G. Then the group ring RG is a Frobenius extension of the group ring RH. If H is normal in G then, by example 3.9 in [KL03], the ring extension $RH \subseteq RG$ has depth 2. (Actually, R was supposed to be a field in [KL03], but the argument given there also applies in this greater generality.)

As a partial converse, it was proved in [KK06] that H is normal in G whenever H is a subgroup of a finite group G such that the complex group algebra $\mathbb{C}G$ is a ring extension of the complex group algebra $\mathbb{C}H$ of depth two. The proof used complex characters. It remained open whether a similar result holds for more general coefficient rings. In Section 1 below, we will provide a positive solution to this open problem.

In Section 2, we will consider, more generally, the same problem for Hopf algebras over R. We will show that, under suitable additional assumptions, a Hopf algebra H over R is a ring extension of depth 2 of a Hopf subalgebra K if and only if K is normal in H. We will also show that this is equivalent to H being a Hopf Galois extension of K for a naturally arising Hopf algebra $\bar{H}$. These results indicate that the depth two property is a suitable ring-theoretic analogue of the concept of normality in group theory and Hopf algebra theory. Finally, in Section 3, we will point out how our main results of Section 1 and Section 2 on the depth two property have the same flavor as a similar result for the more familiar concept of relative projectivity.

1 Depth 2 for Group Algebra Extensions

Throughout this section, R is a non-zero commutative ring, G is a group and $H \leqslant G$ is a subgroup of finite index. We denote the group ring of G over R by RG and the trivial RG-module by R_G. We recall that the *core* K of H in G is the largest normal subgroup of G contained in H. Thus $K = \bigcap_{g\in G} gHg^{-1}$ is the intersection of all conjugates of H in G. Since H has finite index in G, the factor group G/K is finite.

The partitioning of G into its double cosets D with respect to H and H yields a direct sum decomposition $RG = \bigoplus_{D\in H\backslash G/H} RD$ of RG into (RH, RH)-bimodules. For $D \in H\backslash G/H$, we define $p_D\colon RG \to RD$ as the corresponding projection map.

For every left RG-module M we define the RH-module homomorphism

$$\pi_M\colon RG \otimes_{RH} M \to \mathrm{Res}^G_H(M)^{H\backslash G/H}\,, \quad a\otimes m \mapsto (p_D(a)m)_{D\in H\backslash G/H}\,.$$

The collection of these homomorphisms forms a natural transformation

$$\pi := (\pi_M)\colon \mathrm{Res}^G_H \mathrm{Ind}^G_H \mathrm{Res}^G_H \to (\mathrm{Res}^G_H)^{H\backslash G/H}$$

between functors from ${}_{RG}\mathsf{Mod}$ to ${}_{RH}\mathsf{Mod}$. For $M = RG$, we obtain the map

$$\pi_{RG}\colon RG\otimes_{RH} RG \to RG^{H\backslash G/H}\,, \quad a\otimes b \mapsto (p_D(a)b)_{D\in H\backslash G/H}$$

which is an (RH, RG)-bimodule homomorphism.

Similarly, for every left RH-module N, we define the RG-module homomorphism

$$\pi'_N\colon RG\otimes_{RH} RG\otimes_{RH} N \to (RG\otimes_{RH} N)^{H\backslash G/H}\,,$$
$$a\otimes b\otimes n \mapsto (ap_D(b)\otimes n)_{D\in H\backslash G/H}\,.$$

These homomorphisms define a natural transformation

$$\pi' := (\pi'_N)\colon \mathrm{Ind}_H^G\mathrm{Res}_H^G\mathrm{Ind}_H^G \to (\mathrm{Ind}_H^G)^{H\backslash G/H}$$

of functors from ${}_{RH}\mathsf{Mod}$ to ${}_{RG}\mathsf{Mod}$. For $N = RH$ we compose π'_{RH} with the obvious canonical isomorphisms to obtain the map

$$\pi'_{RH}\colon RG\otimes_{RH} RG \to RG^{H\backslash G/H}\,,\quad a\otimes b\mapsto (ap_D(b))_{D\in H\backslash G/H}\,,$$

which is an (RG, RH)-bimodule homomorphism, again denoted by π'_{RH}.

1.1 Remark (a) It is easy to see that π_M and π'_N are epimorphisms for every $M\in {}_{RG}\mathsf{Mod}$ and every $N\in {}_{RH}\mathsf{Mod}$.

(b) It is also easy to verify that π_{RG} (resp. π'_{RH}) is an isomorphism if and only if π_M (resp. π'_N) is an isomorphism for every $M\in {}_{RG}\mathsf{Mod}$ (resp. $N\in {}_{RH}\mathsf{Mod}$).

(c) If H is normal in G then π_{RG} and π'_{RH} are isomorphisms. In fact, let $D = HgH = gH$ and define $\iota_D\colon RG\to RG\otimes_{RH} RG$ by $\iota_D(a) := g\otimes g^{-1}a$ for $a\in RG$. Then ι_D is independent of the choice of g in D and the sum of the maps ι_D provides a two-sided inverse of π_{RG}. Similarly, the maps $\iota'_D\colon RG\to RG\otimes_{RH} RG$, defined by $\iota'_D(a) := ag^{-1}\otimes g$ for $a\in RG$, lead to an inverse of π'_{RH}.

Note that if $\mathcal{F}$ and $\mathcal{G}$ are functors from a category C to an abelian category A such that $\mathcal{F}(C)\mid\mathcal{G}(C)$ for every object $C\in\mathsf{C}$ then it does in general not follow that $\mathcal{F}\mid\mathcal{G}$. However, in the situation of the theorem below this will be the case in an even stronger sense (see (iv)⇒(ii)⇒(vii)).

