

# On the Depth of Subgroups and Group Algebra Extensions

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## Abstract

We investigate notions of depth for inclusions of rings  $B \subseteq A$ , in particular for group algebra extensions  $RH \subseteq RG$  for finite groups  $H \leq G$  and a non-zero commutative ring  $R$ . A group-theoretic (or combinatorial) notion of depth for  $H$  in  $G$  is defined and used to show that  $RH \subseteq RG$  has always finite depth. We compare the depths of  $H \leq G$  and  $RH \subseteq RG$ , and investigate how the depth varies with  $R$ .

## 1 Introduction and Motivation

Given an associative, unitary ring  $A$  and a subring  $B \subseteq A$  with  $1_A = 1_B$ , we call  $B \subseteq A$  a *ring extension*. We set  $T_1(B, A) := A$  which is an  $(A, A)$ -bimodule via left and right multiplication with elements in  $A$ . Moreover, for  $i \geq 1$ , we define, inductively, an  $(A, A)$ -bimodule  $T_{i+1}(B, A) := A \otimes_B T_i(B, A)$ . Via restriction, we may view  $T_i(B, A)$  as  $(B, A)$ -bimodule and as  $(A, B)$ -bimodule, respectively. We will denote these restrictions by  $T_i^l(B, A)$  and  $T_i^r(B, A)$ , respectively. Furthermore, for  $i \geq 1$ , we denote  $T_i(B, A)$ , viewed as  $(B, B)$ -bimodule, by  $T_i'(B, A)$ . In addition, we define  $T_0'(B, A)$  to be the  $(B, B)$ -bimodule  $B$  itself.

With this notation, the ring extension  $B \subseteq A$  is said to have *left and right depth*  $2i + 1$ , for some  $i \geq 0$ , if there is some  $m \in \mathbb{N}$  with  $T_{i+1}'(B, A) \mid mT_i'(B, A)$ , that is,  $T_{i+1}'(B, A)$  is isomorphic to a direct summand of a direct sum of  $m$  copies of  $T_i'(B, A)$ . Furthermore, the ring extension  $B \subseteq A$  is said to have *left depth*  $2i$  (respectively *right depth*  $2i$ ), for some  $i \geq 1$ , if there is some  $m \in \mathbb{N}$  with  $T_{i+1}^l(B, A) \mid mT_i^l(B, A)$  (respectively  $T_{i+1}^r(B, A) \mid mT_i^r(B, A)$ ). In the case that  $B \subseteq A$  has both left and right depth  $d$  it is said to have *depth*  $d$ .

Observe that if  $B \subseteq A$  has depth  $d$  then it also has depth  $d + 1$ . Therefore, we are usually interested in the *minimal depth*  $d(B, A)$  of the ring extension  $B \subseteq A$ . That is,  $d(B, A)$  is defined as the least integer  $d \geq 1$  such that  $B \subseteq A$  has depth  $d$  provided such an integer exists; otherwise, we set  $d(B, A) := \infty$ . Similarly, we define the minimal right depth  $d^r(B, A)$  and the minimal left depth  $d^l(B, A)$  of  $B$  in  $A$ .

In this article we concentrate on the case where  $A$  and  $B$  are group algebras. That is, given a commutative ring  $R \neq 0$ , a finite group  $G$ , and a subgroup  $H$  of  $G$ , we consider the group algebras  $B := RH \subseteq RG =: A$ . As has been shown in [4], the ring extension  $RH \subseteq RG$  has finite minimal depth provided that  $R$  is an algebraically closed field of characteristic 0. One of the aims of this paper is to prove that  $RH \subseteq RG$  has always finite minimal depth, regardless of  $R$ .

In order to investigate the depth of a group algebra extension  $RH \subseteq RG$ , it will be useful to define a notion of *combinatorial depth* for the inclusion of finite groups  $H \leq G$ . Denoting by  $d_c(H, G)$  the minimal combinatorial depth of  $H$  in  $G$ , we show that

$$d(RH, RG) \leq d_c(H, G) \leq 2|G : N_G(H)| < \infty,$$

for any non-zero commutative ring  $R$ . We also examine the dependence of  $d(RH, RG)$  on the coefficient ring  $R$ , and we compute the minimal ring-theoretic and combinatorial depths in a number of examples.

It would be interesting to know whether the finiteness of the minimal depth is also valid for other types of extensions. For instance for extensions of finite-dimensional Hopf algebras over a field  $K$ , or extensions of the form

$$K[X_1, \dots, X_n]^G \subseteq K[X_1, \dots, X_n],$$

where  $K[X_1, \dots, X_n]^G$  is the ring of polynomial invariants of a finite group  $G$ . At present, we do not have an example of a ring extension which is not of finite minimal depth, although we certainly expect that there are many such examples.

The notion of depth relevant to this paper has its origins in the theory of von Neumann algebras. For background information, see, for instance, [6] and [8]. Group algebra extensions of depth 1 and 2, respectively, have been studied in [2] and [1], respectively, and extensions of semisimple algebras of arbitrary finite minimal depth appeared in [4]. A notion of depth 3 for ring extensions was introduced and studied in detail by L. Kadison in [9].

From now on, all rings are supposed to be associative and unitary.

The present paper is organized as follows: in Section 2 below, we begin by fixing the notation needed throughout. In Section 3 we then define and investigate our notion of combinatorial depth of a group inclusion. After that, in Section 4, we show how the combinatorial depth and the ring-theoretic depth are related, thereby proving that, for any finite groups  $H \leq G$  and any commutative ring  $R \neq 0$ , the group algebra extension  $RH \subseteq RG$  has always finite minimal depth. In Section 5 we compute the precise values of the minimal combinatorial depth and the minimal ring-theoretic depth in various examples. We close with an appendix which gives a category-theoretic characterization of the depth of a ring extension. Moreover, we show that, in the case where  $B \subseteq A$  is a ring extension of finite-dimensional semisimple algebras over an algebraically closed field, our notion of depth is equivalent to the one introduced in [4].

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## 2 Preliminaries and General Notation

Throughout this article,  $R \neq 0$  denotes a commutative ring,  $G$  and  $H$  denote finite groups, and all groups appearing are supposed to be finite.

For  $H \leq G$ , we want to examine the minimal depth  $d(RH, RG)$  of the ring extension  $RH \subseteq RG$ . We will, from now on, also write  $d_R(H, G)$  rather than  $d(RH, RG)$ . As mentioned in the introduction, we intend to prove that  $d_R(H, G)$  is always finite. Our proof of this result requires a notion of combinatorial depth of the group  $H$  in the group  $G$ . In order to define this notion, we will make use of a number of known facts concerning the theory of finite bisets which we will now briefly summarize. For further details, we refer the reader to [3].

**Definition 2.1.** Let  $G$  and  $H$  be arbitrary. A  $(G, H)$ -biset is a finite set  $X$ , endowed with a left  $G$ -action and a right  $H$ -action, such that

$$g \cdot (x \cdot h) = (g \cdot x) \cdot h,$$

for  $g \in G$ ,  $h \in H$ ,  $x \in X$ . If  $X$  and  $Y$  are  $(G, H)$ -bisets and if  $f : X \rightarrow Y$  is a map such that, for any  $g \in G$ ,  $h \in H$ , and  $x \in X$ , we have

$$f(g \cdot x) = g \cdot f(x) \quad \text{and} \quad f(x \cdot h) = f(x) \cdot h,$$

then we call  $f$  a *homomorphism* of  $(G, H)$ -bisets.

**Remark 2.2.** (a) Let  $M$  be a finitely generated left  $RG$ -module which is free over  $R$ . Recall that its dual module  $M^* := \text{Hom}_R(M, R)$  becomes a right  $RG$ -module with

$$(f \cdot g)(m) := f(gm), \quad \text{for } f \in M^*, g \in G, m \in M.$$

(b) Let  $X$  be a  $(G, H)$ -biset. Then  $X$  can be considered both as left  $(G \times H)$ -set and as right  $(H \times G)$ -set, via

$$(g, h) \cdot x := gxh^{-1} \quad \text{and} \quad x \cdot (h, g) := g^{-1}xh,$$

for  $x \in X, g \in G, h \in H$ . In particular,  $RX$  is both a left permutation  $R[G \times H]$ -module and a right permutation  $R[H \times G]$ -module. If, conversely,  $Y$  is a left  $(G \times H)$ -set and if  $Z$  is a right  $(H \times G)$ -set then we can regard both  $Y$  and  $Z$  as  $(G, H)$ -bisets with

$$g \cdot y \cdot h := (g, h^{-1})y \quad \text{and} \quad g \cdot z \cdot h := z(h, g^{-1}),$$

for  $g \in G, h \in H, y \in Y$ , and  $z \in Z$ . In this way, we may freely identify  $(G, H)$ -bisets with left  $(G \times H)$ -sets or right  $(H \times G)$ -sets.

(c) Given a  $(G, H)$ -biset  $X$ , we define  $X^\circ$  to be the  $(H, G)$ -biset which is, as a set, equal to  $X$ , with action defined by

$$h \cdot x^\circ \cdot g := (g^{-1}xh^{-1})^\circ,$$

for  $x \in X, g \in G$ , and  $h \in H$ . Here  $x^\circ$  denotes an element  $x \in X$ , viewed as an element in  $X^\circ$ . We call  $X^\circ$  the *opposite biset* of  $X$ . In the case that  $G = H$ , we call  $X$  *symmetric* if  $X$  and  $X^\circ$  are isomorphic as  $(G, G)$ -bisets.

Note that, by what we have just mentioned above,  $X^\circ$  now carries a left  $(H \times G)$ -set structure as well as a right  $(G \times H)$ -set structure, and we have

$$\begin{aligned} (h, g) \cdot x^\circ &= hx^\circ g^{-1} = (gxh^{-1})^\circ = (x \cdot (h^{-1}, g^{-1}))^\circ = ((g, h) \cdot x)^\circ, \\ x^\circ \cdot (g, h) &= h^{-1}x^\circ g = (g^{-1}xh)^\circ = ((g^{-1}, h^{-1}) \cdot x)^\circ = (x \cdot (h, g))^\circ. \end{aligned}$$

In particular,  $RX^\circ$  is both a right permutation  $R[G \times H]$ -module and a left permutation  $R[H \times G]$ -module.

(e) For any subgroup  $U$  of  $G \times H$ , we define its *dual group*

$$U^* := \{(x, y) \in H \times G \mid (y, x) \in U\} \leq H \times G,$$

and we get an isomorphism of  $(H, G)$ -bisets  $(G \times H/U)^\circ \cong H \times G/U^*$ .

(f) The *coproduct*  $X \uplus Y$  of two  $(G, H)$ -bisets  $X$  and  $Y$  is defined as the disjoint union of the underlying sets, with the obvious actions of  $G$  and  $H$ .

(g) Generalizing the above concept, we will also identify arbitrary  $(RG, RH)$ -bimodules with left  $R[G \times H]$ -modules or with right  $R[H \times G]$ -modules.

With these conventions, we observe the following two facts. The proof of the first one is left to the reader.

**Lemma 2.3.** Let  $X = \{x_1, \dots, x_n\}$  be a  $(G, H)$ -biset, and let  $RX$  be the corresponding left permutation  $R[G \times H]$ -module. Let further  $\{x_1^*, \dots, x_n^*\}$  be the basis of  $RX^*$  which is dual to the basis  $X$  of  $RX$ . Then the  $R$ -linear map

$$f : RX^\circ \longrightarrow (RX)^*, \quad x_i^\circ \longmapsto x_i^* \quad (i = 1, \dots, n)$$

defines an isomorphism of right  $R[G \times H]$ -modules, or equivalently, of  $(RH, RG)$ -bimodules.

**Lemma 2.4.** Let  $X$  be a symmetric  $(G, G)$ -biset and assume that  $H \leq G$ . Let further  $X^1$  and  $X^r$ , respectively, be the  $(H, G)$ -biset and the  $(G, H)$ -biset, respectively, obtained from  $X$  by restriction. Then  $X^1 \cong (X^r)^\circ$  as  $(H, G)$ -bisets, and  $X^r \cong (X^1)^\circ$  as  $(G, H)$ -bisets.

**Proof.** For any  $(G, G)$ -biset  $X$ , we have  $(X^\circ)^r = (X^1)^\circ$ , as  $(G, H)$ -bisets. Thus if  $X$  is a symmetric  $(G, G)$ -biset then

$$X^r = \text{Res}_{G \times H}^{G \times G}(X) \cong \text{Res}_{G \times H}^{G \times G}(X^\circ) = (X^\circ)^r = (X^1)^\circ,$$

as  $(G, H)$ -bisets. Analogously, we have an isomorphism  $X^1 \cong (X^r)^\circ$  of  $(H, G)$ -bisets.  $\square$

**Remark/Definition 2.5.** Let  $G, H, K$ , and  $L$  be groups.

(a) For any subgroup  $U$  of  $G$  and any  $g \in G$ , we set

$${}^gU := gUg^{-1} \quad \text{and} \quad U^g := g^{-1}Ug,$$

and write  $\text{Core}_G(U) := \bigcap_{g \in G} U^g$  for the *core* of  $U$  in  $G$ .

(b) For subgroups  $U$  of  $G \times H$  and  $V$  of  $H \times K$ , we define a subgroup  $U * V$  of  $G \times K$  as

$$U * V := \{(g, k) \in G \times K \mid \exists h \in H : (g, h) \in U, (h, k) \in V\}$$

Then, for any  $g \in G$ , any  $k \in K$ , and groups  $U \leq G \times H$ ,  $V \leq H \times K$ ,  $W \leq K \times L$ , we have

$$\begin{aligned} (U * V) * W &= U * (V * W), & (U * V)^* &= V^* * U^*, \\ ({}^{g,1}(U * V)) &= ({}^{g,1}U * V), & (U * V)^{(1,k)} &= U * V^{(1,k)}. \end{aligned}$$

(c) We denote by  $p_1 : G \times H \longrightarrow G$  and  $p_2 : G \times H \longrightarrow H$  the canonical projections.

