NILPOTENT BLOCKS REVISITED

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The main purpose of this note is to give a self-contained proof of Puig's structure theorem for nilpotent blocks. Puig's original proof appeared in [8]. Textbook accounts were given by Külshammer in [6] and by Thévenaz in [9]. The present proof makes use of simplifications discovered by Külshammer, Okuyama and Watanabe [7]. The exposition is based upon lectures given by the author in Budapest in 2000.

1. Separable extensions

Let F be a field, and let A be an (associative unitary) F-algebra. An A-bimodule M will always be required to satisfy

$$\alpha m = m\alpha \quad \text{for} \quad \alpha \in F,\ m \in M.$$

Then the *centralizer*

$$M^A := \{m \in M : am = ma \ \text{ for } \ a \in A\}$$

is a vector space over F.

1.1 Definition. An *extension* of A is an F-algebra B, together with a unitary homomorphism of F-algebras $\beta : A \to B$.

Often A will be a subalgebra of B, and $\beta : A \to B$ will be the inclusion map. For an arbitrary extension of F-algebras $\beta : A \to B$, every (left) B-module N *restricts* to an A-module such that

$$an := \beta(a)n \quad \text{for} \quad a \in A,\ n \in N.$$

In a similar way, every B-bimodule M restricts to an A-bimodule; in this case we clearly have an inclusion of vector spaces $M^B \subseteq M^A$.

1.2 Lemma. *Let $\beta : A \to B$ be an extension of F-algebras, and let*

$$\mu : B \otimes_A B \longrightarrow B,\ x \otimes y \longmapsto xy,$$

be the morphism of B-bimodules induced by the multiplication in B. Then the following assertions are equivalent:
(1) μ splits as a morphism of B-bimodules;
(2) There exists an element $w \in (B \otimes_A B)^B$ such that $\mu(w) = 1_B$.

Proof. (1) $\Rightarrow$ (2). Let $\nu : B \to B \otimes_A B$ be a morphism of B-bimodules such that $\mu \circ \nu = \mathrm{id}_B$. Since $1_B \in Z(B) = B^B$, we have $w := \nu(1_B) \in (B \otimes_A B)^B$ and $\mu(w) = \mu(\nu(1_B)) = \mathrm{id}_B(1_B) = 1_B$.

(2) $\Rightarrow$ (1). Let $w \in (B \otimes_A B)^B$ such that $\mu(w) = 1_B$. Then the map

$$\nu : B \longrightarrow B \otimes_A B,\ x \longmapsto xw = wx,$$

is a morphism of B-bimodules, and $\mu(\nu(x)) = x\mu(\nu(1_B)) = x\mu(w) = x1_B = x$ for $x \in B$.

If the conditions above are satisfied then $\beta : A \to B$ is called a *separable* extension of F-algebras.

1.3 Proposition. (Higman) *Let $\beta : A \to B$ be a separable extension of F-algebras. Then the following holds:*
(i) A short exact sequence of B-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

splits if its restriction to A splits.
(ii) A B-module P is projective (injective) if its restriction to A is.

Proof. (i) Let $w \in (B \otimes_A B)^B$ such that $\mu(w) = 1_B$, and write $w = \sum_{j=1}^k x_j \otimes y_j$ with $x_1, y_1, \ldots, x_k, y_k \in B$. Moreover, let $h \in \mathrm{Hom}_A(N, M)$ such that $g \circ h = \mathrm{id}_N$. Now $\mathrm{Hom}_F(N, M)$ is a B-bimodule with

$$(x\phi y)(n) = x\phi(yn) \quad \text{for} \quad x, y \in B,\ \phi \in \mathrm{Hom}_F(N, M),\ n \in N.$$

Moreover,

$$H : B \otimes_A B \longrightarrow \mathrm{Hom}_F(N, M),\ x \otimes y \longmapsto xhy,$$

is a morphism of B-bimodules. Thus

$$h' := H(w) = \sum_{j=1}^k x_j h y_j \in \mathrm{Hom}_F(N, M)^B = \mathrm{Hom}_B(N, M),$$

and

$$g(h'(n)) = g(\sum_{j=1}^k x_j h(y_j n)) = \sum_{j=1}^k x_j g(h(y_j n)) = \sum_{j=1}^k x_j y_j n = \mu(w)n = 1_B n = n$$

for $n \in N$.

(ii) is an easy consequence of (i).

A *morphism* between two extensions $\beta : A \to B$ and $\gamma : A \to C$ of A is a homomorphism of F-algebras $\phi : B \to C$ which is also a morphism of A-bimodules.