1.2 Theorem *The following are equivalent:*

(i) *The ring extension $RH\subseteq RG$ has left depth 2.*

(i') *The ring extension $RH\subseteq RG$ has right depth 2.*

(ii) *There exists a positive integer k such that $\mathrm{Res}_H^G\mathrm{Ind}_H^G\mathrm{Res}_H^G\mid(\mathrm{Res}_H^G)^k$ as functors from ${}_{RG}\mathsf{Mod}$ to ${}_{RH}\mathsf{Mod}$.*

(ii') *There exists a positive integer l such that $\mathrm{Ind}_H^G\mathrm{Res}_H^G\mathrm{Ind}_H^G\mid(\mathrm{Ind}_H^G)^l$ as functors from ${}_{RH}\mathsf{Mod}$ to ${}_{RG}\mathsf{Mod}$.*

(iii) *For every left RG-module M there exists a positive integer k (possibly depending on M) such that*

$$\mathrm{Res}_H^G\mathrm{Ind}_H^G\mathrm{Res}_H^G(M)\mid(\mathrm{Res}_H^G(M))^k\quad\text{in } {}_{RH}\mathsf{Mod}.$$

(iii') *For every left RH-module N there exists a positive integer l (possibly depending on N) such that*

$$\mathrm{Ind}_H^G\mathrm{Res}_H^G\mathrm{Ind}_H^G(N)\mid(\mathrm{Ind}_H^G(N))^l\quad\text{in } {}_{RG}\mathsf{Mod}.$$

(iv) *There exists a positive integer k such that*

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G \mathrm{Res}_H^G(R_G) \mid (\mathrm{Res}_H^G(R_G))^k \quad \textit{in } {}_{RH}\mathsf{Mod}.$$

(iv') *There exists a maximal ideal I of R satisfying: For every simple left RH-module N which is annihilated by I and on which the core K of H in G acts trivially, there exists a positive integer l such that*

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(N) \mid (\mathrm{Ind}_H^G(N))^l \quad \textit{in } {}_{RG}\mathsf{Mod}.$$

(v) *H is normal in G.*

(vi) *The homomorphism $\pi_{RG}\colon RG \otimes_{RH} RG \to RG^{H\backslash G/H}$ of (RH, RG)-bimodules is an isomorphism.*

(vi') *The homomorphism $\pi'_{RH}\colon RG \otimes_{RH} RG \to RG^{H\backslash G/H}$ of (RG, RH)-bimodules is an isomorphism.*

(vii) *The natural transformation $\pi\colon \mathrm{Res}_H^G \mathrm{Ind}_H^G \mathrm{Res}_H^G \to (\mathrm{Res}_H^G)^{H\backslash G/H}$ between functors from ${}_{RG}\mathsf{Mod}$ to ${}_{RH}\mathsf{Mod}$ is an isomorphism.*

(vii') *The natural transformation $\pi'\colon \mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G \to (\mathrm{Ind}_H^G)^{H\backslash G/H}$ between functors from ${}_{RH}\mathsf{Mod}$ to ${}_{RG}\mathsf{Mod}$ is an isomorphism.*

Proof In a first part of the proof we establish the chain of implications and equivalences (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(vi)⇔(vii)⇒(i).

(i)⇒(ii): By (i), there exist a positive integer k and (RH, RG)-bimodule homomorphisms $i\colon RG \otimes_{RH} RG \to RG^k$ and $p\colon RG^k \to RG \otimes_{RH} RG$ such that $p \circ i = \mathrm{id}$. For every $M \in {}_{RG}\mathsf{Mod}$, these homomorphisms induce left RH-module homomorphisms

$$i_M\colon RG \otimes_{RH} M \cong RG \otimes_{RH} RG \otimes_{RG} M \to (RG)^k \otimes_{RG} M \cong \mathrm{Res}_H^G(M)^k$$

and

$$p_M\colon \mathrm{Res}_H^G(M)^k \cong (RG)^k \otimes_{RG} M \to RG \otimes_{RH} RG \otimes_{RG} M \cong RG \otimes_{RH} M$$

which are functorial in M and satisfy $p_M \circ i_M = \mathrm{id}$. This implies (ii).

(ii)⇒(iii)⇒(iv): This is trivial.

(iv)⇒(v): The hypothesis (iv) implies that $\mathrm{Res}_H^G \mathrm{Ind}_H^G(R_H) \mid (R_H)^k$. Using the Mackey decomposition formula ([CR81, Theorem 10.13]) we obtain that for all $g \in G$, the permutation RH-module $R[H/H \cap gHg^{-1}]$ is a direct summand of $(R_H)^k$. Thus, H acts trivially on $R[H/H \cap gHg^{-1}]$. But this implies that $H = gHg^{-1}$.

(v)⇒(vi)⇔(vii): This was already observed in Remark 1.1.

(vi)⇒(i): This is trivial.

In the second part of the proof we establish the chain of implications and equivalences (i')⇒(ii')⇒(iii')⇒(iv')⇒(v)⇒(vi')⇔(vii')⇒(i'). All implications, except for (iv')⇒(v), can be proved in the same way as their correspondents in the first part of the proof.

(iv')⇒(v): Set $F := R/I$. The hypothesis (iv') implies that $\mathrm{Ind}_{\bar H}^{\bar G} \mathrm{Res}_{\bar H}^{\bar G} \mathrm{Ind}_{\bar H}^{\bar G}(N) \mid (\mathrm{Ind}_{\bar H}^{\bar G}(N))^l$ for every simple $F\bar H$-module N where $\bar H :=$

H/K and $\bar{G} := G/K$ are finite groups. Thus , by Lemma 1.3 below, $p := \mathrm{char}(F)$ does not divide $|\bar{H}|$, and $F_{\bar{H}}$ is the only simple $F\bar{H}$-module, up to isomorphism. Hence $\bar{H} = 1$, and $H = K$ is normal in G. $\square$