(d) Suppose that  $U \leq H$  and that  $\varphi : U \longrightarrow G$  is a group monomorphism. Then we define a *twisted diagonal subgroup*

$$\Delta_\varphi(U) := \{(\varphi(u), u) \mid u \in U\}$$

of  $G \times H$ . In the case that  $G = H$  and  $\varphi$  is induced by conjugation with an element  $g$  belonging to some overgroup of  $G$ , we also write  $\Delta_g(U)$  rather than  $\Delta_\varphi(U)$ . In particular, we obtain  $\Delta_1(U) = \Delta(U) := \{(x, x) \mid x \in U\}$ .

(e) Given a  $(G, H)$ -biset  $X$  and an  $(H, K)$ -biset  $Y$ , the cartesian product  $X \times Y$  becomes a  $(G, K)$ -biset in the obvious way. Moreover, we have a left  $H$ -action on  $X \times Y$ , defined by

$$h \cdot (x, y) := (xh^{-1}, hy),$$

for  $x \in X$ ,  $y \in Y$ , and  $h \in H$ . The  $H$ -orbit of an element  $(x, y) \in X \times Y$  will be denoted by  $[x, y]$ , and the set of all  $H$ -orbits on  $X \times Y$  will be denoted by  $X \times_H Y$ . Note that the  $(G, K)$ -biset structure of  $X \times Y$  induces a  $(G, K)$ -biset structure of  $X \times_H Y$ . Note further that if  $X_1$  and  $X_2$  are  $(G, H)$ -bisets and if  $Y_1$  and  $Y_2$  are  $(H, K)$ -bisets then we have an isomorphism of  $(G, K)$ -bisets:

$$(X_1 \uplus X_2) \times_H (Y_1 \uplus Y_2) \cong (X_1 \times_H Y_1) \uplus (X_1 \times_H Y_2) \uplus (X_2 \times_H Y_1) \uplus (X_2 \times_H Y_2).$$

**Proposition 2.6** (Bouc, [3], Prop. 1). *Let  $G$ ,  $H$ , and  $K$  be groups. Let further  $U \leq G \times H$ , and let  $V \leq H \times K$ . Then one has an isomorphism of  $(G, K)$ -bisets:*

$$(G \times H/U) \times_H (H \times K/V) \cong \bigsqcup_{\substack{p_2(U)hp_1(V) \in \\ p_2(U) \setminus H/p_1(V)}} G \times K/U *^{(h,1)} V.$$

The assertions of the next proposition are easy consequences of the above definitions. We leave the proof to the reader.

**Proposition 2.7.** *Let  $G$ ,  $H$ ,  $K$ , and  $L$  be groups, let  $X$  be a  $(G, H)$ -biset, let  $Y$  be an  $(H, K)$ -biset, and let  $Z$  be a  $(K, L)$ -biset. Then*

$$\begin{aligned} (X \times_H Y)^\circ &\cong Y^\circ \times_H X^\circ, \text{ as } (K, G)\text{-bisets, and} \\ (X \times_H Y) \times_K Z &\cong X \times_H (Y \times_K Z), \text{ as } (G, L)\text{-bisets.} \end{aligned}$$

Moreover, we have an isomorphism of  $(RG, RK)$ -bimodules

$$f : R[X \times_H Y] \longrightarrow RX \otimes_{RH} RY,$$

with  $f([x, y]) = x \otimes y$ , for any  $x \in X$  and any  $y \in Y$ .

**Notation 2.8.** Let  $n \geq 1$ , and let  $G_1, \dots, G_{n+1}$  be groups. For  $1 \leq i \leq n$ , let further  $X_i$  be a  $(G_i, G_{i+1})$ -biset, and let  $x_i \in X_i$ . Then we define

$$[x_1, \dots, x_n] := [\dots [[x_1, x_2], x_3], \dots, x_n] \in X_1 \times_{G_2} X_2 \times_{G_3} \cdots \times_{G_n} X_n.$$

In the next two sections we introduce our notion of combinatorial depth of a group  $H$  in an overgroup  $G$ , and we will show how this combinatorial depth of  $H$  in  $G$  is related to the ring-theoretic depth of the group algebra extension  $RH \subseteq RG$ .

### 3 Combinatorial Depth of Group Inclusions

Throughout this section  $H$  denotes a subgroup of the finite group  $G$ .

**Remark/Definition 3.1.** We define  $\Theta_1(H, G)$  to be the  $(G, G)$ -biset  $G$ , with  $G$  acting on itself via left and right multiplication, respectively. Then, for  $i \geq 1$ , we define a  $(G, G)$ -biset

$$\Theta_{i+1}(H, G) := \Theta_i(H, G) \times_H G.$$

For  $i \geq 1$ , we may view  $\Theta_i(H, G)$  as an  $(H, H)$ -biset via restriction; this restriction will be denoted by  $\Theta'_i(H, G)$ . In addition, we define  $\Theta'_0(H, G)$  as the  $(H, H)$ -biset  $H$ . Moreover, for  $i \geq 1$ , we write  $\Theta_i^1(H, G)$  and  $\Theta_i^r(H, G)$ , respectively, for  $\Theta_i(H, G)$  viewed as  $(H, G)$ -biset and  $(G, H)$ -biset, respectively. With this notation, for any  $i \geq 1$ ,

$$\Theta_i^r(H, G) \times_H \Theta_i^1(H, G) \cong \Theta_{i+1}(H, G) \cong \Theta_i^r(H, G) \times_H \Theta_1^1(H, G),$$

as  $(G, G)$ -bisets.

If  $X$  is a  $(G, H)$ -biset then  $X$  and  $X \times_H H$  are isomorphic as  $(G, H)$ -bisets. Similarly, if  $Y$  is an  $(H, G)$ -biset then  $H \times_H Y$  and  $Y$  are isomorphic as  $(H, G)$ -bisets. Moreover,  $G = H \uplus (G \setminus H)$  is a decomposition of  $(H, H)$ -bisets. Thus, for  $i \geq 1$ , we have decompositions

$$\Theta_{i+1}^r(H, G) \cong \Theta_i^r(H, G) \uplus (\Theta_i^r(H, G) \times_H (G \setminus H)), \quad (1)$$

as  $(G, H)$ -bisets,

$$\Theta_{i+1}^1(H, G) \cong \Theta_i^1(H, G) \uplus ((G \setminus H) \times_H \Theta_i^1(H, G)), \quad (2)$$

as  $(H, G)$ -bisets, and

$$\Theta'_i(H, G) \cong \Theta'_{i-1}(H, G) \uplus (\Theta'_{i-1}(H, G) \times_H (G \setminus H)), \quad (3)$$

as  $(H, H)$ -bisets. This yields chains

$$\Theta_1^r(H, G) \hookrightarrow \Theta_2^r(H, G) \hookrightarrow \Theta_3^r(H, G) \hookrightarrow \dots \quad (4)$$

$$\Theta_1^l(H, G) \hookrightarrow \Theta_2^l(H, G) \hookrightarrow \Theta_3^l(H, G) \hookrightarrow \dots \quad (5)$$

$$\Theta'_0(H, G) \hookrightarrow \Theta'_1(H, G) \hookrightarrow \Theta'_2(H, G) \hookrightarrow \dots \quad (6)$$

of monomorphisms of  $(G, H)$ -,  $(H, G)$ -, and  $(H, H)$ -bisets, motivating the following definition.

**Definition 3.2.** (a) If there exist some  $i \geq 1$ , some  $m \in \mathbb{N}$ , and a  $(G, H)$ -biset monomorphism  $\Theta_{i+1}^r(H, G) \hookrightarrow m \cdot \Theta_i^r(H, G)$  then we say that  $H$  has *right (combinatorial) depth  $2i$*  in  $G$ . Here,  $m \cdot \Theta_i^r(H, G)$  denotes the coproduct of  $m$  copies of  $\Theta_i^r(H, G)$ .

(b) If there exist some  $i \geq 1$ , some  $m \in \mathbb{N}$ , and an  $(H, G)$ -biset monomorphism  $\Theta_{i+1}^l(H, G) \hookrightarrow m \cdot \Theta_i^l(H, G)$  then we say that  $H$  has *left (combinatorial) depth  $2i$*  in  $G$ .

(c) If there exist some  $i \geq 0$ , some  $m \in \mathbb{N}$ , and an  $(H, H)$ -biset monomorphism  $\Theta'_{i+1}(H, G) \hookrightarrow m \cdot \Theta'_i(H, G)$  then we say that  $H$  has *left and right (combinatorial) depth  $2i + 1$*  in  $G$ .

If  $H$  has both left and right combinatorial depth  $d \in \mathbb{N}$  in  $G$  then we say that  $H$  has *(combinatorial) depth  $d$*  in  $G$ .

In a first step, we will show that a subgroup  $H$  of a group  $G$  has left (combinatorial) depth  $d$  in  $G$  if and only if it has right (combinatorial) depth  $d$  in  $G$ .

**Lemma 3.3.** *Let  $i \geq 1$ . Then  $\Theta_i(H, G)$  is a symmetric  $(G, G)$ -biset. In particular,  $\Theta_i^r(H, G) \cong (\Theta_i^l(H, G))^\circ$  as  $(G, H)$ -bisets, and  $\Theta_i^l(H, G) \cong (\Theta_i^r(H, G))^\circ$  as  $(H, G)$ -bisets.*

**Proof.** We argue by induction on  $i$ . Since we have isomorphisms

$$G \cong G \times G / \Delta(G) = G \times G / \Delta(G)^* \cong (G \times G / \Delta(G))^\circ \cong G^\circ,$$

of  $(G, G)$ -bisets, we conclude that  $\Theta_1(H, G)$  is a symmetric  $(G, G)$ -biset. Thus, by Lemma 2.4,  $\Theta_1^l(H, G) \cong (\Theta_1^r(H, G))^\circ$  as  $(H, G)$ -bisets, and  $\Theta_1^r(H, G) \cong (\Theta_1^l(H, G))^\circ$  as  $(G, H)$ -bisets. So we may from now on suppose that  $i \geq 2$ . By definition,

$$\Theta_i(H, G) \cong \Theta_{i-1}^r(H, G) \times_H \Theta_1^l(H, G) \cong \Theta_1^r(H, G) \times_H \Theta_{i-1}^l(H, G),$$

as  $(G, G)$ -bisets. Hence, by Proposition 2.7 and our inductive hypothesis, we also deduce that

$$\Theta_i(H, G)^\circ \cong (\Theta_1^l(H, G))^\circ \times_H (\Theta_{i-1}^r(H, G))^\circ \cong \Theta_1^r(H, G) \times_H \Theta_{i-1}^l(H, G) \cong \Theta_i(H, G),$$

as  $(G, G)$ -bisets. Therefore,  $\Theta_i(H, G)$  is a symmetric  $(G, G)$ -biset, and the second assertion again follows from Lemma 2.4.  $\square$

**Lemma 3.4.** *Let  $d \in \mathbb{N}$ . Then  $H$  has left depth  $d$  in  $G$  if and only if  $H$  has right depth  $d$  in  $G$ .*

**Proof.** Suppose that  $H$  has left depth  $d$  in  $G$ . In order to show that  $H$  has also right depth  $d$  in  $G$ , we only need to consider the case where  $d$  is even. That is,  $d = 2i$ , for some  $i \in \mathbb{N}$ . Then there are some  $m \in \mathbb{N}$  and an  $(H, G)$ -biset monomorphism

$$\Theta_{i+1}^1(H, G) \hookrightarrow m \cdot \Theta_i^1(H, G).$$

Using Lemma 3.3, we then also have a  $(G, H)$ -biset monomorphism

$$\Theta_{i+1}^r(H, G) \cong (\Theta_{i+1}^1(H, G))^\circ \hookrightarrow (m \cdot \Theta_i^1(H, G))^\circ \cong m \cdot \Theta_i^1(H, G)^\circ \cong m \cdot \Theta_i^r(H, G),$$

so that  $H$  has right depth  $2i$  in  $G$ . Analogously, one shows the converse.  $\square$

**Remark/Definition 3.5.** (a) As a consequence of Lemma 3.4, we need not distinguish between left and right depth of  $H$  in  $G$ , and may, from now on, only speak of (combinatorial) depth of  $H$  in  $G$ .

(b) Suppose that  $H$  has depth  $d \geq 1$  in  $G$ . Then  $H$  has also depth  $d + 1$  in  $G$ . For if  $d = 2i$  for some  $i \geq 1$ , then there are some  $m \geq 1$  and a  $(G, H)$ -biset monomorphism  $\Theta_{i+1}^r(H, G) \hookrightarrow m \cdot \Theta_i^r(H, G)$  which of course restricts to an  $(H, H)$ -biset monomorphism  $\Theta_{i+1}'(H, G) \hookrightarrow m \cdot \Theta_i'(H, G)$ . If  $d = 2i + 1$  for some  $i \geq 0$ , then there are some  $m \geq 1$  and an  $(H, H)$ -biset monomorphism  $\Theta_{i+1}'(H, G) \hookrightarrow m \cdot \Theta_i'(H, G)$  which then gives rise to a  $(G, H)$ -biset monomorphism  $\Theta_{i+2}^r(H, G) \cong G \times_H \Theta_{i+1}'(H, G) \hookrightarrow m \cdot (G \times_H \Theta_i'(H, G)) \cong m \cdot \Theta_{i+1}^r(H, G)$ .