1.4 Theorem. (Wedderburn-Malcev) *Let $\beta : A \to B$ be a separable extension of F-algebras, and let $\gamma : A \to C$ be an arbitrary extension of F-algebras. Moreover, let I be a nilpotent ideal of C, and let $\rho : B \to C/I$ be a unitary morphism of extensions. Suppose that ρ lifts to a morphism of A-bimodules $\tau_0 : B \to C$. Then ρ lifts to a unitary morphism of extensions $\tau : B \to C$.*

Proof. Let $\nu : C \to C/I$, $c \mapsto c + I$, denote the canonical map. Then $\nu \circ \gamma : A \to C \to C/I$ is also an extension of A, and we wish to show that there exists a morphism of extensions $\tau : B \to C$ such that $\nu \circ \tau = \rho$. This means that

$$\tau(x) + I = \rho(x) = \tau_0(x) + I \quad \text{for} \quad x \in B.$$

Let $w \in (B \otimes_A B)^B$ such that $\mu(w) = 1_B$, and write $w = \sum_{j=1}^k x_j \otimes y_j$ with $x_1, y_1, \ldots, x_k, y_k \in B$. Since I is nilpotent, it suffices to construct morphisms of A-bimodules $\tau_0, \tau_1, \tau_2, \ldots : B \to C$ such that
(i) $\tau_{n+1}(x) + I^{2^n} = \tau_n(x) + I^{2^n}$,
(ii) $\tau_n(xy) + I^{2^n} = \tau_n(x)\tau_n(y) + I^{2^n}$
for $x, y \in B$ and $n = 0, 1, 2, \ldots$. The existence of τ_0 follows from our hypothesis. Suppose that $\tau_0, \ldots, \tau_n$ have already been constructed successfully. Then

$$\tau_n(1_B)^2 + I^{2^n} = \tau_n(1_B) + I^{2^n}$$

by (ii), and

$$\tau_n(1_B) + I = \tau_{n-1}(1_B) + I = \ldots = \tau_0(1_B) + I = \rho(1_B) = 1_{C/I} = 1_C + I$$

by (i). This implies that $\tau_n(1_B) + I^{2^n} = 1_C + I^{2^n}$. We consider the following morphisms of A-bimodules:

$$\theta : B \otimes_A B \longrightarrow I^{2^n},\ x \otimes y \longmapsto \tau_n(xy) - \tau_n(x)\tau_n(y),$$

$$\lambda : B \otimes_A B \otimes_A B \longrightarrow I^{2^n},\ x \otimes y \otimes z \longmapsto \theta(x \otimes y)\tau_n(z),$$

$$\eta : B \longrightarrow I^{2^n},\ x \longmapsto \lambda(x \otimes w) = \sum_{j=1}^{k} \theta(x \otimes x_j)\tau_n(y_j).$$

Easy computations show that

$$\tau_n(x)\theta(y \otimes z) + \theta(x \otimes yz) = \theta(xy \otimes z) + \theta(x \otimes y)\tau_n(z)$$

and

$$\begin{aligned}
&\tau_n(x)\eta(y) - \eta(xy) + \eta(x)\tau_n(y) + I^{2^{n+1}} \\
&= \sum_{j=1}^{k}[\tau_n(x)\theta(y \otimes x_j)\tau_n(y_j) - \theta(xy \otimes x_j)\tau_n(y_j) + \theta(x \otimes x_j)\tau_n(y_j)\tau_n(y)] + I^{2^{n+1}} \\
&= \sum_{j=1}^{k}[\theta(x \otimes y)\tau_n(x_j)\tau_n(y_j) - \theta(x \otimes yx_j)\tau_n(y_j) + \theta(x \otimes x_j)\tau_n(y_j y)] + I^{2^{n+1}} \\
&= \theta(x \otimes y)\sum_{j=1}^{k}\tau_n(x_j y_j) - \lambda(x \otimes yw) + \lambda(x \otimes wy) + I^{2^{n+1}} \\
&= \theta(x \otimes y)\tau_n(1) + I^{2^{n+1}} = \theta(x \otimes y) + I^{2^{n+1}} = \tau_n(xy) - \tau_n(x)\tau_n(y) + I^{2^{n+1}}
\end{aligned}$$

for $x, y, z \in B$. Hence

$$\tau_{n+1} : B \longrightarrow C,\ x \longmapsto \tau_n(x) + \eta(x),$$

is a morphism of A-bimodules satisfying (i) and (ii).