1.3 Lemma *Let F be a field of characteristic $p \geq 0$, and let H be a subgroup of a finite group G with trivial core. Moreover, let N be a simple FH-module such that $\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(N) \mid (\mathrm{Ind}_H^G(N))^l$ for some positive integer l. Then $|H|$ is not divisible by p, and $N \cong F_H$.*

Proof The hypothesis and the Mackey decomposition formula imply that $\mathrm{Ind}_{H\cap gHg^{-1}}^G \mathrm{Res}_{H\cap gHg^{-1}}^H(N) \mid (\mathrm{Ind}_H^G(N))^l$ for all $g \in G$. Applying $\mathrm{Ind}_H^G \mathrm{Res}_H^G$ repeatedly and again using the Mackey decomposition formula, we obtain by induction on r that

$$\mathrm{Ind}_{H\cap g_1Hg_1^{-1}\cap\cdots\cap g_rHg_r^{-1}}^G \mathrm{Res}_{H\cap g_1Hg_1^{-1}\cap\cdots\cap g_rHg_r^{-1}}^H(N) \mid (\mathrm{Ind}_H^G(N))^{l^r} ,$$

for any $g_1, \ldots, g_r \in G$. Choosing $g_1, \ldots, g_r$ in such a way that $H \cap g_1Hg_1^{-1} \cap \cdots \cap g_rHg_r^{-1} = 1$ we deduce:

$$FG \mid FG^{\dim N} \cong \mathrm{Ind}_1^G \mathrm{Res}_1^H(N) \mid (\mathrm{Ind}_H^G(N))^m$$

where $m := l^r$. Since $\mathrm{Hom}_{FG}(FG, F_G) \neq 0$ we conclude that

$$0 \neq \mathrm{Hom}_{FG}(\mathrm{Ind}_H^G(N)^m, F_G) \cong \mathrm{Hom}_{FG}(\mathrm{Ind}_H^G(N), F_G)^m \cong \mathrm{Hom}_{FH}(N, F_H)^m,$$

by using Frobenius reciprocity. But now Schur's Lemma implies that $N \cong F_H$. So now we have $FG \mid (\mathrm{Ind}_H^G(F_H))^m$. Let $\sigma := \sum_{g\in G} g$. Since $\sigma FG \neq 0$ we also have $0 \neq \sigma \mathrm{Ind}_H^G(F_H)$. Thus there exists $x \in G$ such that

$$0 \neq \sigma(x \otimes 1) = \sigma \otimes 1 = |H| \sum_{gH\in G/H} g \otimes 1,$$

and we conclude that p does not divide $|H|$. $\square$.

1.4 Remark (a) It does not suffice to require the property in (iv') only for the trivial RH-module $N = R_H$ in order to derive $H \trianglelefteq G$: Let G be a non-abelian finite simple group (for instance, $G = A_5$, the alternating group of degree 5), let H be a subgroup of order 2 and let $R = \mathbb{C}$. Then, H is not normal in G, but $\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(\mathbb{C}_H) \mid (\mathrm{Ind}_H^G(\mathbb{C}_H))^l$ for some positive integer l, since $\mathrm{Ind}_H^G(\mathbb{C}_H)$ has every irreducible $\mathbb{C}G$-module as constituent. In fact, assume that there exists an irreducible character χ of G not occurring in the permutation character $\mathrm{ind}_H^G(1_H)$. Then 1_H is not a constituent of $\chi|_H$ by Frobenius reciprocity. This implies that $\chi|_H$ is a multiple of the only other non-principal irreducible character of H, and further that H is contained in $Z(\chi)$, the center of χ. Since $Z(\chi)$ is normal in G and G is simple, we have $Z(\chi) = G$. This implies that χ has degree

1. Since G is non-abelian and simple, the derived subgroup of G is equal to G. Thus, the trivial character of G is the only character of G of degree 1. Hence, χ is the trivial character. But the trivial character is always a constituent of $\mathrm{ind}_H^G(1_H)$. This is a contradiction.

(b) Since the condition (iv') in Theorem 1.2 looks very technical compared to the other, we we want to mention that it is equivalent to the following:

($*$) *There exist positive integers l, m such that*

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(R_H) \mid (\mathrm{Ind}_H^G(R_H))^l$$

and

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G(A_H) \mid (\mathrm{Ind}_H^G(A_H))^m\,,$$

where A_H denotes the kernel of the augmentation map $RH \to R$.

In fact, (iii') clearly implies ($*$). So it suffices to show that ($*$) implies (v). Let I be a maximal ideal of R, so that $F := R/I$ is a field. As before, K denotes the core of H in G. Then $\bar{H} := H/K$ and $\bar{G} := G/K$ are finite groups. Moreover, ($*$) implies easily that

$$\mathrm{Ind}_{\bar{H}}^{\bar{G}} \mathrm{Res}_{\bar{H}}^{\bar{G}} \mathrm{Ind}_{\bar{H}}^{\bar{G}}(F_{\bar{H}}) \mid (\mathrm{Ind}_{\bar{H}}^{\bar{G}}(F_{\bar{H}}))^l$$

and

$$\mathrm{Ind}_{\bar{H}}^{\bar{G}} \mathrm{Res}_{\bar{H}}^{\bar{G}} \mathrm{Ind}_{\bar{H}}^{\bar{G}}(\bar{A}_{\bar{H}}) \mid (\mathrm{Ind}_{\bar{H}}^{\bar{G}}(\bar{A}_{\bar{H}}))^m$$

where $\bar{A}_{\bar{H}}$ is the kernel of the augmentation map $F\bar{H} \to F$. By Lemma 1.3, the first property implies that $|\bar{H}|$ is not divisible by the characteristic of F. If $\bar{H} \neq 1$ then the proof of Lemma 1.3 leads to the contradiction $\mathrm{Hom}_{F\bar{H}}(\bar{A}_{\bar{H}}, F_{\bar{H}}) \neq 0$. So we must have $\bar{H} = 1$, and $H = K$ is normal in G.