(c) For  $i \geq 1$  we set

$$\mathcal{K}_i(H, G) := \mathcal{K}_i := \{\text{Stab}_{G \times H}(\theta) \mid \theta \in \Theta_i^r(H, G)\},$$

and for  $i \geq 0$  we set

$$\mathcal{K}'_i(H, G) := \mathcal{K}'_i := \{\text{Stab}_{H \times H}(\theta) \mid \theta \in \Theta_i'(H, G)\}.$$

Recall that  $\Theta_i'(H, G) = \Theta_i^r(H, G) = \Theta_i(H, G)$  as sets, for  $i \geq 1$ . By (4) and (6) we obtain

$$\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_3 \subseteq \cdots \quad \text{and} \quad \mathcal{K}'_0 \subseteq \mathcal{K}'_1 \subseteq \mathcal{K}'_2 \subseteq \cdots. \quad (7)$$

We also set

$$\mathcal{K}_\infty(H, G) := \mathcal{K}_\infty := \bigcup_{i \geq 1} \mathcal{K}_i \quad \text{and} \quad \mathcal{K}'_\infty(H, G) := \mathcal{K}'_\infty := \bigcup_{i \geq 0} \mathcal{K}'_i.$$

Since the set of point stabilizers of a  $G$ -set determines the isomorphism classes of its orbits, we immediately obtain for  $i \geq 1$ :

$$H \text{ has depth } 2i \text{ in } G \text{ if and only if } \mathcal{K}_{i+1} \subseteq \mathcal{K}_i, \text{ and} \quad (8)$$

$$H \text{ has depth } 2i - 1 \text{ in } G \text{ if and only if } \mathcal{K}'_i \subseteq \mathcal{K}'_{i-1}. \quad (9)$$

Since the set of subgroups of  $G \times H$  is a finite set, there exists some  $i \geq 1$  such that  $\mathcal{K}_i = \mathcal{K}_{i+1}$ . Thus,  $H$  has depth  $2i$  in  $G$  for some  $i \geq 1$ . This implies that there exists a smallest positive integer  $d$  such that  $H$  has depth  $d$  in  $G$ . This integer is called the *minimal (combinatorial) depth of  $H$  in  $G$*  and it is denoted by  $d_c(H, G)$ .

**Proposition 3.6.** (a) For  $i \geq 1$  the following are equivalent:

- (i)  $d_c(H, G) \leq 2i$ , i.e.,  $H$  has depth  $2i$  in  $G$ .
- (ii)  $\mathcal{K}_i = \mathcal{K}_{i+1}$ .
- (iii)  $\mathcal{K}_i = \mathcal{K}_\infty$ .

(b) For  $i \geq 0$  the following are equivalent:

- (i)  $d_c(H, G) \leq 2i + 1$ , i.e.,  $H$  has depth  $2i + 1$  in  $G$ .
- (ii)  $\mathcal{K}'_i = \mathcal{K}'_{i+1}$ .
- (iii)  $\mathcal{K}'_i = \mathcal{K}'_\infty$ .

**Proof.** We only prove part (a). The proof of part (b) is similar. The equivalence of (i) and (ii) follows from (7) and (8). By (7), (iii) implies (ii). Finally, Remark 3.5(b) together with the equivalence of (i) and (ii) show that (ii) implies (iii).  $\square$

**Notation 3.7.** For any  $n \geq 0$ , and any  $x_1, \dots, x_n \in G$ , we define

$$\begin{aligned} H_{x_1, \dots, x_n} &:= H \cap H^{x_1} \cap H^{x_2} \cap \dots \cap H^{x_n} \quad \text{and} \\ \tilde{H}_{x_1, \dots, x_n} &:= H \cap H^{x_n} \cap H^{x_{n-1}x_n} \cap \dots \cap H^{x_1 \dots x_n}. \end{aligned}$$

In the case that  $n = 0$ , we have  $H_{x_1, \dots, x_n} := H$  and  $\tilde{H}_{x_1, \dots, x_n} := H$  by convention.

For  $i \geq 0$  we set

$$\mathcal{U}_i(H, G) := \mathcal{U}_i := \{H_{x_1, \dots, x_i} \mid x_1, \dots, x_i \in G\} \quad \text{and} \quad \mathcal{U}_\infty(H, G) := \mathcal{U}_\infty := \bigcup_{i \geq 0} \mathcal{U}_i.$$

Note that, by convention,  $\mathcal{U}_0 = \{H\}$ , and that  $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$  for  $i \geq 0$ .

**Lemma 3.8.** (a) Let  $i \geq 1$  and let  $x_1, \dots, x_i \in G$ , so that  $[x_1, \dots, x_i] \in \Theta_i(H, G) = G \times_H \dots \times_H G$ . Then one has:

- (i)  $\text{Stab}_{G \times H}([x_1, \dots, x_i]) = \Delta_{x_1 \dots x_i}(\tilde{H}_{x_2, \dots, x_i})$ .
- (ii)  $\text{Stab}_{H \times H}([x_1, \dots, x_i]) = \Delta_{x_1 \dots x_i}(\tilde{H}_{x_1, \dots, x_i})$ .

(b) One has the following explicit descriptions for the sets  $\mathcal{K}'_i$  and  $\mathcal{K}_i$ :

- (i) For  $i \geq 1$  one has  $\mathcal{K}_i = \{\Delta_g(U) \mid g \in G, U \in \mathcal{U}_{i-1}\}$ .
- (ii)  $\mathcal{K}'_0 = \{\Delta_h(H) \mid h \in H\}$ .
- (iii) For  $i \geq 1$  one has  $\mathcal{K}'_i = \{\Delta_{g_1}(H_{g_1, \dots, g_i}) \mid g_1, \dots, g_i \in G\}$ .

**Proof.** (a) An element  $(g, h) \in G \times H$  is contained in  $\text{Stab}_{G \times H}([x_1, \dots, x_i])$  if and only if  $[gx_1, x_2, \dots, x_{i-1}, x_i h^{-1}] = [x_1, \dots, x_i]$ , that is, if and only if there exist  $h_1, \dots, h_{i-1} \in H$  such that

$$gx_1 h_1^{-1} = x_1; \quad h_{j-1} x_j h_j^{-1} = x_j, \quad \text{for } j = 2, \dots, i-1; \quad h_{i-1} x_i h^{-1} = x_i.$$

Thus  $(g, h) \in \text{Stab}_{G \times H}([x_1, \dots, x_i])$  if and only if

$$h \in H \cap x_i^{-1} H x_i \cap x_i^{-1} x_{i-1}^{-1} H x_{i-1} x_i \cap \dots \cap x_i^{-1} \dots x_2^{-1} H x_2 \dots x_i$$

and  $g = x_1 \dots x_i h x_i^{-1} \dots x_1^{-1}$ , or equivalently, if  $(g, h) \in \Delta_{x_1 \dots x_i}(\tilde{H}_{x_2, \dots, x_i})$ . This proves (i).

For  $(g, h) \in G \times H$ , we have  $(g, h) \in \text{Stab}_{H \times H}([x_1, \dots, x_i])$  if and only if

$$(g, h) \in \text{Stab}_{G \times H}([x_1, \dots, x_i]) = \Delta_{x_1 \dots x_i}(\tilde{H}_{x_2, \dots, x_i})$$

and  $g \in H$ . This is equivalent to  $(g, h) \in \Delta_{x_1 \cdots x_i}(\tilde{H}_{x_2, \dots, x_i} \cap H^{x_1 \cdots x_i}) = \Delta_{x_1 \cdots x_i}(\tilde{H}_{x_1, \dots, x_i})$ , and (ii) is proved.

(b) Parts (i) and (iii) follow immediately from the equations

$$\Delta_{x_1 \cdots x_i}(\tilde{H}_{x_2, \dots, x_i}) = \Delta_{g_1}(H_{g_2, \dots, g_i}) \quad \text{and} \quad \Delta_{x_1 \cdots x_i}(\tilde{H}_{x_1, \dots, x_i}) = \Delta_{g_1}(H_{g_1, \dots, g_i}),$$

whenever  $x_1, \dots, x_i \in G$  and  $g_1, \dots, g_i \in G$  determine each other by the equations

$$x_i = g_i, \quad x_{i-1}x_i = g_{i-1}, \quad x_{i-2}x_{i-1}x_i = g_{i-2}, \quad \dots, \quad x_1x_2 \cdots x_i = g_1.$$

Part (ii) follows immediately from the definition of  $\mathcal{K}'_0$ . □

The following theorem gives explicit conditions for  $H$  having depth  $d$  in  $G$ .

**Theorem 3.9.** (a) For  $i \geq 1$  the following are equivalent:

(i)  $d_c(H, G) \leq 2i$ , i.e.,  $H$  has depth  $2i$  in  $G$ .

(ii)  $\mathcal{U}_{i-1} = \mathcal{U}_i$ .

(iii)  $\mathcal{U}_{i-1} = \mathcal{U}_\infty$ .

(iv) For any  $x_1, \dots, x_i \in G$ , there exist  $y_1, \dots, y_{i-1} \in G$  such that  $H_{x_1, \dots, x_i} = H_{y_1, \dots, y_{i-1}}$ .

(b) One has  $d_c(H, G) = 1$  if and only if for every  $x \in G$  there exists some  $y \in H$  such that  $xhx^{-1} = yhy^{-1}$  for all  $h \in H$ .

(c) Let  $i > 1$ . The following are equivalent:

(i)  $d_c(H, G) \leq 2i - 1$ , i.e.,  $H$  has depth  $2i - 1$  in  $G$ .

(ii) For any  $x_1, \dots, x_i \in G$ , there exist  $y_1, \dots, y_{i-1} \in G$  such that  $H_{x_1, \dots, x_i} = H_{y_1, \dots, y_{i-1}}$  and  $x_1hx_1^{-1} = y_1hy_1^{-1}$  for all  $h \in H_{x_1, \dots, x_i}$ .

**Proof.** (a) This follows immediately from Proposition 3.6(a) and Lemma 3.8(b)(i).

(b) This follows immediately from Proposition 3.6(b) and Lemma 3.8(b)(ii).

(c) This follows immediately from Proposition 3.6(b) and Lemma 3.8(b)(iii). □

**Remark 3.10.** Theorem 3.9 can be interpreted as follows. The poset structure of  $\mathcal{U}_\infty$  determines an integer  $i \geq 1$  such that  $d_c(H, G) \in \{2i - 1, 2i\}$ . The integer  $i$  is minimal such that  $\mathcal{U}_{i-1} = \mathcal{U}_\infty$ . More precisely, if  $\mathcal{U}_\infty = \{H\}$  then  $i = 1$ . If  $|\mathcal{U}_\infty| > 1$  let  $U_1, \dots, U_m$  denote the maximal elements in the poset  $\mathcal{U}_\infty \setminus \{H\}$  and choose  $i$  minimal such that every element in  $\mathcal{U}_\infty$  can be expressed as the meet of  $i - 1$  elements from  $\{U_1, \dots, U_m\}$ . In order to find the precise value of  $d_c(H, G)$ , one has to check the stronger condition in Theorem 3.9(c)(ii) for this  $i$  (or 3.9(b) in the case that  $i = 1$ ), which involves also the conjugation maps on subgroups.

The following theorem gives further bounds for the integer  $d_c(H, G)$ . For a finite non-empty partially ordered set  $\mathcal{X}$ , we define the largest integer  $n$  such that there exists a chain  $x_1 < \dots < x_n$  in  $\mathcal{X}$  as the *depth* of  $\mathcal{X}$ .

**Theorem 3.11.** Let  $\delta$  denote the depth of the partially ordered set  $\mathcal{U}_\infty$ , and let  $K := \text{Core}_G(H)$ . Moreover, let  $\delta_*$  denote the smallest positive integer  $k$  such that  $K$  can be written as the intersection of  $k$  conjugates of  $H$  in  $G$ , and let  $\delta^*$  denote the smallest positive integer  $k$  such that the intersection of any  $k$  pairwise distinct  $G$ -conjugates of  $H$  is equal to  $K$ . Then one has:

(a)  $2\delta_* - 1 \leq d_c(H, G) \leq 2\delta$ .

(b)  $\delta_* \leq \delta \leq \delta^* \leq |G : N_G(H)|$ .

(c) If  $\delta_* = \delta$  and  $K \leq Z(G)$  then  $d_c(H, G) = 2\delta - 1$ .

**Proof.** (a) We first show that  $d_c(H, G) \leq 2\delta$ , using Theorem 3.9(a)(ii). Let  $x_1, \dots, x_\delta \in G$  and set  $U_i := H_{x_1, \dots, x_i}$  for  $i = 0, \dots, \delta$ . Then  $U_\delta \leq U_{\delta-1} \leq \dots \leq U_1 \leq U_0 = H$  is a chain in  $\mathcal{U}_\infty$  consisting of  $\delta + 1$  subgroups. Thus, there exists  $i \in \{1, \dots, \delta\}$  such that  $U_{i-1} = U_i$ . If we define  $y_1, \dots, y_{\delta-1} \in G$  by omitting  $x_i$  from  $x_1, \dots, x_\delta$  then  $H_{x_1, \dots, x_\delta} = H_{y_1, \dots, y_{\delta-1}}$ , as desired. Next we show that  $d_c(H, G) \geq 2\delta_* - 1$ . If  $\delta_* = 1$  then this is trivial, as  $d_c(H, G) \geq 1$  by definition. Assume that  $\delta_* > 1$  and that  $d_c(H, G) \leq 2(\delta_* - 1)$ . Then, by Theorem 3.9(a), one has  $\mathcal{U}_{\delta_*-2} = \mathcal{U}_\infty$ . Thus,  $K \in \mathcal{U}_\infty$  can be written as the intersection of some set of  $\delta_* - 1$   $G$ -conjugates of  $H$ . This is a contradiction.

(b) Clearly,  $\delta_* \leq |G : N_G(H)|$ , and by (a) we also know that  $\delta_* \leq \delta$ .