We now turn to group algebras. Let H be a subgroup of a finite group G, and let FH and FG denote the corresponding group algebras. The following result is not needed for the proof of the main result of this paper, but it motivates some of the later arguments.

1.5 Proposition. (MASCHKE) *The inclusion map $FH \to FG$ is a separable extension of F-algebras if and only if the index $|G : H|$ is not divisible by the characteristic p of F.*

Proof. Suppose first that $FH \to FG$ is a separable extension. The map

$$g : F[G/H] \longrightarrow F,\ \sum_{xH \in G/H} \alpha_{xH} xH \longmapsto \sum_{xH \in G/H} \alpha_{xH},$$

is a morphism of FG-modules. It induces a short exact sequence of FG-modules

$$\mathcal{E} : 0 \longrightarrow \mathrm{Ker}(g) \longrightarrow F[G/H] \longrightarrow F \longrightarrow 0.$$

The morphism of FH-modules $h : F \to F[G/H]$, $\alpha \mapsto \alpha H$, shows that the restriction of $\mathcal{E}$ to FH splits. Hence, by Proposition 1.3, $\mathcal{E}$ splits as well. Hence there is a trivial FG-submodule U of $F[G/H]$ such that $F[G/H] = U \oplus \mathrm{Ker}(g)$. But the only trivial FG-submodule of $F[G/H]$ is $F\sum_{xH \in G/H} xH$. We conclude that

$$0 \neq g(\sum_{xH \in G/H} xH) = |G : H| 1_F.$$

Thus $|G : H|$ is not divisible by $p = \mathrm{char}(F)$.

Now suppose, conversely, that $|G:H|$ is not divisible by $p = \mathrm{char}(F)$, and let T be a transversal for G/H. Then

$$w := |G:H|^{-1} \sum_{t \in T} t \otimes t^{-1} \in FG \otimes_{FH} FG$$

is independent of T. This implies that

$$gw = |G:H|^{-1} \sum_{t \in T} gt \otimes t^{-1} g^{-1} g = wg \quad \text{for} \quad g \in G.$$

Hence $w \in (FG \otimes_{FH} FG)^{FG}$ and $\mu(w) = |G:H|^{-1} \sum_{t \in T} tt^{-1} = 1$.

For any subgroup H of G, we have a unitary subalgebra

$$(FG)^H = (FG)^{FH} = \{x \in FG : hx = xh \text{ for } h \in H\}$$

of FG, and for any subgroup I of H, we have a *relative trace map*

$$\mathrm{Tr}_I^H : (FG)^I \longrightarrow (FG)^H, \quad x \longmapsto \sum_{hI \in H/I} hxh^{-1}.$$

Standard properties imply that the image $(FG)_I^H$ of Tr_I^H is an ideal of $(FG)^H$.

Now let b be a primitive idempotent in $Z(FG)$. Then the F-algebra $B := FGb = bFG$ is called a *block* of FG. We denote by P a *defect group* of B. By definition, P is a subgroup of G, minimal with respect to the condition $b \in (FG)_P^G$. It is well-known that P is a p-subgroup of G where $p = \mathrm{char}(F)$. The following result is related to Proposition 1.5 and not really necessary for the remainder of the paper.

1.6 Proposition. *In the situation above, the map $FP \to B$, $x \mapsto xb$, is a separable extension of F-algebras.*

Proof. Let $x \in (FG)^P$ such that $\mathrm{Tr}_P^G(x) = b$, and let T be a transversal for G/P. Then $w := \sum_{t \in T} txb \otimes bt^{-1} \in B \otimes_{FP} B$ is independent of T. Then, for $g \in G$, we have

$$gw = \sum_{t \in T} gtxb \otimes bt^{-1} g^{-1} g = wg.$$

Hence $w \in (B \otimes_{FP} B)^B$, and $\mu(w) = \sum_{t \in T} txbt^{-1} = \mathrm{Tr}_P^G(xb) = \mathrm{Tr}_P^G(x)b = b$.

In this section, we closely followed the exposition in [7].

2. Brauer pairs

As the name indicates, Brauer pairs were first used by R. Brauer [2]. The concept was extended by Alperin and Broué [1]. A somewhat different but equivalent approach was given by Broué and Puig [3], and this is the approach which we will be using here.

