The Depth of Subgroups of PSL$(2, q)$ II

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Abstract

This paper continues [4], where the ordinary depths of the subgroups of $G := \mathrm{PSL}(2, q)$ with q a prime power are determined. Now, we will pay attention to the minimal combinatorial depths of the subgroups of G. It turns out that most of the subgroups have a minimal combinatorial depth ≤ 7, and often it coincides with the ordinary depth of this subgroup in G. However, if $H \leq G$ is of order p^{f-1}, where $q = p^f$ with p a prime and $f \geq 3$, then H has minimal combinatorial depth $2f - 2$ in G, and the normalizer of H in G has minimal combinatorial depth $2f - 1$ in G.

1 Introduction

Let G be a finite group and H be a subgroup of G. The ordinary depth of H in G was defined in [2] based on earlier notions of the depth in connection with von-Neumann algebras [5] and the development of its meaning to Frobenius extensions [7]. Since the ordinary depth of H in G is, in fact, the depth of the inclusion of group algebras $\mathbb{C}H \subseteq \mathbb{C}G$, this led to the definition of the depth of group algebra extensions $RH \subseteq RG$, where R is a non-zero commutative ring [1]. In this context, the combinatorial depth of H in G was introduced.

We will now give a short overview of the definition of the combinatorial depth as well as some known properties, taking the results of [1]. To define the combinatorial depth of H in G, some facts about bisets are necessary. Let J, K and L be finite groups, X be a (J, K)-biset and Y be a (K, L)-biset. The Cartesian product $X \times Y$ is equipped with a natural (J, L)-biset structure. Furthermore, K acts on $X \times Y$ by $k \cdot (x, y) := (xk^{-1}, ky)$ for $x \in X$, $y \in Y$ and $k \in K$. The set of K-orbits arising from this action is denoted by $X \times_K Y$. Again, we have a natural (J, L)-biset structure, which is induced from the one on $X \times Y$.

Let $\Theta_1(H, G)$ be the (G, G)-biset G, where G acts on itself via left and right multiplication, and $\Theta_{i+1}(H, G) := \Theta_i(H, G) \times_H G$ for $i \geq 1$. Since H is a subgroup of G, $\Theta_i(H, G)$ can also be seen as (H, H)-, (H, G)- or (G, H)-biset. The respective restrictions are denoted by $\Theta'_i(H, G)$, $\Theta^l_i(H, G)$ and $\Theta^r_i(H, G)$. In addition, $\Theta'_0(H, G) := H$ as (H, H)-biset. Then, H is said to have combinatorial depth $2i$ in G, if $\Theta^r_{i+1}(H, G)$ resp. $\Theta^l_{i+1}(H, G)$ is isomorphic to a direct summand of $a \cdot \Theta^r_i(H, G)$ resp. $a \cdot \Theta^l_i(H, G)$ for an $i \geq 1$ and some $a \in \mathbb{N}$. Moreover, H is said to have combinatorial depth $2i + 1$, if $\Theta'_{i+1}(H, G)$ is a direct summand of $a \cdot \Theta'_i(H, G)$ for an $i \geq 0$ and some $a \in \mathbb{N}$.

It is not difficult to show that if H has combinatorial depth i in G, then it also has combinatorial depth $i + 1$ in G. Thus, we ask for the smallest positive integer $d_c(H, G)$, called the minimal combinatorial depth of H in G, such that H has combinatorial depth $d_c(H, G)$ in G. Let $H^x := x^{-1}Hx$ denote the conjugation of H with an element $x \in G$ and define $H_{x_1,\ldots,x_i} := H \cap H^{x_1} \cap \ldots \cap H^{x_i}$ for $x_1, \ldots, x_i \in G$. The determination of $d_c(H, G)$ can be done with the help of the sets

$$\mathcal{U}_i := \mathcal{U}_i(H, G) := \{H_{x_1,\ldots,x_i} : x_1, \ldots, x_i \in G\} \quad \text{and} \quad \mathcal{U}_\infty := \mathcal{U}_\infty(H, G) := \bigcup_{i \geq 0} \mathcal{U}_i,$$

where $i \geq 0$ and $\mathcal{U}_0 := \{H\}$, as described in the following theorem.