Next we show that  $\delta \leq \delta^*$ . Let  $K = U_\delta < U_{\delta-1} < \dots < U_1 = H$  be a chain (of maximal length) in  $\mathcal{U}_\infty$ . We will construct, inductively, elements  $x_1, \dots, x_\delta \in G$  such that, for all  $i = 1, \dots, \delta$ , the groups  $H^{x_1}, \dots, H^{x_i}$  are pairwise distinct and  $U_i = H^{x_1} \cap \dots \cap H^{x_i}$ . We start by setting  $x_1 := 1$ . Next assume that  $x_1, \dots, x_i \in G$  with the desired property are already constructed for  $1 \leq i < \delta$ . Since  $U_{i+1} < U_i$  and  $U_{i+1}$  can be written as the intersection of all  $G$ -conjugates of  $H$  which contain  $U_{i+1}$ , there exists  $x_{i+1} \in G$  such that  $U_{i+1} \leq U_i \cap H^{x_{i+1}} < U_i$ . This implies that  $H^{x_{i+1}}$  is distinct from each of the groups  $H^{x_1}, \dots, H^{x_i}$ . Moreover, the maximality of the length of the chain  $U_\delta < \dots < U_1$  implies that  $U_{i+1} = U_i \cap H^{x_{i+1}}$ . Now, having constructed  $x_1, \dots, x_\delta \in G$  with the desired property, we observe that the intersection  $U_{\delta-1}$  of  $\delta - 1$  pairwise distinct conjugates of  $H$  is strictly larger than  $K$ . This implies that  $\delta^* > \delta - 1$ .

(c) If  $\delta_* = \delta = 1$  then  $H$  is normal in  $G$  and  $H = K \leq Z(G)$  by assumption. So Theorem 3.9(b) then implies the result. Assume now that  $\delta_* = \delta > 1$ . We will apply the criterion in Theorem 3.9(c). Let  $x_1, \dots, x_\delta \in G$ . Set  $U_i := H_{x_1, \dots, x_i}$  for  $i = 0, \dots, \delta$ . Then  $U_\delta \leq U_{\delta-1} \leq \dots \leq U_1 \leq U_0 = H$  is a chain in  $\mathcal{U}_\infty$ . By the definition of  $\delta$  there exists  $i \in \{1, \dots, \delta\}$  with  $U_i = U_{i-1}$ . If  $i > 1$  then we can define the sequence  $y_1, \dots, y_{\delta-1} \in G$  by omitting  $x_i$  from  $x_1, \dots, x_\delta$  and have  $y_1 = x_1$  and  $H_{x_1, \dots, x_\delta} = H_{y_1, \dots, y_{\delta-1}}$ , as desired. Therefore, we may assume that  $U_\delta < U_{\delta-1} < \dots < U_1$ . In this case, the maximality of  $\delta$  forces  $U_1 = H$  and  $U_\delta = K$ . In this case we can define  $y_1, \dots, y_{\delta-1}$  as the sequence  $x_2, \dots, x_\delta$  and obtain  $H_{x_1, \dots, x_\delta} = K = H_{y_1, \dots, y_{\delta-1}}$ . Moreover,  $x_1$  and  $y_1$  act identically on  $K$ , since  $K \leq Z(G)$ . Hence  $d_c(H, G) \leq 2\delta - 1$  and, by (a), we also have  $2\delta - 1 = 2\delta_* - 1 \leq d_c(H, G)$ .  $\square$

In [2] and [1], respectively, the first and third author established group-theoretic characterizations of group algebra extensions  $RH \subseteq RG$  of depth 1 and 2, respectively. We close this section by giving analogous interpretations of group inclusions  $H \leq G$  of small combinatorial depth. Recall that a subgroup  $H$  of some group  $G$  is called a *trivial-intersection* (TI) subgroup of  $G$  provided that  $H \cap H^g = 1$  whenever  $g \in G \setminus N_G(H)$ .

**Theorem 3.12.** *Let  $K := \text{Core}_G(H)$ . One has:*

- (a)  $d_c(H, G) = 1$  if and only if  $G = HC_G(H)$ .
- (b)  $d_c(H, G) \leq 2$  if and only if  $H$  is normal in  $G$ .
- (c) If  $H$  is TI in  $G$  then  $d_c(H, G) \leq 3$ .
- (d) If  $H/K$  is TI in  $G/K$  then  $d_c(H, G) \leq 4$ . If, moreover,  $K \leq Z(G)$  then  $d_c(H, G) \in \{1, 3\}$ .

**Proof.** Assertions (a)–(c) follow immediately from Theorem 3.9. To prove (d), suppose that  $H/K$  is TI in  $G/K$ . Then  $\mathcal{U}_\infty = \{K, H\} = \mathcal{U}_1$ . Thus,  $d_c(H, G) \leq 4$ , by Theorem 3.11(a). Note that if  $H/K$  is TI in  $G/K$  then  $\delta_* = \delta \in \{1, 2\}$ . Therefore, if additionally  $K \leq Z(G)$ , then Theorem 3.11 (c) implies  $d_c(H, G) \in \{1, 3\}$ . (Note that (c) is also a consequence of (b) and (d)).  $\square$

**Examples 3.13.** (a) The following example shows that the bound  $d_c(H, G) \leq 2\delta$  from Theorem 3.11(a) is not sharp, in the sense that it can happen that  $d_c(H, G) \leq 2(\delta - 1)$ .

Consider the wreath product  $G := C_2 \wr \mathfrak{S}_4$  of the group  $C_2 := \langle (1, 2) \rangle$  of order 2 with the symmetric group  $\mathfrak{S}_4$  of degree 4. We may regard  $G$  as a subgroup of the symmetric group  $\mathfrak{S}_8$  in the usual way. Let  $H := \langle (1, 2), (3, 4) \rangle \leq G$ . The normalizer  $N_G(H)$  is isomorphic to  $(C_2 \wr C_2) \times (C_2 \wr C_2)$  so that  $|G : N_G(H)| = 6$ . Thus, there are precisely six conjugates of  $H$  in  $G$ : any such group is generated by two of the transpositions  $(1, 2), (3, 4), (5, 6), (7, 8)$ . The core  $K := \text{Core}_G(H)$  is trivial, and can be written as the intersection of any four distinct  $G$ -conjugates of  $H$ . But the intersection of the three  $G$ -conjugates  $\langle (1, 2), (3, 4) \rangle, \langle (3, 4), (5, 6) \rangle, \langle (3, 4), (7, 8) \rangle$  is not trivial. Thus,  $\delta^* = 4$ . Clearly,  $\delta_* = 2$  and  $\delta = 3$ . So Theorem 3.11(a) and (b) give  $3 \leq d_c(H, G) \leq 6$ .

Using Theorem 3.9(c), one can show that in fact  $d_c(H, G) = 3$ : let  $1 =: x_0, x_1, x_2 \in G$ . If  $H = H^{x_2}$  or  $H^{x_1} = H^{x_2}$  then we define  $y_1 := x_1$  which satisfies the conditions in Theorem 3.9(c)(ii). Otherwise we either have  $H^{x_a} \neq H^{x_b}$  for any  $0 \leq a < b \leq 2$ , or we have  $H^{x_2} \neq H = H^{x_1} \neq H^{x_2}$ . So, in particular,  $|H \cap H^{x_1} \cap H^{x_2}| \leq 2$ . If  $H \cap H^{x_1} \cap H^{x_2} = 1$  then we set  $y_1 := (1, 5)(2, 6)(3, 7)(4, 8)$  which again satisfies the conditions in Theorem 3.9(c)(ii). Suppose now that  $|H \cap H^{x_1} \cap H^{x_2}| = 2$ . If  $H \neq H^{x_1}$  then  $H \cap H^{x_1} = H \cap H^{x_1} \cap H^{x_2}$ , and we set  $y_1 := x_1$ . Otherwise we are in the case where  $H = H^{x_1}$ , and we then have  $H \cap H^{x_1} \cap H^{x_2} \in \{\langle (1, 2) \rangle, \langle (3, 4) \rangle\}$ . We may suppose that  $H \cap H^{x_1} \cap H^{x_2} = \langle (3, 4) \rangle$ ; the case where  $H \cap H^{x_1} \cap H^{x_2} = \langle (1, 2) \rangle$  can be treated similarly. If  $x_1 \in C_G(\langle (3, 4) \rangle)$  then we define  $y_1 := (1, 5)(2, 6)$ . If  $x_1 \in N_G(H) \setminus C_G(\langle (3, 4) \rangle)$  then we must have  $x_1(3, 4)x_1^{-1} = (1, 2)$ , and we set  $y_1 := (1, 7, 5, 3)(2, 8, 6, 4)$ . In either case,  $y_1$  satisfies both conditions in Theorem 3.9(c)(ii). Since  $H$  is not normal in  $G$ , we thus get  $d_c(H, G) = 3$ .

This example also shows that the converse of Theorem 3.12 (d) does not hold, in general. For  $H/K \cong H$  is not a TI-subgroup of  $G/K \cong G$ .

(b) Let  $G := \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^{-1} \rangle$  and  $H := \langle a^4, b \rangle$ . Then  $G$  is a dihedral group of order 16, and  $H$  is a Klein four group. In this case, the core  $K$  of  $H$  in  $G$  equals the centre  $Z(G) = \langle a^4 \rangle$  of  $G$ . Moreover,  $H$  is not TI in  $G$ , but the factor group  $H/K$  is TI in  $G/K$ . Thus we must have  $d_c(H, G) = 3$ .

(c) The following example shows that we cannot drop the assumption  $\text{Core}_G(H) \leq Z(G)$  in the second statement in Theorem 3.12(d). Consider  $G := \mathfrak{S}_4$ , the symmetric group of degree 4. Let further  $H$  be a Sylow 2-subgroup of  $G$ . Then the core  $K$  of  $H$  in  $G$  is the normal Klein four subgroup of  $\mathfrak{S}_4$ , so that  $K \not\leq Z(G)$ . Moreover,  $G/K \cong \mathfrak{S}_3$ ,  $H/K \cong \mathfrak{S}_2$ , and  $H/K$  is a TI-subgroup of  $G/K$ . By Theorem 3.12, we must have  $3 \leq d_c(H, G) \leq 4$ . Using Theorem 3.9(c)(ii), we can deduce that  $d_c(P, \mathfrak{S}_4) = 4$ . In fact, if we choose  $x_1 = 1$  and  $x_2 \in G \setminus H$  then there exists no  $y_1 \in G$  satisfying the conditions in Theorem 3.9(c)(ii).

## 4 Depth of Group Algebra Extensions

Throughout this section  $G$  denotes a finite group,  $H$  denotes a subgroup of  $G$ , and  $R$  denotes a commutative ring with  $R \neq 0$ . In the course of this section we determine how the combinatorial depth of  $H$  in  $G$  is related to the depth of the ring extension  $RH \subseteq RG$ .

**Theorem 4.1.** *Let  $d \in \mathbb{N}$ . The ring extension  $RH \subseteq RG$  has left depth  $d$  if and only if it has right depth  $d$ . Moreover, we have  $d_R(H, G) \leq d_c(H, G)$ . In particular,  $RH$  has finite minimal depth in  $RG$ .*

**Proof.** Suppose that, for some  $d \in \mathbb{N}$ , the group algebra  $RH$  has left depth  $d$  in  $RG$ . If  $d$  is odd then  $RH$  has automatically right depth  $d$  in  $RG$  as well. If  $d = 2i$ , for some  $i \geq 1$ , then there is some  $m \in \mathbb{N}$  such that

$$T_{i+1}^1(RH, RG) \mid m \cdot T_i^1(RH, RG). \quad (10)$$

Note that, for any  $j \geq 1$ , the  $(RH, RG)$ -bimodule  $T_j^1(RH, RG)$  is isomorphic to the permutation bimodule  $R\Theta_j^1(H, G)$ . This follows from Proposition 2.7. So, in consequence of Lemma 2.3 and Lemma 3.3, we obtain, for any  $j \geq 1$ , an isomorphism  $(\Theta_j^1(H, G))^\circ \cong \Theta_j^1(H, G)$  of  $(G, H)$ -bisets and thus an isomorphism

$$(T_j^1(RH, RG))^* \cong (R\Theta_j^1(H, G))^* \cong R(\Theta_j^1(H, G))^\circ \cong R\Theta_j^1(H, G) \cong T_j^1(RH, RG)$$

of  $(RG, RH)$ -bimodules. Hence, by (10), we also have

$$T_{i+1}^r(RH, RG) \mid m \cdot T_i^r(RH, RG),$$

and  $RH$  has right depth  $2i = d$  in  $RG$ . Analogously, we deduce that if  $RH$  has right depth  $d \in \mathbb{N}$  in  $RG$  then  $RH$  has also left depth  $d$  in  $RG$ . In particular,  $d_R^l(H, G) = d_R^r(H, G) = d_R(H, G)$ .

Now let  $d := d_c(H, G)$  be the minimal combinatorial depth of  $H$  in  $G$ . In the case that  $d = 2i+1$  for some  $i \geq 0$ , there exist  $m \in \mathbb{N}$  and an  $(H, H)$ -biset monomorphism  $\Theta'_{i+1}(H, G) \hookrightarrow m \cdot \Theta'_i(H, G)$ . From this we deduce that

$$T'_{i+1}(RH, RG) \cong R\Theta'_{i+1}(H, G) \mid m \cdot R\Theta'_i(H, G) \cong m \cdot T'_i(RH, RG),$$

as  $(RH, RH)$ -bimodules so that  $RH$  has depth  $2i + 1 = d$  in  $RG$ . Similarly one deals with the case where  $d = 2i$  for some  $i \geq 1$ . Therefore,

$$d_R(H, G) \leq d = d_c(H, G),$$

and, together with the finiteness of  $d_c(H, G)$ , cf. Remark 3.5(c), the assertion of the theorem follows.  $\square$

An obvious question now is to what extent the minimal depth  $d_R(H, G)$  of a group algebra extension  $RH \subseteq RG$  depends on the ring  $R$ . For instance, given a unitary subring  $R'$  of  $R$ , how are  $d_R(H, G)$  and  $d_{R'}(H, G)$  related? Proposition 4.4 below gives answers to some of these questions.