In the following, let F be a field of characteristic $p > 0$, and let G be a finite group. A *Brauer pair* (P, e) of FG consists of a p-subgroup P of G and a primitive idempotent e in $Z(FC_G(P))$. We recall the *Brauer homomorphism*

$$\mathrm{Br}_P : (FG)^P \longrightarrow FC_G(P), \; \sum_{g \in G} \alpha_g g \longmapsto \sum_{g \in C_G(P)} \alpha_g g,$$

a homomorphism of F-algebras defined for every p-subgroup P of G. We also recall the following standard facts:

$$\mathrm{Ker}(\mathrm{Br}_P) = \sum_{Q < P} (FG)_Q^P \quad \text{and} \quad (FG)_1^P = \bigcap_{1 < Q \leq P} \mathrm{Ker}(\mathrm{Br}_Q).$$

2.1 Theorem. (BROUE-PUIG) *Let (P, e) be a Brauer pair of FG, and let Q be a subgroup of P. Then there exists a unique primitive idempotent f in $Z(FC_G(Q))$ such that $\mathrm{Br}_Q(i)f = \mathrm{Br}_Q(i)$ for every primitive idempotent i in $(FG)^P$ with $\mathrm{Br}_P(i)e \neq 0$.*

Proof. Suppose that f_1 and f_2 are primitive idempotents in $Z(FC_G(Q))$ satisfying the condition above. There certainly exists a primitive idempotent i in $(FG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$. Then

$$0 \neq \mathrm{Br}_Q(i) = \mathrm{Br}_Q(i)f_2 = \mathrm{Br}_Q(i)f_1 f_2,$$

so $f_1 f_2 \neq 0$ and $f_1 = f_2$. Thus the uniqueness is proved.

In order to show existence we argue by induction on $|P : Q|$. Supose first that $P = Q$ and that i is a primitive idempotent in $(FG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$. Since $\mathrm{Br}_P(i)$ is a primitive idempotent in $FC_G(P)$ this implies that $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i)$, so the condition is satisfied with $f := e$.

Now suppose that $Q < P$, so that $Q < N_P(Q) =: R$. By induction, there is a primitive idempotent g in $Z(FC_G(R))$ such that $\mathrm{Br}_R(i)g = \mathrm{Br}_R(i)$ for every primitive idempotent i in $(FG)^P$ with $\mathrm{Br}_P(i)e \neq 0$. The Brauer homomorphism Br_R maps $Z(FC_G(Q))^R$ into $Z(FC_G(R))$. Let f be a primitive idempotent in $Z(FC_G(Q))^R$ such that $\mathrm{Br}_R(f)g = g$, and let f' be a primitive idempotent in $Z(FC_G(Q))$ such that $ff' = f'$. Moreover, let T be the stabilizer of f' in R, under the conjugation action. Then $f = \mathrm{Tr}_T^R(f')$. But $f \notin \mathrm{Ker}(\mathrm{Br}_R) = \sum_{S<R}(FG)_S^R$, so $T = R$, and $f = f'$ is a primitive idempotent in $Z(FC_G(Q))$.

Now let i be a primitive idempotent in $(FG)^P$ with $\mathrm{Br}_P(i)e \neq 0$. We wish to show that $\mathrm{Br}_Q(i)(1-f) = 0$. As a first approximation, we show that

$$\mathrm{Br}_S(\mathrm{Br}_Q(i)(1-f)) = 0 \quad \text{whenever} \quad Q < S \leq R.$$

So let S be a subgroup of R with $Q < S$. Since $\mathrm{Br}_R(i) \neq 0$ there exists a primitive idempotent j in $(FG)^R$ such that $ij = j = ji$ and $\mathrm{Br}_R(j) \neq 0$. Then

$$\mathrm{Br}_R(j)g = \mathrm{Br}_R(ji)g = \mathrm{Br}_R(j)\mathrm{Br}_R(i)g = \mathrm{Br}_R(j)\mathrm{Br}_R(i) = \mathrm{Br}_R(ji) = \mathrm{Br}_R(j) \neq 0.$$

By induction, there is a primitive idempotent h in $Z(FC_G(S))$ such that $\mathrm{Br}_S(j)h = \mathrm{Br}_S(j) \neq 0$. Then $0 \neq \mathrm{Br}_S(ji)h = \mathrm{Br}_S(j)\mathrm{Br}_S(i)h$; in particular, we have $\mathrm{Br}_S(i)h \neq 0$.

By induction, there also exists a primitive idempotent h' in $Z(FC_G(S))$ such that $\mathrm{Br}_S(i)h' = \mathrm{Br}_S(i) \neq 0$. Then $0 \neq \mathrm{Br}_S(i)h'h$; in particular, we have $h'h \neq 0$. Thus $h' = h$ and $\mathrm{Br}_S(i)h = \mathrm{Br}_S(i)$.