(c) It is easy to check that the conditions in Theorem 1.2 hold if and only if the natural epimorphism π is split, or equivalently, if the natural epimorphism π' is split.

(d) If one wanted to only prove the equivalence of the three conditions (i), (i') and (v) in Theorem 1.2, there exists a shorter proof. In fact, (v) implies (i) and (i') as was already observed in Remark 1.1. Conversely, (i) implies (v) by very short arguments given in the proof of the Theorem. Finally, that (i') implies (v) can be shown in the same way as in the proof that (i) implies (v), by using right modules instead of left modules. So altogether, one has that the extension $RH \subseteq RG$ has depth 2 if and only if H is normal in G, and the proof of this fact has become almost trivial.

2 Depth 2 for Hopf Algebra Extensions

Throughout this section we assume that R is a commutative ring and that $i\colon K \to H$ is a (not necessarily injective) homomorphism of Hopf algebras over R. We denote the multiplication, unit, comultiplication, counit, and antipode of H by μ_H, η_H, Δ_H, ϵ_H, and S_H, respectively. For K we adopt a similar notation. Unadorned tensor products will always stand for tensor products over

R. Throughout we will make use of the “Sweedler notation” $\sum_{(a)} a_1 \otimes \cdots \otimes a_n$ for the $(n-1)$-fold application of Δ_H to an element $a \in H$. We will consider H as (K,K)-bimodule with the usual structure maps induced by i.

2.1 We recall some basic notions, notations, and facts about Hopf algebras that will be used in this section.

(a) The *left adjoint action* of H on itself is defined as the R-algebra homomorphism

$$\mathrm{ad}_{\mathrm{l}} \colon H \to \mathrm{End}_R(H)\,, \quad a \mapsto (b \mapsto \sum_{(a)} a_1 b S(a_2))\,,$$

and the *right adjoint action* of H on itself is defined as the R-algebra antihomomorphism

$$\mathrm{ad}_{\mathrm{r}} \colon H \to \mathrm{End}_R(H)\,, \quad a \mapsto (b \mapsto \sum_{(a)} S(a_1) b a_2)\,.$$

(b) Generalizing Definition 3.4.1 in [M93] we say that the extension $i \colon K \to H$ of Hopf algebras over R is *left* (resp. right) *normal* if $i(K)$ is stable under the left (resp. right) adjoint action of H, i.e., if

$$(\mathrm{ad}_{\mathrm{l}}(a))(i(K)) \subseteq i(K) \quad (\text{resp. } (\mathrm{ad}_{\mathrm{r}}(a))(i(K)) \subseteq i(K)\,)\,,$$

for all $a \in H$. It is called *normal* if it is left and right normal.

(c) We set

$$K^+ := \ker(\epsilon_K)\,.$$

This is a two-sided ideal of K and one has $K = K^+ \oplus R1_K$. Note also that $i(K^+) = i(K) \cap \ker(\epsilon_H)$. Furthermore, we set

$$I := Hi(K^+)H \quad \text{and} \quad \bar{H} := H/I\,.$$

Note that I is an ideal of H and $\bar{H}$ is again an R-algebra. We denote by $\pi \colon H \to \bar{H}$, $a \mapsto \bar{a}$, the corresponding natural epimorphism of R-algebras. For $x \in K$ we have

$$\overline{i(x)} = \epsilon_K(x)\bar{1}_H\,, \tag{2.1.a}$$

since $x - \epsilon_K(x) \in K^+$. If $Hi(K^+) \subseteq i(K^+)H$ then $I = i(K^+)H$, and if $i(K^+)H \subseteq Hi(K^+)$ then $I = Hi(K^+)$.

(d) Recall from [M93, Definition 1.6.2] that a right H-comodule structure on an R-module M is an R-module homomorphism $\rho \colon M \to M \otimes H$ such that $(\rho \otimes \mathrm{id}_H) \circ \rho = (\mathrm{id}_M \otimes \Delta_H) \circ \rho$ and $\mathrm{can} \circ (\mathrm{id}_M \otimes \epsilon_H) \circ \rho = \mathrm{id}_M$, where $\mathrm{can} \colon M \otimes R \to M$ is the natural isomorphism. Similarly, one defines left H-comodule structures.

Recall from [M93, Definition 4.1.2] that an R-algebra A is called a right H-comodule algebra if it is a right H-comodule whose structure map is an R-algebra homomorphism. Similarly, one defines left H-comodule algebras.

Most parts of the following lemma are well-known. Proofs in the case that R is a field and i is an inclusion can be found in [M93, Lemma 3.4.2]. We can adapt them easily to our situation and include them for the reader's convenience where they cannot be cited verbatim.

2.2 Lemma (a) *The R-algebra structure of $\bar{H}$ can be extended to a unique Hopf algebra structure over R such that $\pi\colon H \to \bar{H}$ is a homomorphism of Hopf algebras over R.*

(b) *H is a right $\bar{H}$-comodule algebra with comodule structure map*

$$\rho := (\mathrm{id}_H \otimes \pi) \circ \Delta_H\colon H \to H \otimes \bar{H}\,, \quad a \mapsto \sum_{(a)} a_1 \otimes \bar{a}_2\,,$$

and its coinvariants

$$H^{\mathrm{co}\bar{H}} := \{h \in H \mid \rho(h) = h \otimes \bar{1}_H\}$$

form an $\mathrm{ad}_{\mathrm{l}}(H)$-stable R-submodule of H.

Similarly, H is a left $\bar{H}$-comodule algebra with comodule structure map $\rho' := (\pi \otimes \mathrm{id}_H) \circ \Delta_H$ and its coinvariants ${}^{\mathrm{co}\bar{H}}H := \{h \in H \mid \rho'(h) = \bar{1}_H \otimes h\}$ form an $\mathrm{ad}_{\mathrm{r}}(H)$-stable R-submodule of H.