For the reader's convenience we recall the following notions.

**Definition 4.2.** Let  $S$  be an associative, unitary  $R$ -algebra.

- (a)  $R$  is called *semilocal* if  $R$  has only finitely many maximal ideals.
- (b) Suppose that  $S$  and  $R$  satisfy the following condition:  $M' \longrightarrow M \longrightarrow M''$  is an exact sequence of left  $R$ -modules if and only if  $S \otimes_R M' \longrightarrow S \otimes_R M \longrightarrow S \otimes_R M''$  is an exact sequence of left  $S$ -modules. Then  $S$  is called *faithfully flat over  $R$* .

**Remark 4.3.** Let  $S \neq 0$  be a commutative  $R$ -algebra. Suppose that  $M$  and  $N$  are left  $RG$ -modules. If  $M \mid N$  as  $RG$ -modules then  $S \otimes_R M \mid S \otimes_R N$  as  $SG$ -modules. In the case that  $R$  and  $S$  are noetherian,  $R$  is semilocal,  $S$  is faithfully flat over  $R$ , and  $M$  and  $N$  are finitely generated  $RG$ -modules, the converse is also true. This generalized version of the Deuring-Noether Theorem can be found in [7, Prop. 2.5.8].

**Proposition 4.4.** (a) If  $S \neq 0$  is a commutative  $R$ -algebra then  $d_S(H, G) \leq d_R(H, G)$ . In particular,  $d_R(H, G) \leq d_{\mathbb{Z}}(H, G)$  for every commutative ring  $R \neq 0$ . If, moreover,  $R$  and  $S$  are noetherian,  $S$  is faithfully flat over  $R$ , and  $R$  is semilocal, then we have  $d_R(H, G) = d_S(H, G)$ .

(b) Let  $p$  be a prime, and let  $(K, \mathcal{O}, F)$  be a  $p$ -modular system. That is,  $\mathcal{O}$  is a complete discrete valuation ring with quotient field  $K$  of characteristic 0, with maximal ideal  $(\pi)$ , and with residue field  $F = \mathcal{O}/(\pi)$  of characteristic  $p$ . Then

$$d_K(H, G) \leq d_{\mathcal{O}}(H, G) = d_F(H, G) \leq d_c(H, G).$$

**Proof.** (a) Suppose that  $S \neq 0$  is a commutative  $R$ -algebra, and let  $d := d_R(H, G)$ . We consider the case that  $d = 2i + 1$ , for some  $i \geq 0$ , first. Then there is some  $m \in \mathbb{N}$  such that

$$R\Theta'_{i+1}(H, G) \cong T'_{i+1}(RH, RG) \mid m \cdot T'_i(RH, RG) \cong m \cdot R\Theta'_i(H, G)$$

as  $(RH, RH)$ -bimodules. But then we also get

$$\begin{aligned} T'_{i+1}(SH, SG) &\cong S\Theta'_{i+1}(H, G) \cong S \otimes_R R\Theta'_{i+1}(H, G) \mid m \cdot (S \otimes_R R\Theta'_{i+1}(H, G)) \\ &\cong m \cdot (S\Theta'_i(H, G)) \cong m \cdot T'_i(SH, SG) \end{aligned}$$

as  $(SH, SH)$ -bimodules. Thus,  $SH$  has depth  $d$  in  $SG$ . In the case that  $d = 2i$  for some  $i \geq 1$ , replacing  $\Theta'_i(H, G)$  by  $\Theta_i^f(H, G)$  and  $\Theta'_{i+1}(H, G)$  by  $\Theta_{i+1}^f(H, G)$ , we similarly deduce that  $SH$  has depth  $d$  in  $SG$ . So, in any case,  $d_S(H, G) \leq d = d_R(H, G)$ .

In the case that  $R$  and  $S$  are noetherian rings, with  $R$  semilocal and  $S$  faithfully flat over  $R$ , we also get  $d_R(H, G) \leq d_S(H, G)$ , by [7, Prop. 2.5.8]. Note that here we use the fact that, for any  $i \geq 1$ , the permutation  $R[G \times G]$ -module  $T_i(H, G)$  is finitely generated over  $R$ .

(b) Now, consider a  $p$ -modular system  $(K, \mathcal{O}, F)$ . Then we have unitary ring homomorphisms  $\mathcal{O} \hookrightarrow K$  and  $\mathcal{O} \twoheadrightarrow \mathcal{O}/(\pi) = F$  turning both  $K$  and  $F$  into commutative, associative, unitary  $\mathcal{O}$ -algebras. So, by part (a), we have  $d_K(H, G) \leq d_{\mathcal{O}}(H, G)$  as well as  $d_F(H, G) \leq d_{\mathcal{O}}(H, G)$ .

In order to show that  $d_F(H, G) = d_{\mathcal{O}}(H, G)$ , let  $d := d_F(H, G)$ . If  $d = 2i + 1$ , for some  $i \geq 0$ , then there exists  $m \in \mathbb{N}$  with

$$F\Theta'_{i+1}(H, G) \cong T'_{i+1}(FH, FG) \mid m \cdot T'_i(FH, FG) \cong m \cdot F\Theta'_i(H, G)$$

as  $(FH, FH)$ -bimodules. By [10, Thm. 4.8.9], this implies

$$T'_{i+1}(\mathcal{O}H, \mathcal{O}G) \cong \mathcal{O}\Theta'_{i+1}(H, G) \mid m \cdot \mathcal{O}\Theta'_i(H, G) \cong m \cdot T'_i(\mathcal{O}H, \mathcal{O}G)$$

as  $(\mathcal{O}H, \mathcal{O}H)$ -bimodules, so that  $\mathcal{O}H$  has also depth  $d$  in  $\mathcal{O}G$ . If  $d = 2i$ , for some  $i \geq 1$ , then replacing  $\Theta'_i(H, G)$  by  $\Theta_i^f(H, G)$  and  $\Theta'_{i+1}(H, G)$  by  $\Theta_{i+1}^f(H, G)$  in the above argument, we deduce that  $\mathcal{O}H$  has also depth  $d$  in  $\mathcal{O}G$ . Therefore, in any case,  $d_{\mathcal{O}}(H, G) \leq d = d_F(H, G)$ , and the proof of (b) is complete.  $\square$

**Remark/Definition 4.5.** (a) Suppose that  $R$  is a field. In consequence of Proposition 4.4(a) above, the minimal depth of  $RH$  in  $RG$  then only depends on the characteristic of  $R$ . Therefore, whenever  $p$  is a prime or  $p = 0$ , we from now on set  $d_p(H, G) := d_R(H, G)$  where  $R$  is any field of characteristic  $p$ .

(b) Let  $p$  be a prime. By Proposition 4.4, we obtain

$$d_0(H, G) \leq d_p(H, G) \leq d_{\mathbb{Z}}(H, G) \leq d_c(H, G).$$

Let  $R \neq 0$  be again an arbitrary commutative ring. Since  $R$  has a maximal ideal, there exists a field  $F$  which is an  $R$ -algebra. Since  $d_0(H, G) \leq d_p(H, G)$ , we also obtain, independently of the characteristic of  $F$ ,

$$d_0(H, G) \leq d_F(H, G) \leq d_R(H, G) \leq d_{\mathbb{Z}}(H, G) \leq d_c(H, G).$$

## 5 Examples

Given a finite set  $\Omega$ , we denote the symmetric and alternating group on  $\Omega$  by  $\mathfrak{S}(\Omega)$  and  $\mathfrak{A}(\Omega)$ , respectively. In the case that  $\Omega = \{1, \dots, n\}$  for some  $n \in \mathbb{N}$ , we set  $\mathfrak{S}_n := \mathfrak{S}(\Omega)$  and  $\mathfrak{A}_n := \mathfrak{A}(\Omega)$ .

In an appendix to [4], the second and third author determined the depths of the ring extensions  $\mathbb{C}\mathfrak{S}_n \subseteq \mathbb{C}\mathfrak{S}_{n+1}$  and  $\mathbb{C}\mathfrak{A}_n \subseteq \mathbb{C}\mathfrak{A}_{n+1}$ . By [4, Prop. A.2, Prop. A.5], we know that, for  $n \geq 2$ , we have  $d_0(\mathfrak{S}_n, \mathfrak{S}_{n+1}) = d_{\mathbb{C}}(\mathfrak{S}_n, \mathfrak{S}_{n+1}) = 2n - 1$ , and for  $n \geq 3$ , we have  $d_0(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = d_{\mathbb{C}}(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = 2(n - \lceil \sqrt{n} \rceil) + 1$ . Moreover, note that we clearly have  $d_0(\mathfrak{S}_1, \mathfrak{S}_2) = d_0(1, \mathfrak{S}_2) = 1 = 2 \cdot 1 - 1$ , and  $d_0(\mathfrak{A}_2, \mathfrak{A}_3) = d_0(1, \mathfrak{A}_3) = 1 = 2(2 - \lceil \sqrt{2} \rceil) + 1$ .

In the following, we will determine  $d_c(\mathfrak{S}_n, \mathfrak{S}_{n+1})$  and  $d_c(\mathfrak{A}_n, \mathfrak{A}_{n+1})$ . We will see that, in general,  $d_c(\mathfrak{A}_n, \mathfrak{A}_{n+1}) > d_0(\mathfrak{A}_n, \mathfrak{A}_{n+1})$ , whereas  $d_c(\mathfrak{S}_n, \mathfrak{S}_{n+1}) = d_0(\mathfrak{S}_n, \mathfrak{S}_{n+1})$ .

It should be emphasized here that in [4] the depth of an extension of finite-dimensional, semisimple algebras over an algebraically closed field of characteristic 0 was defined via the so-called inclusion matrix of the ring extension. In the appendix to our present paper we will give an interpretation of the depth of an arbitrary ring extension  $B \subseteq A$  using some category theory, thereby proving the equivalence of our notion of depth and that used in [4] in the case where  $A$  and  $B$  are finite-dimensional semisimple algebras over an algebraically closed field.

In Subsection 5.B we will then address the following question: given any ring extensions  $C \subseteq B$  and  $B \subseteq A$ , can the minimal depth  $d(C, A)$  be expressed in terms of the minimal depths  $d(C, B)$  and  $d(B, A)$ ? We will see that this is, in general, not the case, even if the rings in question are group algebras.

### 5.A The Symmetric and Alternating Groups

First we determine the minimal combinatorial depth of  $\mathfrak{S}_n$  in  $\mathfrak{S}_{n+1}$ .

**Proposition 5.1.** *For  $n \geq 1$  and any commutative ring  $R \neq 0$  one has  $d_c(\mathfrak{S}_n, \mathfrak{S}_{n+1}) = d_R(\mathfrak{S}_n, \mathfrak{S}_{n+1}) = 2n - 1$ .*

**Proof.** Suppose that  $x_1, \dots, x_n \in \mathfrak{S}_{n+1}$  are such that the groups  $\mathfrak{S}_n^{x_1}, \dots, \mathfrak{S}_n^{x_n}$  are mutually distinct. Then their intersection  $\mathfrak{S}_n^{x_1} \cap \dots \cap \mathfrak{S}_n^{x_n}$  fixes  $n$  points of  $\{1, \dots, n+1\}$  and therefore

$$\mathfrak{S}_n^{x_1} \cap \dots \cap \mathfrak{S}_n^{x_n} = 1 \leq Z(\mathfrak{S}_{n+1}).$$

This implies that  $\delta^* \leq n$ . Moreover, it is obvious that  $\delta_* = n$ . Thus, by Theorem 3.11(b), we have  $\delta_* = \delta = n$ . Now Theorem 3.11(c) implies  $d_c(\mathfrak{S}_n, \mathfrak{S}_{n+1}) = 2n - 1$ .

On the other hand, by [4, Prop. A.2], we already know that  $d_0(H, G) = 2n - 1$ . By Remark 4.5(b), we have  $d_0(H, G) \leq d_R(H, G) \leq d_c(H, G)$  for any commutative ring  $R \neq 0$ . Thus,  $d_0(H, G) = d_R(H, G) = 2n - 1$ , completing the proof of the proposition.  $\square$

Next, we determine the minimal combinatorial depth of  $\mathfrak{A}_n$  in  $\mathfrak{A}_{n+1}$ .

**Proposition 5.2.** *For  $n \geq 2$ , one has  $d_c(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = 2n - 3$ .*

**Proof.** In the case that  $n = 2$ , we deduce from Theorem 3.12(a) that  $d_c(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = 1$ . So we may suppose that  $n \geq 3$ . Note that the core of  $\mathfrak{A}_n$  in  $\mathfrak{A}_{n+1}$  is trivial. Therefore, by Theorem 3.11(c), it suffices to show that  $\delta_*(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = \delta^*(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = n - 1$ . So suppose that  $x_1, \dots, x_{n-1} \in \mathfrak{A}_{n+1}$  are such that  $\mathfrak{A}_n^{x_1}, \dots, \mathfrak{A}_n^{x_{n-1}}$  are mutually distinct. Then their intersection  $\mathfrak{A}_n^{x_1} \cap \dots \cap \mathfrak{A}_n^{x_{n-1}}$  has  $n - 1$  fixed points on  $\{1, \dots, n + 1\}$ . That is, there are some  $\omega_1, \omega_2 \in \{1, \dots, n + 1\}$  with

$$\mathfrak{A}_n^{x_1} \cap \dots \cap \mathfrak{A}_n^{x_{n-1}} = \mathfrak{A}(\{\omega_1, \omega_2\}) = 1.$$

On the other hand, the intersection of  $n - 2$  conjugates of  $\mathfrak{A}_n$  in  $\mathfrak{A}_{n+1}$  has at most  $n - 2$  fixed points on  $\{1, \dots, n + 1\}$ . Therefore, this intersection is isomorphic to  $\mathfrak{A}_m$  for some  $m \geq 3$ , which is not trivial. This shows that  $\delta_*(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = n - 1 = \delta^*(\mathfrak{A}_n, \mathfrak{A}_{n+1})$  and the proof is complete.  $\square$

**Remark 5.3.** As already indicated at the beginning of this section, the previous proposition and [4, Prop. A.5] show that, for  $n \geq 5$ , we get

$$d_0(\mathfrak{A}_n, \mathfrak{A}_{n+1}) = 2(n - \lceil \sqrt{n} \rceil) + 1 < 2n - 3 = d_c(\mathfrak{A}_n, \mathfrak{A}_{n+1}),$$

in contrast to the case of the symmetric groups examined in Proposition 5.1. Moreover, by Remark 4.5(b), for every  $n \in \mathbb{N}$  and every prime  $p$ , we have

$$d_0(\mathfrak{A}_n, \mathfrak{A}_{n+1}) \leq d_p(\mathfrak{A}_n, \mathfrak{A}_{n+1}) \leq d_c(\mathfrak{A}_n, \mathfrak{A}_{n+1}).$$

At the moment, we do not know the precise value of  $d_p(\mathfrak{A}_n, \mathfrak{A}_{n+1})$ .