Theorem 1.1. *Let $i \geq 1$. Then:*

(i) $d_c(H,G) \leq 2i \Leftrightarrow \mathcal{U}_{i-1} = \mathcal{U}_i \Leftrightarrow \mathcal{U}_{i-1} = \mathcal{U}_\infty \Leftrightarrow$ *For any $x_1, \ldots, x_i \in G$, there exist $y_1, \ldots, y_{i-1} \in G$, such that $H_{x_1,\ldots,x_i} = H_{y_1,\ldots,y_{i-1}}$.*

(ii) $d_c(H,G) \leq 2i-1 \Leftrightarrow$ *For any $x_1, \ldots, x_i \in G$, there exist $y_1, \ldots, y_{i-1} \in G$, such that $H_{x_1,\ldots,x_i} = H_{y_1,\ldots,y_{i-1}}$ and $x_1 h x_1^{-1} = y_1 h y_1^{-1}$ for all $h \in H_{x_1,\ldots,x_i}$.*

(iii) $d_c(H,G) = 1 \Leftrightarrow$ *For any $x \in G$, there exists a $y \in H$, such that $xhx^{-1} = yhy^{-1}$ for all $h \in H$.*

From Theorem 1.1, we deduce immediately $d_c(H,G) \leq 3$ if H is a TI-subgroup in G, $d_c(H,G) \leq 2$ if and only if $H \trianglelefteq G$, and $d_c(H,G) = 1$ if and only if $G = HC_G(H)$, where $C_G(H)$ denotes the centralizer of H in G. Furthermore, it is clear that $d_c(H,G) = d_c(K,G)$ if H and K are conjugate in G. Moreover, if we already know $2i-1 \leq d_c(H,G) \leq 2i$, then the additional condition of Theorem 1.1 (ii) yields that it is enough to look at the subgroups in $\mathcal{U}_{i-1} \setminus \mathcal{U}_{i-2}$ to determine $d_c(H,G)$.
Finally, we note that there is a correlation between the ordinary depth $d_0(H,G)$ and the minimal combinatorial depth: We always have $d_0(H,G) \leq d_c(H,G)$. This fact is extracted from a much more general result shown in [1].
In this article, we look for the minimal combinatorial depths of the subgroups of the finite projective special linear groups $\mathrm{PSL}(2,q)$ of degree 2. As in [4], we need some properties of these groups, which we therefore repeat here. All these facts can be found in [6].
Let $q = p^f$, with p a prime and some positive integer f, and E be the identity of $\mathrm{PSL}(2,q)$. The group $\mathrm{PSL}(2,q)$ acts 2-transitively on the projective line $\mathbb{P}^1(\mathbb{F}_q)$. This action is an essential part of the proof of the following results.

Theorem 1.2. *(i) Let $\mathcal{P}$ be a Sylow p-subgroup of $\mathrm{PSL}(2,q)$. Then, all the elements of $\mathcal{P}$ have a common fixed point and any element of $\mathcal{P} \setminus \{E\}$ has only this fixed point. The Sylow p-subgroups of $\mathrm{PSL}(2,q)$ are TI-subgroups.*

(ii) Let $\mathcal{U} < \mathrm{PSL}(2,q)$ be cyclic of order $(q-1)/k$, $k = \gcd(q-1,2)$. Then, $\mathcal{U}$ has two fixed points and any nontrivial element of $\mathcal{U}$ has no further ones. Moreover, there is no element in $\mathrm{PSL}(2,q) \setminus \mathcal{U}$ that fixes both of these points, so $\mathcal{U}$ is a TI-subgroup. The normalizer of any nontrivial subgroup of $\mathcal{U}$ is a dihedral group of order $2(q-1)/k$.