Now $0 \neq \mathrm{Br}_R(j)g = \mathrm{Br}_R(j)\mathrm{Br}_R(f)g = \mathrm{Br}_R(\mathrm{Br}_Q(j)f)g$. But $\mathrm{Br}_Q(j)$ is a primitive idempotent in $\mathrm{Br}_Q((FG)^R) = FC_G(Q)^R$. Thus $\mathrm{Br}_Q(j)f = \mathrm{Br}_Q(j)$. Hence $\mathrm{Br}_S(j)\mathrm{Br}_S(f)h = \mathrm{Br}_S(j)h \neq 0$; in particular, we have $\mathrm{Br}_S(f)h \neq 0$. Since $\mathrm{Br}_S(f) \in Z(FC_G(S))$ this implies $\mathrm{Br}_S(f)h = h$. Hence

$$\mathrm{Br}_S(\mathrm{Br}_Q(i)(1-f)) = \mathrm{Br}_S(i)(1-\mathrm{Br}_S(f)) = \mathrm{Br}_S(i)h(1-\mathrm{Br}_S(f)) = 0,$$

and our first approximation is proved. We conclude that

$$\mathrm{Br}_Q(i)(1-f) \in \bigcap_{Q<S\leq R} \mathrm{Ker}(\mathrm{Br}_S : (FC_G(Q))^S \longrightarrow FC_G(S)) = (FC_G(Q))_Q^R = \mathrm{Br}_Q((FG)_Q^P).$$

(This last equality is a standard property of the Brauer homomorphism.) Hence $\mathrm{Br}_Q(i)(1-f) = (1-f)\mathrm{Br}_Q(i) \in \mathrm{Br}_Q(i(FG)_Q^P i)$. But $i(FG)_Q^P i$ is a proper ideal of the local F-algebra $i(FG)^P i$; thus it is nilpotent. Since $\mathrm{Br}_Q(i)(1-f)$ is an idempotent we obtain $\mathrm{Br}_Q(i)(1-f) = 0$.

In the situation of Theorem 2.1 we write $(Q, f) \leq (P, e)$. The following result is an easy consequence.

2.2 Corollary. *Let (P, e), (Q, f) and (R, g) be Brauer pairs of FG. Then the following holds:*

(i) $(Q, f) \leq (P, e)$ if and only if $Q \leq P$ and $\mathrm{Br}_P(i)e \neq 0 \neq \mathrm{Br}_Q(i)f$ for some primitive idempotent i in $(FG)^P$.

(ii) If $(R, g) \leq (Q, f)$ and $(Q, f) \leq (P, e)$ then $(R, g) \leq (P, e)$.

It follows that $\leq$ is a partial order on the set of Brauer pairs of FG. Moreover, G acts by conjugation on this set, and the action of G is compatible with the partial order $\leq$.

Now let b be a primitive idempotent in $Z(FG)$, and let $B = FGb = bFG$ be the corresponding block. A Brauer pair (P, e) of FG is called a *Brauer B-pair* if $(1, b) \leq (P, e)$. This means that $\mathrm{Br}_P(b)e \neq 0$.

Every Brauer pair (P, e) of FG is a Brauer B-pair for a unique block B of FG, and then every Brauer pair (Q, f) of FG with $(Q, f) \leq (P, e)$ is also a Brauer B-pair.

2.3 Theorem. (Sylow) *Let B be a block of FG with defect group P. Then there exists a Brauer B-pair (P, e). Moreover, if (Q, f) is an arbitrary Brauer B-pair then $(Q, f) \leq (P, e)^x$ for some $x \in G$.*

Proof. Let b be the primitive idempotent in $Z(FG)$ such that $B = FGb$. Then $\mathrm{Tr}_P^G(a) = b$ for some $a \in (FG)^P$. We write $b = \sum_{i \in I} i$ with pairwise orthogonal primitive idempotents i in $(FG)^P$. Then

$$b = b\mathrm{Tr}_P^G(a) = \mathrm{Tr}_P^G(ba) = \sum_{i \in I} \mathrm{Tr}_P^G(ia) \in \sum_{i \in I} \mathrm{Tr}_P^G((FG)^P i (FG)^P).$$

Each summand $\mathrm{Tr}_P^G((FG)^P i (FG)^P)$ is an ideal in $Z(FG)$. Rosenberg's Lemma implies that b is contained in $\mathrm{Tr}_P^G((FG)^P i (FG)^P)$ for some $i \in I$. If $i \in (FG)_Q^P$ for some proper subgroup Q of P then $b \in (FG)_Q^G$ contradicting the choice of P. Thus $i \notin (FG)_Q^P$ for every proper subgroup Q of P. Rosenberg's Lemma implies that $i \notin \sum_{Q<P} (FG)_Q^P = \mathrm{Ker}(\mathrm{Br}_P)$. Let e be a primitive idempotent in $Z(FC_G(P))$ such that $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i) \neq 0$. Then (P, e) is a Brauer B-pair.