(c) *The map*

$$\beta\colon H \otimes_K H \to H \otimes \bar{H}\,, \quad a \otimes b \mapsto \sum_{(b)} ab_1 \otimes \bar{b}_2 = (a \otimes \bar{1}_H)\rho(b)\,,$$

is well-defined and an (H, K)-bimodule homomorphism. Here, $H \otimes_K H$ is viewed as (H, K)-bimodule via $c(a \otimes b)x := ca \otimes bi(x)$ and $H \otimes \bar{H}$ is viewed as (H, K)-bimodule via $c(a \otimes \bar{b})x := cai(x) \otimes \bar{b}$, for $a, b, c \in H$ and $x \in K$.

Similarly, $H \otimes_K H$ and $\bar{H} \otimes H$ are (K, H)-bimodules, and the map

$$\beta'\colon H \otimes_K H \to \bar{H} \otimes H, \quad a \otimes b \mapsto \sum_{(a)} \bar{a}_1 \otimes a_2 b,$$

is a well-defined homomorphism of (K, H)-bimodules.

Proof (a) There can be at most one such Hopf algebra structure on $\bar{H}$, since, by the surjectivity of π, the maps $\Delta_{\bar{H}}$, $\epsilon_{\bar{H}}$, and $S_{\bar{H}}$ are uniquely determined by Δ_H, ϵ_H, and S_H, respectively.

In order to define $\Delta_{\bar{H}}$, we consider the R-algebra homomorphism $(\pi \otimes \pi) \circ \Delta_H\colon H \to H \otimes H \to \bar{H} \otimes \bar{H}$ and show that I is contained in its kernel. It suffices to show that $(\pi \otimes \pi)(\Delta_H(i(K^+))) = 0$. So let $x \in K^+$. Then $\Delta_K(x) = \sum_{j=1}^n (y_j + r_j 1_K) \otimes (z_j + s_j 1_K)$ with elements $y_j, z_j \in K^+$ and $r_j, s_j \in R$ $(j = 1, \ldots, n)$, by the decomposition of K in 2.1(c). Applying $\epsilon_K \otimes \mathrm{id}_K$ we obtain $x = \sum_{j=1}^n (r_j z_j + r_j s_j 1_K)$. The same decomposition of K shows that $\sum_{j=1}^n r_j s_j 1_K = 0$ and that

$$\Delta_K(x) = \sum_{j=1}^n ((y_j \otimes z_j) + (r_j 1_K \otimes z_j) + (y_j \otimes s_j 1_K)) \in K \otimes K\,.$$

It follows that $(\pi\otimes\pi)(\Delta_H(i(x))) = (\pi\otimes\pi)(i\otimes i)\Delta_K(x) = 0$. Thus there exists a unique R-algebra homomorphism $\Delta_{\bar H}\colon \bar H \to \bar H\otimes\bar H$ such that $\Delta_{\bar H}\circ\pi = (\pi\otimes\pi)\circ\Delta_H$.

Since $\epsilon_H(I) \subseteq \epsilon_H(H)\epsilon_H(i(K^+))\epsilon_H(H) = R\epsilon_K(K^+)R = 0$, there exists a unique R-algebra homomorphism $\epsilon_{\bar H}\colon \bar H \to R$ such that $\epsilon_{\bar H}\circ\pi = \epsilon_H$.

In order to define $S_{\bar H}$, we show that $S_H(I)\subseteq I$. Using that S_H is an anti ring homomorphism, this follows immediately from $S_H(i(K^+)) = i(S_K(K^+)) \subseteq i(K^+)$, where the last inclusion is a consequence of $\epsilon_K\circ S_K = \epsilon_K$. Therefore, there exists a unique R-module homomorphism $S_{\bar H}\colon \bar H\to\bar H$ such that $S_{\bar H}\circ\pi = \pi\circ S_H$.

It is now a straightforward verification that the Hopf algebra axioms hold for these structure maps of $\bar H$, using the surjectivity of π and the validity of the desired equations for H.

(b) Every homomorphism of Hopf algebras leads to a right and a left comodule algebra structure of the first one over the second one in the way indicated in the Lemma. The statement concerning the coinvariants follows from the same chain of equations as in the proof of Lemma 3.4.2(b) in [M93].

(c) In order to show that β is well-defined we need to show that

$$\sum_{(b)} ai(x)b_1\otimes\bar b_2 = \sum_{(b),(x)} ai(x_1)b_1\otimes\overline{i(x_2)b_2}\,,$$

for all $a,b\in H$ and all $x\in K$. First note that Equation (2.1.a) implies

$$\sum_{(x)} i(x_1)\otimes\overline{i(x_2)} = i(x)\otimes\bar 1_H$$

for all $x\in K$. This implies the desired equation:

$$\begin{aligned}\sum_{(b),(x)} ai(x_1)b_1\otimes\overline{i(x_2)b_2} &= (a\otimes\bar 1_H)(\sum_{(x)} i(x_1)\otimes\overline{i(x_2)})(\sum_{(b)} b_1\otimes\bar b_2)\\ &= (a\otimes\bar 1_H)(i(x)\otimes\bar 1_H)(\sum_{(b)} b_1\otimes\bar b_2) = \sum_{(b)} ai(x)b_1\otimes\bar b_2\,.\end{aligned}$$

Obviously, β is a left H-module homomorphism. The following sequence of equations shows that β is also a right K-module homomorphism: For $a,b\in H$ and $x\in K$ we have

$$\begin{aligned}\beta((a\otimes b)x) = \beta(a\otimes bi(x)) &= \sum_{(b),(x)} ab_1i(x_1)\otimes\overline{b_2i(x_2)}\\ &= (a\otimes\bar 1_H)(\sum_{(b)} b_1\otimes\bar b_2)(\sum_{(x)} i(x_1)\otimes\overline{i(x_2)})\\ &= (a\otimes\bar 1_H)(\sum_{(b)} b_1\otimes\bar b_2)(i(x)\otimes\bar 1_H)\\ &= \sum_{(b)} ab_1i(x)\otimes\bar b_2 = \beta(a\otimes b)x\,.\end{aligned}$$

Similarly, one shows the statement on β'. ☐

The following proposition is well-known, see for instance [M93, Lemma 3.4.2(1)] for a proof in the case that R is a field and i is an inclusion. This proof also works in our situation and we adopt it for the reader's convenience.