## 5.B Wreath Products

Suppose that  $C \subseteq B$  and  $B \subseteq A$  are arbitrary ring extensions. Moreover, let  $K \leq H \leq G$  be groups. We want to investigate the following question:

**Question 5.4.** (a) Can one express or bound  $d(C, A)$  in terms of  $d(C, B)$  and  $d(B, A)$ ?

(b) Can one express or bound  $d_c(K, G)$  in terms of  $d_c(K, H)$  and  $d_c(H, G)$ ?

**Example 5.5.** As mentioned earlier, we will show that this is, in general, not the case. To this end, we fix the following notation, for the remainder of this subsection: let  $p$  be a prime, let  $C_p = \langle (1, \dots, p) \rangle$  be a cyclic group of order  $p$ , and let  $G := C_2 \wr C_p$  be the wreath product of  $C_2$  and  $C_p$ . As usual, we may identify  $G$  with the subgroup  $\langle (1, 2), (1, 3, 5, \dots, 2p - 1)(2, 4, 6, \dots, 2p) \rangle$  of  $\mathfrak{S}_{2p}$ . The *base group* of  $G$  is isomorphic to a direct product of  $p$  copies of  $C_2$ , and will be denoted by  $H$ . That is,  $H = \langle (1, 2), (3, 4), \dots, (2p - 1, 2p) \rangle$ . Moreover, we denote by  $K$  the subgroup of  $H$  generated by the transpositions  $(1, 2), (3, 4), \dots, (2p - 3, 2p - 2)$ . Thus  $C_G(H) = H = C_H(K)$ , and  $H \trianglelefteq G$ , so that, by Theorem 3.12(a) and (b), we get

$$d_c(K, H) = 1 \quad \text{and} \quad d_c(H, G) = 2.$$

Furthermore, [1, Thm. 1.2] and [2, Thm. 1.8] imply  $d_0(K, H) = 1$  and  $d_0(H, G) = 2$ . Thus, for any commutative ring  $R \neq 0$ , we have

$$d_R(K, H) = 1 \quad \text{and} \quad d_R(H, G) = 2,$$

by Remark 4.5(b). In order to determine  $d_c(K, G)$  we use Theorem 3.11(c). First note that the core of  $K$  in  $G$  is trivial. Moreover, note that  $K$  has precisely  $p$  conjugate subgroups in  $G$  and that the intersection of any  $p - 1$ , mutually distinct, such conjugates is a subgroup of order 2. Thus,  $\delta_* = \delta^* = p$ , and by Theorem 3.11(b) we obtain  $\delta_* = \delta$ . Now Theorem 3.11(c) implies that

$$d_c(K, G) = 2p - 1.$$

Finally, we want to determine  $d_0(K, G)$ . For this, we recall what the ordinary irreducible characters of  $G$ ,  $H$ , and  $K$  look like. Given an ordinary irreducible character  $\psi$  of  $H$ , we can write  $\psi = \psi_1 \times \cdots \times \psi_p$  where, for  $i = 1, \dots, p$ , the character  $\psi_i$  is an irreducible character of  $C_2$ . Thus  $\psi_i$  is either the trivial or the sign character. Similarly, an irreducible character  $\eta$  of  $K$  is of the form  $\eta = \eta_1 \times \cdots \times \eta_{p-1}$  where  $\eta_j$  is either the trivial character or the sign character of  $C_2$ , for  $j = 1, \dots, p - 1$ .

Suppose that  $\psi = \psi_1 \times \cdots \times \psi_p$  is an irreducible character of  $H$  and that  $g \in G$  with  $g = \sigma h$ , for some  $h \in H$  and  $\sigma \in \langle (1, \dots, p) \rangle$ . Then

$${}^g\psi := \psi_{\sigma^{-1}(1)} \times \cdots \times \psi_{\sigma^{-1}(p)}$$

is again an irreducible character of  $H$ , and we obtain an action of  $G$  on the set  $\text{Irr}(H)$  of irreducible characters of  $H$ . A character  $\psi \in \text{Irr}(H)$  is fixed under this  $G$ -action if and only if  $\psi_1 = \dots = \psi_p$  in which case  $\psi$  extends to an irreducible character of  $G$ . More precisely, there are  $p$  distinct irreducible characters of  $G$  which extend  $\psi$ . Otherwise the inertial group of  $\psi$  in  $G$  equals  $H$ , and  $\psi$  induces irreducibly to a character of  $G$ . Furthermore, we then have  $\text{Ind}_H^G(\psi) = \text{Ind}_H^G({}^g\psi)$ , for any  $g \in G$ .

Lastly, note that an irreducible character  $\psi = \psi_1 \times \cdots \times \psi_p$  of  $H$  restricts to the irreducible character  $\psi_1 \times \cdots \times \psi_{p-1}$  of  $K$ .

For convenience, we will identify a character  $\psi = \psi_1 \times \cdots \times \psi_p \in \text{Irr}(H)$  with the sequence  $(i_1, \dots, i_p)$  where, for  $k = 1, \dots, p$ , we have  $i_k = 1$  if  $\psi_k$  is the trivial character, and  $i_k = -1$  if  $\psi_k$  is the sign character. We will use an analogous notation for the irreducible characters of  $K$ .

**Proposition 5.6.** *With the notation as in Example 5.5, for any commutative ring  $R \neq 0$ , we have  $d_R(K, G) = 2p - 1$ .*

**Proof.** There are precisely  $p$  conjugates of  $K$  in  $G$ ; each of these is generated by  $p - 1$  of the transpositions  $(1, 2), (3, 4), \dots, (2p - 1, 2p)$ . The intersection of these groups is the trivial group. Therefore,  $d_0(K, G) \leq 2p - 1$ , by [4, Thm. 6.9]. In order to show that  $d_0(K, G) = 2p - 1$  we consider the bipartite graph  $\Gamma$  with vertices

$$\mathcal{V} = \text{Irr}(G) \cup \text{Irr}(K)$$

and edges

$$\mathcal{E} = \{ \{ \chi, \eta \} \mid \chi \in \text{Irr}(G), \eta \in \text{Irr}(K), \langle \text{Res}_K^G(\chi), \eta \rangle \neq 0 \}.$$

This is a connected graph. Suppose that  $\eta = (i_1, \dots, i_{p-1}) \in \text{Irr}(K)$  and  $\eta' = (j_1, \dots, j_{p-1}) \in \text{Irr}(K)$  are such that  $\langle \eta', \text{Res}_K^G(\text{Ind}_K^G(\eta)) \rangle \neq 0$ . That is, in  $\Gamma$  there is a path of length 2 from  $\eta$  to  $\eta'$ . We claim that

$$|\{1 \leq k \leq p - 1 \mid j_k = -1\}| \leq |\{1 \leq k \leq p - 1 \mid i_k = -1\}| + 1. \quad (11)$$

If  $i_1 = \dots = i_{p-1} = -1$  then this is clearly true. If  $i_1 = \dots = i_{p-1} = 1$  then  $\text{Ind}_K^H(\eta) = (1, \dots, 1) + (1, \dots, 1, -1)$ . Moreover,  $\text{Res}_K^G(\text{Ind}_H^G((1, \dots, 1))) = p\eta$ , and

$$\text{Res}_H^G(\text{Ind}_H^G((1, \dots, 1, -1))) = (1, \dots, 1, -1) + (1, \dots, 1, -1, 1) + \dots + (-1, 1, \dots, 1).$$

Thus  $|\{1 \leq k \leq p-1 \mid j_k = -1\}| \leq 1 = |\{1 \leq k \leq p-1 \mid i_k = -1\}| + 1$ .

We may now suppose that  $(1, \dots, 1) \neq \eta \neq (-1, \dots, -1)$ . Then we have

$$\text{Ind}_K^H(\eta) = (i_1, \dots, i_{p-1}, 1) + (i_1, \dots, i_{p-1}, -1).$$

Let  $\psi = (l_1, \dots, l_p) \in \text{Irr}(H)$  be such that  $\langle \text{Res}_K^H(\psi), \eta' \rangle \neq 0 \neq \langle \psi, \text{Res}_H^G(\text{Ind}_K^G(\eta)) \rangle$ . If  $\langle \psi, \text{Res}_H^G(\text{Ind}_H^G((i_1, \dots, i_{p-1}, 1))) \rangle \neq 0$  then  $|\{1 \leq k \leq p \mid l_k = -1\}| = |\{1 \leq k \leq p-1 \mid i_k = -1\}|$ , and if  $\langle \psi, \text{Res}_H^G(\text{Ind}_H^G((i_1, \dots, i_{p-1}, -1))) \rangle \neq 0$  then  $|\{1 \leq k \leq p \mid l_k = -1\}| = |\{1 \leq k \leq p-1 \mid i_k = -1\}| + 1$ . Since  $\eta' = (l_1, \dots, l_{p-1})$ , this implies (11).

As is easily seen, there is a path of length  $2p-2$  from  $(1, \dots, 1)$  to  $(-1, \dots, -1)$  in  $\Gamma$ . The intermediate vertices belonging to  $\text{Irr}(K)$  are those labelled by the characters  $(1, \dots, 1, -1)$ ,  $(1, \dots, 1, -1, -1), \dots, (1, -1, \dots, -1)$ . On the other hand, by the inequality (11), there cannot be a path of length less than  $2p-2$  from  $(1, \dots, 1)$  to  $(-1, \dots, -1)$ . In consequence of [4, Thm. 3.13] we obtain  $d_0(K, G) \geq 2p-1$ . Now, since also  $d_c(K, G) = 2p-1$ , Remark 4.5(a) implies  $d_0(K, G) = 2p-1$ . Finally, Remark 4.5(b) implies that  $d_R(K, G) = 2p-1$  for every commutative ring  $R \neq 0$ .  $\square$

**Remark 5.7.** Proposition 5.6 above and the considerations in Example 5.5 give negative answers to both parts of Question 5.4.

## A Appendix – Some Category Theory

Let  $B \subseteq A$  be any ring extension. In the case that  $A$  and  $B$  are group algebras, in [1, Thm. 1.2] characterizations of  $B$  having depth 2 in  $A$  were given in terms of induction and restriction functors. We will now prove an analogue for arbitrary ring extensions  $B \subseteq A$  of arbitrary finite depth  $d > 1$ . This will then facilitate the proofs of Theorem A.8 and Theorem A.9 below which show, in particular, that our notion of depth and the notion of depth used in [4] are equivalent in the case where  $A$  and  $B$  are finite-dimensional semisimple algebras over an algebraically closed field.

**Remark A.1.** (a) Suppose that  $\mathcal{C}$  and  $\mathcal{D}$  are abelian categories, and let  $\Phi : \mathcal{C} \rightarrow \mathcal{D}$  and  $\Psi : \mathcal{C} \rightarrow \mathcal{D}$  be functors. Then we obtain a functor  $\Phi \oplus \Psi : \mathcal{C} \rightarrow \mathcal{D}$  which is defined in the obvious way. Moreover, for any  $n \in \mathbb{N}$ , we denote the functor  $\bigoplus_{i=1}^n \Phi$  by  $n\Phi$ .

If there are natural transformations  $\bar{\varphi} : \Phi \rightarrow \Psi$  and  $\bar{\psi} : \Psi \rightarrow \Phi$  with  $\bar{\psi} \circ \bar{\varphi} = \text{id}_\Phi$  then we write  $\Phi \mid \Psi$ .

(b) For any ring  $A$ , we denote the category of left  $A$ -modules by  $A\text{-Mod}$ , and we denote the category of right  $A$ -modules by  $\text{Mod-}A$ .