Now let (Q, f) be an arbitrary Brauer B-pair. Then $\mathrm{Br}_Q(b)f \neq 0$. Mackey's formula implies that

$$b \in \mathrm{Tr}_P^G((FG)^P i (FG)^P) \subseteq \sum_{x \in G} \mathrm{Tr}_{Q \cap P^x}^Q((FG)^{Q \cap P^x} i^x (FG)^{Q \cap P^x}).$$

Thus there is $x \in G$ such that $Q \subseteq P^x$ and $\mathrm{Br}_Q((FG)^Q i^x (FG)^Q f) \neq 0$. Hence $0 \neq \mathrm{Br}_Q(i^x)f$, and $(Q, f) \leq (P, e)^x$.

3. Source algebras

Let F be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. Moreover, let b be a primitive idempotent in $Z(FG)$, and let P be a defect group of the corresponding block $B = FGb$. By Theorem 2.3 and its proof, there exist a primitive idempotent i in $(FG)^P$ and a Brauer B-pair (P, e) such that

$$b \in \mathrm{Tr}_P^G((FG)^P i (FG)^P) \subseteq FGiFG \quad \text{and} \quad \mathrm{Br}_P(i)e \neq 0.$$

It follows that $i = ib \in B$ and $B = FGbFG = FGiFG = BiB$. This implies that the F-algebras B and iBi are Morita equivalent. We call i a *source idempotent* and $iBi = iFGi$ a *source algebra* of B. It is not difficult to show that i is essentially unique, and that every primitive idempotent i' in $(FG)^P b$ with $\mathrm{Br}_P(i') \neq 0$ is a source idempotent of B. However, these facts are not needed in the following.

3.1 Proposition. *In the situation above, the map*

$$FP \longrightarrow iBi, \; x \longmapsto xi = ix,$$

is a separable extension of F-algebras.

Proof. We write $b = \sum_{j=1}^k \mathrm{Tr}_P^G(x_j i y_j)$ with $x_1, y_1, \ldots, x_k, y_k \in (FG)^P$. Each summand $\mathrm{Tr}_P^G(x_j i y_j)$ is contained in the local F-algebra $Z(B)$. Since $b \notin J(Z(B))$ we must have $c := \mathrm{Tr}_P^G(x_j i y_j) \notin J(Z(B))$ for some $j \in \{1, \ldots, k\}$. Thus c has an inverse $c^{-1} \in Z(B)$. Then

$$b = cc^{-1} = \mathrm{Tr}_P^G(x_j i y_j)c^{-1} = \mathrm{Tr}_P^G(x_j i y_j c^{-1}).$$

This shows that $b = \mathrm{Tr}_P^G(xiy)$ for suitable $x, y \in (FG)^P$. We choose a transversal T for G/P and set $v := \sum_{t\in T} txi \otimes iyt^{-1} \in FG \otimes_{FP} FG$. Then v is independent of T. Thus, for $g \in G$, we have

$$gv = \sum_{t\in T} gtxi \otimes iyt^{-1}g^{-1}g = vg$$

for $g \in G$. This means that $v \in (FG \otimes_{FP} FG)^{FG}$ and

$$\mu(v) = \sum_{t\in T} txiyt^{-1} = \mathrm{Tr}_P^G(xiy) = b.$$

We consider $iFGi \otimes_{FP} iFGi$ as a subset of $FG \otimes_{FP} FG$ and set $w := ivi \in (iFGi \otimes_{FP} iFGi)^{iFGi}$. Then $\mu(w) = i\mu(v)i = ibi = i$, and the result follows.