2.3 Proposition *Assume that the extension $i\colon K \to H$ of Hopf algebras is left (resp. right) normal. Then $Hi(K^+) \subseteq i(K^+)H$ (resp. $i(K^+)H \subseteq Hi(K^+)$).*

Proof We only show that left normality implies $Hi(K^+) \subseteq i(K^+)H$. The opposite statement is proved similarly. Let $a \in H$ and $x \in K^+$. Then

$$ai(x) = \sum_{(a)} a_1 i(x)\epsilon_H(a_2) = \sum_{(a)} a_1 i(x) S_H(a_2) a_3 = \sum_{(a)} (\mathrm{ad}_l(a_1))(i(x)) a_2 \,.$$

It suffices to show that, for $a \in H$ and $x \in K^+$, one has $(\mathrm{ad}_l(a))(i(x)) \in i(K^+)$. But

$$\epsilon_H((\mathrm{ad}_l(a))(i(x))) = \epsilon_H(\sum_{(a)} a_1 i(x) S_H(a_2)) = \sum_{(a)} \epsilon_H(a_1)\epsilon_K(x)\epsilon_H(S_H(a_2)) = 0\,.$$

Thus, since $i\colon K \to H$ is left normal,

$$(\mathrm{ad}_l(a))(i(x)) \subseteq i(K) \cap \ker(\epsilon_H) = i(K^+)\,,$$

cf. 2.1(c). ☐

The construction of the inverse of β in the proof of the following proposition is standard (cf. for instance the proof of [M93, Proposition 3.4.3]).

2.4 Proposition *If $Hi(K^+) \subseteq i(K^+)H$ (resp. $i(K^+)H \subseteq Hi(K^+)$) then the homomorphism β (resp. β') from Lemma 2.2(c) is an isomorphism.*

Proof We only prove one version of the proposition. The opposite version is proved similarly. Note that the hypothesis implies $I = i(K^+)H$. We define the map

$$\gamma\colon H \otimes \bar{H} \to H \otimes_K H\,, \quad a \otimes \bar{b} \mapsto \sum_{(b)} aS_H(b_1) \otimes b_2\,.$$

In order to see that γ is well-defined, i.e., independent of the choice of b in its residue class modulo $I = i(K^+)H$, we need to show that

$$\sum_{(x),(b)} aS_H(i(x_1)b_1) \otimes i(x_2)b_2 = 0\,,$$

for all $a, b \in H$ and all $x \in K^+$. But

$$\sum_{(x),(b)} aS_H(i(x_1)b_1) \otimes i(x_2)b_2 = \sum_{(x),(b)} aS_H(b_1)S_H(i(x_1))i(x_2) \otimes b_2$$
$$= \sum_{(b)} aS_H(b_1)i(\epsilon_K(x)1_K) \otimes b_2 = 0\,.$$

Finally, we show that $\gamma \circ \beta = \mathrm{id}$ and $\beta \circ \gamma = \mathrm{id}$. In fact, for $a, b \in H$ we have

$$\gamma(\beta(a \otimes b)) = \sum_{(b)} \gamma(ab_1 \otimes \bar{b}_2) = \sum_{(b)} ab_1 S_H(b_2) \otimes b_3$$
$$= \sum_{(b)} a\epsilon_H(b_1) \otimes b_2 = \sum_{(b)} a \otimes \epsilon_H(b_1)b_2 = a \otimes b\,,$$

and

$$\beta(\gamma(a \otimes \bar{b})) = \sum_{(b)} \beta(aS_H(b_1) \otimes b_2) = \sum_{(b)} aS_H(b_1)b_2 \otimes \bar{b}_3$$
$$= (a \otimes \bar{1}_H)(\sum_{(b)} S_H(b_1)b_2 \otimes \bar{b}_3) = (a \otimes \bar{1}_H)(\sum_{(b)} \epsilon_H(b_1)1_H \otimes \bar{b}_2)$$
$$= (a \otimes \bar{1}_H)(1_H \otimes \overline{\sum_{(b)} \epsilon_H(b_1)b_2}) = (a \otimes \bar{1}_H)(1_H \otimes \bar{b}) = a \otimes \bar{b}\,.$$

□

Using the right $\bar{H}$-comodule algebra structure of H from Lemma 2.2(b) and generalizing Definition 8.1.1 in [M93], we call the extension $i\colon K \to H$ of Hopf algebras *right* $\bar{H}$*-Galois* if $\beta\colon H \otimes_K H \to H \otimes \bar{H}$ is an isomorphism and if $i(K) = H^{\mathrm{co}\bar{H}}$. Similarly, $i\colon K \to H$ is called *left* $\bar{H}$*-Galois* if the map β' is an isomorphism and the coinvariants ${}^{\mathrm{co}\bar{H}}H$ of the left $\bar{H}$-comodule structure of H are equal to $i(K)$.

For a proof of the following two propositions in the case that R is a field (which goes back to ideas from [Sch92]) see [M93, Proposition 3.4.3]. The same proof still works in our more general situation.

2.5 Proposition *Assume that β (resp. β') is an isomorphism and that H is faithfully flat as left or as right K-module. Then the Hopf algebra extension $i\colon K \to H$ is right (resp. left) $\bar{H}$-Galois.*

Proof We only show the *right* Galois statement. The left statement can be shown in a similar way.