(c) Let  $B \subseteq A$  be any ring extension, and let  $i \geq 1$ . Recall that we have defined the  $(A, A)$ -bimodule  $T_i(B, A)$  as the  $i$ -fold tensor product of  $A$  over  $B$ . Moreover, as before, we have the  $(A, B)$ -bimodule  $T_i^r(B, A)$ , the  $(B, A)$ -bimodule  $T_i^l(B, A)$ , and the  $(B, B)$ -bimodule  $T_i'(B, A)$  all of which are obtained from  $T_i(B, A)$  via appropriate restrictions. We also have

- $T'_i(B, A) \otimes_B - = (\text{Res}_B^A \text{Ind}_B^A)^i$  as functors from  $B\text{-Mod}$  to  $B\text{-Mod}$ ,
- $T_i^l(B, A) \otimes_A - = (\text{Res}_B^A \text{Ind}_B^A)^{i-1} \text{Res}_B^A$  as functors from  $A\text{-Mod}$  to  $B\text{-Mod}$ ,
- $T_i^r(B, A) \otimes_B - = (\text{Ind}_B^A \text{Res}_B^A)^{i-1} \text{Ind}_B^A$  as functors from  $B\text{-Mod}$  to  $A\text{-Mod}$ ,
- $- \otimes_B T'_i(B, A) = (\text{Res}_B^A \text{Ind}_B^A)^i$  as functors from  $\mathbf{Mod}\text{-}B$  to  $\mathbf{Mod}\text{-}B$ ,
- $- \otimes_B T_i^l(B, A) = (\text{Ind}_B^A \text{Res}_B^A)^{i-1} \text{Ind}_B^A$  as functors from  $\mathbf{Mod}\text{-}B$  to  $\mathbf{Mod}\text{-}A$ ,
- $- \otimes_A T_i^r(B, A) = (\text{Res}_B^A \text{Ind}_B^A)^{i-1} \text{Res}_B^A$  as functors from  $\mathbf{Mod}\text{-}A$  to  $\mathbf{Mod}\text{-}B$ .

**Proposition A.2.** *Let  $B \subseteq A$  be a ring extension, and let  $i \in \mathbb{N}$ . Then the following are equivalent:*

(i)  $B$  has depth  $2i + 1$  in  $A$ .

(ii) There is some  $m \in \mathbb{N}$  such that  $(\text{Res}_B^A \text{Ind}_B^A)^{i+1} \mid m(\text{Res}_B^A \text{Ind}_B^A)^i$  as functors from  $B\text{-Mod}$  to  $B\text{-Mod}$ .

(ii') There is some  $m \in \mathbb{N}$  such that  $(\text{Res}_B^A \text{Ind}_B^A)^{i+1} \mid m(\text{Res}_B^A \text{Ind}_B^A)^i$  as functors from  $\mathbf{Mod}\text{-}B$  to  $\mathbf{Mod}\text{-}B$ .

**Proof.** We verify the equivalence of (i) and (ii). The equivalence of (i) and (ii') is proved analogously. Suppose that  $B$  has depth  $2i + 1$  in  $A$  so that there is some  $m \in \mathbb{N}$  with  $T'_{i+1}(B, A) \mid mT'_i(B, A)$  as  $(B, B)$ -bimodules. Hence, there are  $(B, B)$ -bimodule homomorphisms

$$\varphi : T'_{i+1}(B, A) \longrightarrow mT'_i(B, A), \quad \text{and} \quad \psi : mT'_i(B, A) \longrightarrow T'_{i+1}(B, A)$$

with  $\psi \circ \varphi = \text{id}_{T'_{i+1}(B, A)}$ . Then  $\varphi$  and  $\psi$  induce, for any left  $B$ -module  $N$ , left  $B$ -module homomorphisms

$$\varphi_N : [(\text{Res}_B^A \text{Ind}_B^A)^{i+1}](N) \cong T'_{i+1}(B, A) \otimes_B N \longrightarrow (mT'_i(B, A)) \otimes_B N \cong [m(\text{Res}_B^A \text{Ind}_B^A)^i](N),$$

$$\psi_N : [m(\text{Res}_B^A \text{Ind}_B^A)^i](N) \cong (mT'_i(B, A)) \otimes_B N \longrightarrow T'_{i+1}(B, A) \otimes_B N \cong [(\text{Res}_B^A \text{Ind}_B^A)^{i+1}](N)$$

such that  $\psi_N \circ \varphi_N = \text{id}_{[(\text{Res}_B^A \text{Ind}_B^A)^{i+1}](N)}$ . In fact,  $\bar{\varphi} := (\varphi_N)$  and  $\bar{\psi} := (\psi_N)$  are natural transformations between functors from  $B\text{-Mod}$  to  $B\text{-Mod}$  with  $\bar{\psi} \circ \bar{\varphi} = \text{id}_{(\text{Res}_B^A \text{Ind}_B^A)^{i+1}}$ . Hence we get

$$(\text{Res}_B^A \text{Ind}_B^A)^{i+1} \mid m(\text{Res}_B^A \text{Ind}_B^A)^i$$

as functors from  $B\text{-Mod}$  to  $B\text{-Mod}$ , proving (ii).

Assuming (ii), we have natural transformations  $\bar{\varphi} := (\varphi_N)$  and  $\bar{\psi} := (\psi_N)$  between  $(\text{Res}_B^A \text{Ind}_B^A)^{i+1}$  and  $m(\text{Res}_B^A \text{Ind}_B^A)^i$ , as functors from  $B\text{-Mod}$  to  $B\text{-Mod}$ , such that  $\bar{\psi} \circ \bar{\varphi} = \text{id}_{(\text{Res}_B^A \text{Ind}_B^A)^{i+1}}$ . Then

$$\varphi_B : T'_{i+1}(B, A) \otimes_B B \longrightarrow (mT'_i(B, A)) \otimes_B B,$$

$$\psi_B : (mT'_i(B, A)) \otimes_B B \longrightarrow T'_{i+1}(B, A) \otimes_B B$$

are left  $B$ -module homomorphisms with  $\psi_B \circ \varphi_B = \text{id}_{T'_{i+1}(B, A)}$ . For  $b \in B$ , let  $\mu_b : B \rightarrow B$ ,  $b' \mapsto b'b$  be right multiplication with  $b$ . This is of course a morphism in  $B\text{-Mod}$ , and we thus get a commutative diagram

$$\begin{array}{ccc} T'_{i+1}(B, A) \otimes_B B & \xrightarrow{\varphi_B} & mT'_i(B, A) \otimes_B B \\ \downarrow \text{id} \otimes \mu_b & & \downarrow \text{id} \otimes \mu_b \\ T'_{i+1}(B, A) \otimes_B B & \xrightarrow{\varphi_B} & mT'_i(B, A) \otimes_B B \end{array}$$

That is, for any  $x \in T'_{i+1}(B, A)$  and any  $b, b' \in B$ , we have

$$\begin{aligned}\varphi_B((x \otimes b')b) &= \varphi_B(x \otimes b'b) = \varphi_B((\text{id} \otimes \mu_b)(x \otimes b')) = (\text{id} \otimes \mu_b)(\varphi_B(x \otimes b')) \\ &= \varphi_B(x \otimes b')b.\end{aligned}$$

Therefore,  $\varphi_B$  is in fact a  $(B, B)$ -bimodule homomorphism. Similarly, we deduce that also  $\psi_B$  is a  $(B, B)$ -bimodule homomorphism. Thus

$$T'_{i+1}(B, A) \cong T'_{i+1}(B, A) \otimes_B B \mid mT'_i(B, A) \otimes_B B \cong mT'_i(B, A)$$

as  $(B, B)$ -bimodules. This proves (i), and the proof of the proposition is complete.  $\square$

The next proposition gives a category-theoretic characterization of even depth. The proof is very similar to that of Proposition A.2 above, and we therefore omit it here.

**Proposition A.3.** *Let  $B \subseteq A$  be a ring extension, and let  $i \in \mathbb{N}$ . Then the following are equivalent:*

- (i)  $B$  has left (respectively right) depth  $2i$  in  $A$ .
- (ii) There is some  $m \in \mathbb{N}$  such that  $(\text{Res}_B^A \text{Ind}_B^A)^i \text{Res}_B^A \mid m(\text{Res}_B^A \text{Ind}_B^A)^{i-1} \text{Res}_B^A$  as functors from  $A\text{-Mod}$  to  $B\text{-Mod}$  (respectively, from  $\mathbf{Mod}\text{-}A$  to  $\mathbf{Mod}\text{-}B$ ).
- (ii') There is some  $m \in \mathbb{N}$  such that  $(\text{Ind}_B^A \text{Res}_B^A)^i \text{Ind}_B^A \mid m(\text{Ind}_B^A \text{Res}_B^A)^{i-1} \text{Ind}_B^A$  as functors from  $\mathbf{Mod}\text{-}B$  to  $\mathbf{Mod}\text{-}A$  (respectively, from  $B\text{-Mod}$  to  $A\text{-Mod}$ ).

**Remark A.4.** Now suppose that  $A_1$  and  $A_2$  are finite-dimensional semisimple algebras over an algebraically closed field  $K$ . Let  $D_1, \dots, D_r$  be representatives for the isomorphism classes of simple left  $A_1$ -modules. Denote the category of finitely generated left  $A_1$ -modules by  $A_1\text{-mod}$ , and suppose further that

$$\Phi : A_1\text{-mod} \longrightarrow A_2\text{-Mod} \quad \text{and} \quad \Psi : A_1\text{-mod} \longrightarrow A_2\text{-Mod}$$

are  $K$ -linear functors such that, for each  $i \in \{1, \dots, r\}$ , there are homomorphisms  $\varphi_i \in \text{Hom}_{A_2}(\Phi(D_i), \Psi(D_i))$  and  $\psi_i \in \text{Hom}_{A_2}(\Psi(D_i), \Phi(D_i))$  with  $\psi_i \circ \varphi_i = \text{id}_{\Phi(D_i)}$ .

(a) We define  $\mathcal{C}'$  to be the full subcategory of  $A_1\text{-mod}$  with objects  $D_1, \dots, D_r$ . Moreover, we denote by  $\Phi'$  and  $\Psi'$  the restrictions of  $\Phi$  and  $\Psi$ , respectively, to  $\mathcal{C}'$ . We claim that  $\varphi' := (\varphi_i)_{i=1, \dots, r}$  and  $\psi' := (\psi_i)_{i=1, \dots, r}$  are natural transformations between  $\Phi'$  and  $\Psi'$  such that  $\psi' \circ \varphi' = \text{id}_{\Phi'}$ . To show this, let  $i, j \in \{1, \dots, r\}$ , and let  $f \in \text{Hom}_{A_1}(D_i, D_j)$ . We need to verify that the following diagrams commute:

$$\begin{array}{ccccc} \Phi'(D_i) & \xrightarrow{\varphi_i} & \Psi'(D_i) & & \Psi'(D_i) & \xrightarrow{\psi_i} & \Phi'(D_i) \\ \downarrow \Phi'(f) & & \downarrow \Psi'(f) & , & \downarrow \Psi'(f) & & \downarrow \Phi'(f) \\ \Phi'(D_j) & \xrightarrow{\varphi_j} & \Psi'(D_j) & & \Psi'(D_j) & \xrightarrow{\psi_j} & \Phi'(D_j). \end{array}$$

If  $i \neq j$  then  $f = 0$ , by Schur's Lemma. Thus, in this case, we get  $\Phi'(f) = \Phi(f) = 0$  as well as  $\Psi'(f) = \Psi(f) = 0$ , and there is nothing to prove. In the case that  $i = j$ , by Schur's Lemma again, there is some  $\lambda \in K$  such that  $f = \lambda \text{id}_{D_i}$ . We therefore have

$$\begin{aligned}\Psi'(f) \circ \varphi_i &= \Psi(f) \circ \varphi_i = (\lambda \Psi(\text{id}_{D_i})) \circ \varphi_i = (\lambda \text{id}_{\Psi(D_i)}) \circ \varphi_i = \lambda \varphi_i = \varphi_i \circ (\lambda \text{id}_{\Phi(D_i)}) \\ &= \varphi_i \circ \Phi(f) = \varphi_i \circ \Phi'(f).\end{aligned}$$

This shows that  $\varphi'$  is a natural transformation. Similarly, we deduce that also  $\psi'$  is a natural transformation, and we obviously have  $\psi' \circ \varphi' = \text{id}_{\Phi'}$ .

(b) Next we define  $\mathcal{C}''$  to be the full subcategory of  $A_1\text{-mod}$  whose objects are precisely the left  $A_1$ -modules belonging to the following set:

$$\bigcup_{s \geq 0} \bigcup_{i_1, \dots, i_s \in \{1, \dots, r\}} \{D_{i_1} \oplus \dots \oplus D_{i_s}\}.$$

The restrictions of the functors  $\Phi$  and  $\Psi$ , respectively, to  $\mathcal{C}''$  will be denoted by  $\Phi''$  and  $\Psi''$ , respectively. Then the natural transformations  $\varphi'$  and  $\psi'$  constructed in part (a) can be extended to natural transformations  $\varphi'' : \Phi'' \rightarrow \Psi''$  and  $\psi'' : \Psi'' \rightarrow \Phi''$  such that  $\psi'' \circ \varphi'' = \text{id}_{\Phi''}$ . To see this, let  $M := D_{i_1} \oplus \dots \oplus D_{i_s}$  be an object in  $\mathcal{C}''$ . For  $k \in \{1, \dots, s\}$ , let  $\pi_k : M \rightarrow D_{i_k}$  and  $\iota_k : D_{i_k} \rightarrow M$  be the corresponding projection and injection, respectively. We define

$$\varphi_M := \varphi_{i_1, \dots, i_s} := \sum_{k=1}^s \Psi(\iota_k) \circ \varphi_{i_k} \circ \Phi(\pi_k) : \Phi(M) \rightarrow \Psi(M),$$

and we set  $\varphi'' := (\varphi_M)_{M \in \mathcal{C}''}$ . Similarly, we define  $\psi_M : \Psi(M) \rightarrow \Phi(M)$  and  $\psi'' := (\psi_M)_{M \in \mathcal{C}''}$ . The straightforward verification that  $\varphi''$  and  $\psi''$  are natural transformations and that  $\psi'' \circ \varphi'' = \text{id}_{\Phi''}$  is left to the reader.

(c) Lastly, we consider a fixed finitely generated left  $A_1$ -module  $Z$ . We want to show that there are left  $A_2$ -module homomorphisms  $\varphi_Z : \Phi(Z) \rightarrow \Psi(Z)$  and  $\psi_Z : \Psi(Z) \rightarrow \Phi(Z)$  such that  $\psi_Z \circ \varphi_Z = \text{id}_{\Phi(Z)}$  and such that, for every  $f \in \text{Hom}_{A_1}(Z, Z)$ , we have  $\Psi(f) \circ \varphi_Z = \varphi_Z \circ \Phi(f)$  and  $\Phi(f) \circ \psi_Z = \psi_Z \circ \Psi(f)$ . In the case that  $Z$  is an object in the category  $\mathcal{C}''$  defined above, this is true, as we have shown in (b).