In the following, we consider FG as an FP-bimodule. The indecomposable direct summands of FG have the form $F[PgP]$ where PgP is a double coset. Now we note that $iFGi$ is a direct summand of the FP-bimodule FG. We write $iFGi = M_1 \oplus \cdots \oplus M_r$ with indecomposable FP-bimodules $M_1, \ldots, M_r$. For $j = 1, \ldots, r$, we choose an element $g_j \in G$ and an isomorphism of FP-bimodules $\phi_j : F[Pg_jP] \to M_j$. Then $x_j := \phi_j(g_j) \in \phi_j(Pg_jP) =: X_j$ and $P_j := P \cap g_jPg_j^{-1} \leq P$. Let e_j be the unique primitive idempotent in $Z(FC_G(P_j))$ such that $(P_j, e_j) \leq (P, e)$.

3.2 Proposition. *In the situation above, we have $(P_j, e_j)^{g_j} \leq (P, e)$ for $j = 1, \ldots, r$.*

Proof. For $u \in P_j$, we have

$$ux_jg_j^{-1}u^{-1} = u\phi_j(g_j)g_j^{-1}u^{-1}g_jg_j^{-1} = \phi_j(u \cdot g_j \cdot g_j^{-1}u^{-1}g_j)g_j^{-1} = x_jg_j^{-1}.$$

Thus $x_jg_j^{-1} \in (FG)^{P_j}$. Now $X := X_1 \cup \cdots \cup X_r$ is an F-basis of $iFGi$ which is invariant under left and right multiplication with elements in P. We can extend X to an F-basis Y of FG with the same property. Then Yg_j^{-1} is an F-basis of FG which is invariant under left and right multiplication with elements in P_j. Since $x_jg_j^{-1} \in Yg_j^{-1} \cap (FG)^{P_j}$ we have $\mathrm{Br}_{P_j}(x_jg_j^{-1}) \neq 0$. Since $x_jg_j^{-1} = i \cdot x_jg_j^{-1} \cdot g_jig_j^{-1}$ there are primitive idempotents i_j in $i(FG)^{P_j}i$ and i'_j in $i(FG)^{g_j^{-1}P_jg_j}i$ such that

$$0 \neq \mathrm{Br}_{P_j}(i_j \cdot x_jg_j^{-1} \cdot g_ji'_jg_j^{-1}) = \mathrm{Br}_{P_j}(i_j)\mathrm{Br}_{P_j}(x_jg_j^{-1})\mathrm{Br}_{P_j}(g_ji'_jg_j^{-1}).$$

Thus the primitive idempotents $\mathrm{Br}_{P_j}(i_j)$ and $\mathrm{Br}_{P_j}(g_ji'_jg_j^{-1})$ are contained in the same block of $FC_G(P_j)$. Now $(P_j, e_j) \leq (P, e)$ implies that

$$\mathrm{Br}_{P_j}(i_j)e_j = \mathrm{Br}_{P_j}(i_ji)e_j = \mathrm{Br}_{P_j}(i_j)\mathrm{Br}_{P_j}(i)e_j = \mathrm{Br}_{P_j}(i_j)\mathrm{Br}_{P_j}(i) = \mathrm{Br}_{P_j}(i_ji) = \mathrm{Br}_{P_j}(i_j).$$

Hence $\mathrm{Br}_{P_j}(i_j)$ and $\mathrm{Br}_{P_j}(g_ji'_jg_j^{-1})$ are contained in $FC_G(P_j)e_j$. Hence

$$0 \neq \mathrm{Br}_{P_j}(g_ji'_jg_j^{-1})e_j = \mathrm{Br}_{P_j}(g_ji'_jg_j^{-1})\mathrm{Br}_{P_j}(g_jig_j^{-1})e_j;$$

in particular, we have $\mathrm{Br}_{P_j}(g_jig_j^{-1})e_j \neq 0$. Since also $\mathrm{Br}_{g_jPg_j^{-1}}(g_jig_j^{-1})g_jeg_j^{-1} \neq 0$ we get $(P_j, e_j) \leq g_j(P, e)g_j^{-1}$.

The block B of FG is called *nilpotent* if the following condition is satisfied:

(*) Whenever (Q, f) is a Brauer B-pair such that $(Q, f) \leq (P, e)$ and whenever $g \in G$ is such that $(Q, f)^g \leq (P, e)$ then there are elements $c \in C_G(Q)$ and $u \in P$ such that $g = cu$.

From now on we suppose that our block B of FG is nilpotent. Then, by Proposition 3.2, we may assume that $g_j \in C_G(P_j)$ for $j = 1, \ldots, r$. The map

$$FP \longrightarrow FP \otimes_F iFGi,\ u \longmapsto u \otimes ui\ \ (u \in P),$$

is an extension of F-algebras. Moreover, the map

$$\sigma : FP \otimes_F iFGi \longrightarrow iFGi,\ u \otimes x \longmapsto x \quad (u \in P,\ x \in iFGi),$$

is a morphism of extensions of FP. Moreover,

$$I := \mathrm{Ker}(\sigma) = J(FP) \otimes_F iFGi$$

is a nilpotent ideal of $FP \otimes_F iFGi$.