The only thing that needs to be proved is that $i(K) = H^{\mathrm{co}\bar{H}}$. If H is faithfully flat as left K-module the second theorem in §13.1 in [W79] implies

$$i(K) = \ker(i_1 - i_2)\,, \tag{2.5.a}$$

where the R-module homomorphisms $i_1, i_2 \colon H \to H \otimes_K H$ are defined by $i_1(a) := a \otimes 1_H$ and $i_2(a) := 1_H \otimes a$, for $a \in H$. If H is faithfully flat as right H-module, the "right module version" of the theorem cited above implies Equation (2.5.a) as well. The homomorphism $\beta \colon H \otimes_K H \to H \otimes \bar{H}$ satisfies

$$(\beta \circ i_1)(a) = \beta(a \otimes 1_H) = a \otimes \bar{1}_H$$

and

$$(\beta \circ i_2)(a) = \beta(1 \otimes a) = \sum_{(a)} a_1 \otimes \bar{a}_2 = \rho(a)\,,$$

for all $a \in H$, where ρ is the comodule structure map of H over $\bar{H}$ introduced in Lemma 2.2(b). Therefore, with Equation (2.5.a), we have

$$i(K) = \ker(i_1 - i_2) = \ker(\beta \circ (i_1 - i_2)) = H^{\mathrm{co}\bar{H}}\,.$$

□

2.6 Proposition *Assume that the Hopf algebra extension $i \colon K \to H$ is right (resp. left) $\bar{H}$-Galois. Then the extension $i \colon K \to H$ is left (resp. right) normal.*

Proof By the right $\bar{H}$-Galois property, the coinvariants $H^{\mathrm{co}\bar{H}}$ are equal to $i(K)$. Now, by Lemma 2.2(b), $i(K) = H^{\mathrm{co}\bar{H}}$ is $\mathrm{ad_l}(H)$-invariant. Similarly, the left $\bar{H}$-Galois property implies that $i(K) = {}^{\mathrm{co}\bar{H}}H$ is $\mathrm{ad_r}(H)$-invariant. □

In the proof of the following proposition we will need the following well-known Lemma. Its proof is omitted.

2.7 Lemma *Let A be a ring, let J be an ideal of A and let M be a left A-module. If M is finitely generated projective as A-module then M/JM is finitely generated projective as A/J-module.*

A proof of the following proposition (in the situation where R is a field) can be found in [KK06, Proposition 3.4] and [KL03, Example 3.4]. Although our situation is more general, the arguments still work. We include them for the convenience of the reader.

2.8 Proposition *Assume that $Hi(K^+) \subseteq i(K^+)H$ (resp. $i(K^+)H \subseteq Hi(K^+)$) and assume that H is finitely generated projective as left (resp. right) K-module. Then the extension $i \colon K \to H$ has right (resp. left) depth 2.*

Proof We only show one version of the proposition. The opposite statement is proved similarly. By Proposition 2.4, $\beta \colon H \otimes_K H \to H \otimes \bar{H}$ (from Lemma 2.2(c)) is an isomorphism of (H, K)-bimodules. Since $Hi(K^+) \subseteq i(K^+)H$, we have $I = i(K^+)H$. By Lemma 2.7, applied to $A = i(K)$, $J = i(K^+)$ and $M = H$, we have $\bar{H} \mid R^n$ in ${}_R\mathsf{Mod}$ for some positive integer n. Altogether, this implies

$$H \otimes_K H \cong H \otimes \bar{H} \mid H \otimes_R R^n \cong H^n$$

in ${}_H\mathsf{Mod}_K$. Therefore, the extension $i\colon K \to H$ has right depth 2. $\square$

Finally, we have the following proposition.

2.9 Proposition *Assume that the extension $i\colon K \to H$ has right (resp. left) depth 2 and assume that H is faithfully flat as left or as right K-module. Then $Hi(K^+) \subseteq i(K^+)H$ (resp. $i(K^+)H \subseteq Hi(K^+)$).*

Proof The right depth 2 condition implies that there exists a positive integer n such that

$$R_H \otimes_H H \otimes_K H \mid (R_H \otimes_H H)^n$$

in Mod_K. Here, R_H denotes the trivial right H-module, i.e., the module defined by the augmentation map $\epsilon_H\colon H \to R$. Note that $R_H \otimes_H H \cong R_K$ as right K-modules. Thus, we have

$$R_K \otimes_K H \mid (R_K)^n$$

in Mod_K. Since K^+ annihilates R_K, $i(K^+)$ annihilates the right H-module $R_K \otimes_K H \cong H/i(K^+)H$. However, the annihilator in H of an H-module is a two-sided ideal of H. Thus, also $Hi(K^+)H$ annihilates $H/i(K^+)H$. On the other hand, if $a \in H$ annihilates $H/i(K^+)H$ then $(1 + i(K^+)H)a = 0$ and $a \in i(K^+)H$. This shows that $Hi(K^+)H = i(K^+)H$ and in particular that $Hi(K^+) \subseteq i(K^+)H$. The opposite statement is proved similarly, using left modules. $\square$

Now, Propositions 2.3, 2.4, 2.5, 2.6, 2.8, and 2.9 imply the following theorem.