So we may suppose that  $Z$  is not in  $\mathcal{C}''$ , and we define  $\mathcal{C}'''$  to be the full subcategory of  $A_1\text{-mod}$  such that any object in  $\mathcal{C}'''$  is either equal to  $Z$  or is already an object in  $\mathcal{C}''$ . We denote the restrictions to  $\mathcal{C}'''$  of  $\Phi$  and  $\Psi$ , respectively, by  $\Phi'''$  and  $\Psi'''$ , respectively. We show that we can extend the natural transformations  $\varphi''$  and  $\psi''$  to natural transformations  $\varphi''' : \Phi''' \rightarrow \Psi'''$  and  $\psi''' : \Psi''' \rightarrow \Phi'''$  such that  $\psi''' \circ \varphi''' = \text{id}_{\Phi'''}$ . For this, let  $M$  be an object in  $\mathcal{C}''$  such that  $Z$  is  $A_1$ -isomorphic to  $M$ , and we fix an  $A_1$ -isomorphism  $h : Z \rightarrow M$ . Moreover, we set

$$\varphi_Z := \Psi(h)^{-1} \circ \varphi_M \circ \Phi(h), \quad \text{and} \quad \psi_Z := \Phi(h)^{-1} \circ \psi_M \circ \Psi(h).$$

We leave the straightforward verification that  $\varphi''' := (\varphi_X)_{X \in \mathcal{C}'''}$  and  $\psi''' := (\psi_X)_{X \in \mathcal{C}'''}$  have the desired properties to the reader.

Specializing  $Z := A_1$ , we observe that  $\Phi(A_1)$  becomes an  $(A_2, A_1)$ -bimodule in the following way: for each  $a \in A_1$ , let  $\mu_a : A_1 \rightarrow A_1$ ,  $b \mapsto ba$  be right multiplication with  $a$ . This is a left  $A_1$ -module homomorphism, and for  $a, a' \in A_1$  we have  $\mu_a \circ \mu_{a'} = \mu_{a'a}$ . For  $x \in \Phi(A_1)$  and  $a \in A_1$ , we now define  $x \cdot a := (\Phi(\mu_a))(x)$ . It is easily checked that this defines a right  $A_1$ -module structure on  $\Phi(A_1)$  and turns  $\Phi(A_1)$  into an  $(A_2, A_1)$ -bimodule. Analogously,  $\Psi(A_1)$  carries an  $(A_2, A_1)$ -bimodule structure. Moreover, from this we now deduce that  $\varphi_{A_1}$  and  $\psi_{A_1}$  are in fact homomorphisms of  $(A_2, A_1)$ -bimodules with  $\psi_{A_1} \circ \varphi_{A_1} = \text{id}_{\Phi(A_1)}$ . To summarize, we have proved the following:

**Lemma A.5.** *Let  $A_1$  and  $A_2$  be finite-dimensional semisimple algebras over an algebraically closed field  $K$ . Let further*

$$\Phi : A_1\text{-mod} \longrightarrow A_2\text{-Mod} \quad \text{and} \quad \Psi : A_1\text{-mod} \longrightarrow A_2\text{-Mod}$$

*be  $K$ -linear functors such that, for each simple left  $A_1$ -module  $D$ , there are some  $\varphi_D \in \text{Hom}_{A_2}(\Phi(D), \Psi(D))$  and some  $\psi_D \in \text{Hom}_{A_2}(\Psi(D), \Phi(D))$  with  $\psi_D \circ \varphi_D = \text{id}_{\Phi(D)}$ . Then there are  $(A_2, A_1)$ -bimodule homomorphisms  $\varphi_{A_1} : \Phi(A_1) \longrightarrow \Psi(A_1)$  and  $\psi_{A_1} : \Psi(A_1) \longrightarrow \Phi(A_1)$  with  $\psi_{A_1} \circ \varphi_{A_1} = \text{id}_{\Phi(A_1)}$ . In particular,  $\Phi(A_1) \mid \Psi(A_1)$ , as  $(A_2, A_1)$ -bimodules.*

**Remark A.6.** Again let  $K$  be an algebraically closed field, and let  $B \subseteq A$  be a ring extension of finite-dimensional semisimple  $K$ -algebras. Let further  $\text{Irr}(A) = \{\chi_1, \dots, \chi_r\}$  and  $\text{Irr}(B) = \{\eta_1, \dots, \eta_s\}$  be the sets of irreducible characters of  $A$  and  $B$ , respectively. Then we define an  $(s \times r)$ -matrix  $\mathcal{M} = (m_{kl})$  with non-negative integer entries as follows: for  $k = 1, \dots, s$  and  $l = 1, \dots, r$ , we set

$$m_{kl} := \langle \eta_k, \text{Res}_B^A(\chi_l) \rangle = \langle \text{Ind}_B^A(\eta_k), \chi_l \rangle.$$

The equality follows from Frobenius Reciprocity. The matrix  $\mathcal{M}$  is called the *inclusion matrix* of the ring extension  $B \subseteq A$ . Then, for  $i \geq 1$ , we define, inductively,  $\mathcal{M}^{2i} := \mathcal{M}^{2i-1}\mathcal{M}^T$ , and  $\mathcal{M}^{2i+1} := \mathcal{M}^{2i}\mathcal{M}$ . In accordance with [4], we say that  $\mathcal{M}$  has depth  $d \geq 2$  if there is some positive integer  $q$  such that  $\mathcal{M}^{d+1} \leq q\mathcal{M}^{d-1}$ . It is shown in [4] that  $\mathcal{M}$  has always finite minimal depth.

For  $i \geq 1$ , we can express the entries in  $\mathcal{M}^i$  in terms of iterated restrictions and inductions of characters: for  $u, v = 1, \dots, s$ , the entry in position  $(u, v)$  in  $\mathcal{M}^2$  is

$$\sum_{t=1}^r \langle \text{Ind}_B^A(\eta_u), \chi_t \rangle \langle \text{Ind}_B^A(\eta_v), \chi_t \rangle = \langle \text{Ind}_B^A(\eta_u), \text{Ind}_B^A(\eta_v) \rangle = \langle \text{Res}_B^A(\text{Ind}_B^A(\eta_u)), \eta_v \rangle.$$

For  $k = 1, \dots, s$  and  $l = 1, \dots, r$ , the entry in position  $(k, l)$  in  $\mathcal{M}^3$  is

$$\begin{aligned} \sum_{t=1}^s (\mathcal{M}^2)_{kt} \mathcal{M}_{tl} &= \sum_{t=1}^s \langle \text{Res}_B^A \text{Ind}_B^A(\eta_k), \eta_t \rangle \langle \text{Ind}_B^A(\eta_t), \chi_l \rangle \\ &= \sum_{t=1}^s \langle \text{Res}_B^A \text{Ind}_B^A(\eta_k), \eta_t \rangle \langle \eta_t, \text{Res}_B^A(\chi_l) \rangle \\ &= \langle \text{Res}_B^A \text{Ind}_B^A(\eta_k), \text{Res}_B^A(\chi_l) \rangle \\ &= \langle \text{Ind}_B^A \text{Res}_B^A \text{Ind}_B^A(\eta_k), \chi_l \rangle = \langle \eta_k, \text{Res}_B^A \text{Ind}_B^A \text{Res}_B^A(\chi_l) \rangle. \end{aligned}$$

In general, we have

**Lemma A.7.** *Retaining the notation from Remark A.6, let  $i \geq 1$ . Then, for  $k, u, v \in \{1, \dots, s\}$  and  $l \in \{1, \dots, r\}$ , the  $(u, v)$ -entry in  $\mathcal{M}^{2i}$  equals*

$$\langle [(\text{Res}_B^A \text{Ind}_B^A)^i](\eta_u), \eta_v \rangle = \langle \eta_u, [(\text{Res}_B^A \text{Ind}_B^A)^i](\eta_v) \rangle,$$

*and the  $(k, l)$ -entry in  $\mathcal{M}^{2i+1}$  equals*

$$\langle [(\text{Ind}_B^A \text{Res}_B^A)^i \text{Ind}_B^A](\eta_k), \chi_l \rangle = \langle \eta_k, [(\text{Res}_B^A \text{Ind}_B^A)^i \text{Res}_B^A](\chi_l) \rangle.$$

**Proof.** We argue with induction on  $i$ . By Remark A.6, we know that the assertion is true for  $i = 1$ . Thus let now  $i \geq 2$ . Let further  $k, u, v \in \{1, \dots, s\}$  and  $l \in \{1, \dots, r\}$ . Then, by induction, we get

$$\begin{aligned}
(\mathcal{M}^{2i})_{uv} &= \sum_{t=1}^r (\mathcal{M}^{2i-1})_{ut} (\mathcal{M}^T)_{tv} = \sum_{t=1}^r \langle [(\text{Ind}_B^A \text{Res}_B^A)^{i-1} \text{Ind}_B^A](\eta_u), \chi_t \rangle \langle \chi_t, \text{Ind}_B^A(\eta_v) \rangle \\
&= \langle [(\text{Ind}_B^A \text{Res}_B^A)^{i-1} \text{Ind}_B^A](\eta_u), \text{Ind}_B^A(\eta_v) \rangle \\
&= \langle [\text{Res}_B^A (\text{Ind}_B^A \text{Res}_B^A)^{i-1} \text{Ind}_B^A](\eta_u), \eta_v \rangle \\
&= \langle [(\text{Res}_B^A \text{Ind}_B^A)^i](\eta_u), \eta_v \rangle = \langle \eta_u, [(\text{Res}_B^A \text{Ind}_B^A)^i](\eta_v) \rangle,
\end{aligned}$$

and from this we then deduce

$$\begin{aligned}
(\mathcal{M}^{2i+1})_{kl} &= \sum_{t=1}^s (\mathcal{M}^{2i})_{kt} \mathcal{M}_{tl} = \sum_{t=1}^s \langle [(\text{Res}_B^A \text{Ind}_B^A)^i](\eta_k), \eta_t \rangle \langle \text{Ind}_B^A(\eta_t), \chi_l \rangle \\
&= \langle [(\text{Res}_B^A \text{Ind}_B^A)^i](\eta_k), \text{Res}_B^A(\chi_l) \rangle \\
&= \langle [\text{Ind}_B^A (\text{Res}_B^A \text{Ind}_B^A)^i](\eta_k), \chi_l \rangle \\
&= \langle [(\text{Ind}_B^A \text{Res}_B^A)^i \text{Ind}_B^A](\eta_k), \chi_l \rangle = \langle \eta_k, [(\text{Res}_B^A \text{Ind}_B^A)^i \text{Res}_B^A](\chi_l) \rangle.
\end{aligned}$$

This proves the lemma.  $\square$

**Theorem A.8.** *Let  $B \subseteq A$  be a ring extension of finite-dimensional semisimple algebras over an algebraically closed field  $K$ . Moreover, let  $i \geq 1$ . Then the following are equivalent:*

- (i)  $B$  has depth  $2i + 1$  in  $A$ .
- (ii) There is some  $m \in \mathbb{N}$  such that, for every simple left  $B$ -module  $D$ , we have

$$[(\text{Res}_B^A \text{Ind}_B^A)^{i+1}](D) \mid m[(\text{Res}_B^A \text{Ind}_B^A)^i](D).$$

- (ii') There is some  $m \in \mathbb{N}$  such that, for every simple right  $B$ -module  $D$ , we have

$$[(\text{Res}_B^A \text{Ind}_B^A)^{i+1}](D) \mid m[(\text{Res}_B^A \text{Ind}_B^A)^i](D).$$

- (iii) The inclusion matrix of  $B \subseteq A$  has depth  $2i + 1$ .

**Proof.** Assertions (ii), (ii') and (iii) are equivalent, by Remark A.6 and Lemma A.7. Moreover, assertion (i) implies assertion (ii), by Proposition A.2. It remains to show that (ii) implies (i). For this, we apply Lemma A.5, with  $A_1 = A_2 = B$ ,  $\Phi = T'_{i+1}(B, A) \otimes_B -$ , and  $\Psi = mT'_i(B, A) \otimes_B -$ . Hence  $T'_{i+1}(B, A) \mid mT'_i(B, A)$ , and  $B$  has depth  $2i + 1$  in  $A$ , as claimed.  $\square$

The next theorem deals with the case of even depth. Its proof is analogous to that of Theorem A.8 just given above. We thus leave it to the reader.

**Theorem A.9.** *Let  $B \subseteq A$  be a ring extension of finite-dimensional semisimple algebras over an algebraically closed field  $K$ . Moreover, let  $i \geq 1$ . Then the following are equivalent:*

- (i)  $B$  has left depth  $2i$  in  $A$ .
- (i')  $B$  has right depth  $2i$  in  $A$ .

(ii) *There is some  $m \in \mathbb{N}$  such that, for every simple left  $A$ -module  $D$ , we have*

$$[(\text{Res}_B^A \text{Ind}_B^A)^i \text{Res}_B^A](D) \mid m[(\text{Res}_B^A \text{Ind}_B^A)^{i-1} \text{Res}_B^A](D).$$

(ii') *There is some  $m \in \mathbb{N}$  such that, for every simple right  $A$ -module  $D$ , we have*

$$[(\text{Res}_B^A \text{Ind}_B^A)^i \text{Res}_B^A](D) \mid m[(\text{Res}_B^A \text{Ind}_B^A)^{i-1} \text{Res}_B^A](D).$$

(iii) *The inclusion matrix of  $B \subseteq A$  has depth  $2i$ .*

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