3.3 Theorem. (Puig) *There exists an extension of F-algebras*

$$\alpha : FP \longrightarrow \mathrm{Mat}(n, F)$$

(i.e. a representation of FP) and an isomorphism of extensions of FP,

$$iFGi \longrightarrow FP \otimes_F \mathrm{Mat}(n, F).$$

Proof. The morphism of extensions $\sigma : FP \otimes_F iFGi \to iFGi$ induces an isomorphism of extensions $\bar{\sigma} : (FP \otimes_F iFGi)/I \to iFGi$. Let $\rho := \bar{\sigma}^{-1}$. We can consider $iFGi$ and $FP \otimes_F iFGi$ as permutation modules over $F[P \times P]$. We claim that, for $j = 1, \ldots, r$, the stabilizers of $x_j \in iFGi$ and of $1 \otimes x_j \in FP \otimes_F iFGi$ coincide. Indeed, let $u, v \in P$ such that $ux_jv = x_j$. Then $ug_jv = g_j$, so $u = g_jv^{-1}g_j^{-1} \in P \cap g_jPg_j^{-1} = P_j$. Since $g_j \in C_G(P_j)$ this implies $u = g_j^{-1}ug_j = v^{-1}$ and $u(1 \otimes x_j)v = uv \otimes ux_jv = 1 \otimes x_j$, and our claim is proved.

It follows that there is a morphism of FP-bimodules

$$\tau_0 : iFGi \longrightarrow FP \otimes_F iFGi$$

such that $\tau_0(x_j) = 1 \otimes x_j$ for $j = 1, \ldots, r$. Then $\bar{\sigma}(\tau_0(x_j) + I) = \sigma(\tau_0(x_j)) = x_j$, i.e. $\tau_0(x_j) + I = \rho(x_j)$ for $j = 1, \ldots, r$. Thus τ_0 lifts ρ.

Since $FP \to iFGi$ is a separable extension, Theorem 1.4 implies that there is a morphism of extensions $\tau : iFGi \to FP \otimes_F iFGi$ lifting ρ. Hence $\sigma(\tau(x)) = \bar{\sigma}(\tau(x) + I) = \bar{\sigma}(\rho(x)) = x$ for $x \in iFGi$; in particular, we have $FP \otimes_F iFGi = \tau(iFGi) \oplus I$.

Let M be a simple $iFGi$-module. Then $FP \otimes_F M$ is an $FP \otimes_F iFGi$-module. We can restrict $FP \otimes_F M$ to $iFGi$ (via τ) and then further to FP (via $u \mapsto ui$). Then FP acts on $FP \otimes_F M$ in such a way that $u(v \otimes m) = uv \otimes ium$ for $u, v \in P$, $m \in M$. Thus $FP \otimes_F M$ is a projective FP-module. Since $FP \to iFGi$ is a separable extension, Proposition 1.3 implies that $FP \otimes_F M$ is a projective $iFGi$-module as well.

On the other hand, $FP \otimes_F M$ has a composition series of length $|P|$, with all composition factors isomorphic (both over $FP \otimes_F iFGi$ and over $iFGi$). This implies that M is the only simple $iFGi$-module, up to isomorphism, and $FP \otimes_F M$ is its projective cover. In particular, $FP \otimes_F M$ is a faithful $iFGi$-module.

But $FP \otimes_F J(iFGi)$ annihilates $FP \otimes_F M$. Thus the map

$$\phi : iFGi \xrightarrow{\tau} FP \otimes_F iFGi \longrightarrow FP \otimes_F (iFGi/J(iFGi))$$

is injective. Comparing dimensions we see that ϕ is even an isomorphism, an isomorphism of extensions of FP. The result follows since $iFGi/J(iFGi) \cong \mathrm{Mat}(n, F)$ for some n.

3.4 Corollary. *Let B be a nilpotent block of FG with defect group P. Then B is isomorphic (as an F-algebra) to* $\mathrm{Mat}(m, FP)$ *for some m.*

Proof. We know that B is Morita equivalent to its source algebra iBi, and by Theorem 3.3 iBi is Morita equivalent to FP. Since FP is a local F-algebra, the result follows.

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