2.10 Theorem *Assume that H is finitely generated projective as left (resp. right) K-module and that H is faithfully flat as left or as right K-module. Then the following are equivalent:*

(i) *The extension $i\colon K \to H$ is left (resp. right) normal.*

(ii) *One has $Hi(K^+) \subseteq i(K^+)H$ (resp. $i(K^+)H \subseteq Hi(K^+)$).*

(iii) *The map β (resp. β') from Lemma 2.2(c) is an isomorphism.*

(iv) *The extension $i\colon K \to H$ is right (resp. left) $\bar{H}$-Galois (where H is endowed with the natural $\bar{H}$-comodule algebra structure, cf. Lemma 2.2(b)).*

(v) *The extension $i\colon K \to H$ has right (resp. left) depth 2.*

2.11 Remark (a) The map $\beta\colon H \otimes_K H \to H \otimes \bar{H}$ plays a similar role as the map $\pi_{RG}\colon RG \otimes_{RH} RG \to RG^{H\backslash G/H}$ in Section 1. Assume that $\bar{H} \mid R^n$ in ${}_R\mathsf{Mod}$ for a positive integer n. Then the map β induces a natural transformation $\iota\colon \mathrm{Ind}_K^H \mathrm{Res}_K^H \mathrm{Ind}_K^H \to (\mathrm{Ind}_K^H)^n$ between functors from ${}_K\mathsf{Mod}$ to ${}_H\mathsf{Mod}$. The conditions in Theorem 2.10 are equivalent to ι being a split natural monomorphism. Similar observations hold for the map β'.

(b) One can also use the propositions in this section to obtain a version of Theorem 2.10 that states that the five two-sided conditions are equivalent. In this case, one does not need to require that H is finitely generated as left *and* right K-module. It suffices to have one of the two properties as hypothesis, since $\bar{H}$ is finitely generated projective as left R-module if and only if it is finitely generated projective as right R-module (see the proof of Proposition 2.8).

(c) If $S_H\colon H \to H$ and $S_K\colon K \to K$ are bijective then each of the five one-sided conditions in Theorem 2.10 is equivalent to its opposite (with no condition on the K-module structure of H).

3 Relative Projectivity

We assume again, as in Section 1, that R is a non-zero commutative ring, G is a group and $H \leqslant G$ is a subgroup of G of finite index. Recall that a left RG-module M is called *(RG, RH)-projective* if every short exact sequence

$$0 \to M' \to M'' \to M \to 0 \quad \text{in} \quad {}_{RG}\mathsf{Mod}$$

which splits after restriction to ${}_{RH}\mathsf{Mod}$, already splits in ${}_{RG}\mathsf{Mod}$ (cf. [CR81, Definition 19.1]). Similarly, one defines (RG, RH)-projectivity for right RG-modules. Recall that, for $M \in {}_{RG}\mathsf{Mod}$, the relative trace map $\mathrm{Tr}_H^G\colon \mathrm{End}_{RH}(M) \to \mathrm{End}_{RG}(M)$ is defined by $(\mathrm{Tr}_H^G(f))(m) := \sum_{gH\in G/H} gf(g^{-1}m)$. It does not depend on the coset representative g in gH. Also recall the following equivalence of statements (sometimes called Higman's criterion, cf. [CR81, Theorem 19.2 and Lemma 19.3]).

3.1 Proposition *Let $M \in {}_{RG}\mathsf{Mod}$. Then the following are equivalent:*

(i) *M is (RG, RH)-projective.*

(ii) *There exists $\gamma \in \mathrm{End}_{RH}(M)$ such that $\mathrm{Tr}_H^G(\gamma) = \mathrm{id}_M$.*

(iii) *$M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$ in ${}_{RG}\mathsf{Mod}$.*

(iv) *$M \mid \mathrm{Ind}_H^G(N)$ for some $N \in {}_{RH}\mathsf{Mod}$.*

(v) *The RG-module epimorphism $\pi_M\colon RG \otimes_{RH} M \to M$, $g \otimes m \mapsto gm$, splits.*

Note that π_M is natural in M and therefore gives rise to a natural transformation

$$\pi\colon \mathrm{Ind}_H^G \mathrm{Res}_H^G \to \mathrm{Id}_{{}_{RG}\mathsf{Mod}}\,.$$

For $M = RG$ we obtain the (RG, RG)-bimodule epimorphism

$$\pi_{RG}\colon RG \otimes_{RH} RG \to RG\,, \quad a \otimes b \mapsto ab\,,$$

which is simply the multiplication map in RG. The following theorem is now similar in spirit to Theorem 1.2.

3.2 Theorem *The following are equivalent:*

(i) *The multiplication map $\pi_{RG}\colon RG \otimes_{RH} RG \to RG$ is a split epimorphism in ${}_{RG}\mathsf{Mod}_{RG}$ (i.e., $RH \subseteq RG$ is a separable ring extension).*

(ii) *Every RG-module is (RG, RH)-projective.*

(iii) *The trivial RG-module R_G is (RG, RH)-projective.*

(iv) *The natural epimorphism* $\pi\colon \mathrm{Ind}_H^G \mathrm{Res}_H^G \to \mathrm{Id}_{_{RG}\mathsf{Mod}}$ *is split.*
(v) $\mathrm{Id}_{_{RG}\mathsf{Mod}} \mid \mathrm{Ind}_H^G \mathrm{Res}_H^G$.
(vi) $[G:H]$ *is a unit in* R.

Proof The implications (i)⇒(iv)⇒(v)⇒(ii)⇒(iii) are trivial or straightforward.

(iii)⇒(vi): By the hypothesis there exists an RG-module homomorphism $\iota\colon R_G \to RG \otimes_{RH} R_H$ such that $\pi_{R_G} \circ \iota = \mathrm{id}_{R_G}$. Let $\sigma := \sum_{gH \in G/H} g \otimes 1 \in RG \otimes_{RH} R_H$. Then $R\sigma$ is the set of G-fixed points of $RG \otimes_{RH} R_H$. Thus, $\iota(1) = r\sigma$ for some $r \in R$. It follows that $1 = \pi_{R_G}(\iota(1)) = \pi_{R_G}(r\sigma) = r[G:H]$. Thus, $[G:H]$ is a unit of R.

(vi)⇒(i): The map

$$\iota\colon RG \to RG \otimes_{RH} RG\,, \quad a \mapsto [G:H]^{-1} \sum_{gH \in G/H} g \otimes g^{-1}a\,,$$

is a well-defined (RG, RG)-bimodule homomorphism and $\pi_{RG} \circ \iota = \mathrm{id}_{RG}$. □

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