(iii) Let $\mathcal{S} < \mathrm{PSL}(2,q)$ be cyclic of order $(q+1)/k$. Then, none of the elements of $\mathcal{S} \setminus \{E\}$ has a fixed point. The normalizer of any nontrivial subgroup of $\mathcal{S}$ is a dihedral subgroup of order $2(q+1)/k$. Furthermore, $\mathcal{S}$ is a TI-subgroup.

(iv) Let $\mathcal{V} \cong \mathfrak{C}_2 \times \mathfrak{C}_2$ be a subgroup of $\mathrm{PSL}(2,q)$ with q odd. If 16 does not divide q^2-1, then $N(\mathcal{V}) \cong \mathfrak{A}_4$ and all subgroups of $\mathrm{PSL}(2,q)$, which are isomorphic to $\mathfrak{A}_4$, are conjugate. However, if 16 divides q^2-1, then $N(\mathcal{V}) \cong \mathfrak{S}_4$. In this case, $\mathrm{PSL}(2,q)$ contains two conjugacy classes of subgroups isomorphic to $\mathfrak{S}_4$.

The subgroups of $\mathrm{PSL}(2,q)$ are known by a theorem of Dickson. A complete list of all subgroups of $\mathrm{PSL}(2,q)$ looks as follows:

1. Elementary-abelian p-groups.
2. Cyclic groups $\mathfrak{C}_z$ of order z, where z divides $(q \pm 1)/k$ with $k = \gcd(q-1,2)$.
3. Dihedral subgroups $\mathfrak{D}_z$ of order $2z$ with z as in 2.
4. Alternating groups $\mathfrak{A}_4$ if $p > 2$ or $p = 2$ and f even.
5. Symmetric groups $\mathfrak{S}_4$ if $q^2 - 1 \equiv 0 \pmod{16}$.
6. Alternating groups $\mathfrak{A}_5$ if $p = 5$ or $q^2 - 1 \equiv 0 \pmod{5}$.

7. Semidirect products $\mathfrak{C}_p^m \rtimes \mathfrak{C}_t$ of elementary-abelian groups of order p^m with cyclic groups of order t, where t divides $p^m - 1$ as well as $(q-1)/k$.

8. Groups $\mathrm{PSL}(2, p^m)$ if $m \mid f$ and $\mathrm{PGL}(2, p^m)$ if $2m \mid f$.

In section 2 we go through this list step by step, skipping the elementary-abelian p-groups as well as the semidirect products with normal p-subgroups. These subgroups are discussed in the last part. On the one hand, they yield the most interesting results, but on the other hand, for some of these subgroups, we can only give bounds of their depths in $\mathrm{PSL}(2, q)$.

2 Calculation of the depths

We introduce some notations used in this paper. As in [4], we write $N_G(X)$ resp. $C_G(X)$ for the normalizer resp. centralizer of a subset X in a group G, and if $G = \mathrm{PSL}(2, q)$, we just write $N(X)$ resp. $C(X)$. Moreover, the conjugation of X by $g \in G$ is denoted by $X^g := g^{-1}Xg$ and we set $X_{g_1,\dots,g_n} := X \cap X^{g_1} \cap \ldots \cap X^{g_n}$ for $g_1, \dots, g_n \in G$. If X is a subgroup of G, we use the sets $\mathcal{U}_i(X, G)$ $(i \in \{0, 1, \dots\} \cup \{\infty\})$ as defined in section 1, just writing $\mathcal{U}_i$ when the meaning of X and G is clear. Finally, E always denotes the identity of $\mathrm{PSL}(2, q)$.
Our interest lies in the minimal combinatorial depths of the subgroups of $\mathrm{PSL}(2, q)$. In the following, we just write "depth" instead of "minimal combinatorial depth".

2.1 Cyclic subgroups

Apart from $\mathfrak{C}_3 \lhd \mathrm{PSL}(2, 2)$, the cyclic subgroups, which do not contain elements of order p, are nonnormal TI-subgroups of $\mathrm{PSL}(2, q)$. Thus, their depth in $\mathrm{PSL}(2, q)$ is 3. Since $\mathfrak{C}_3$ is self-centralizing in $\mathrm{PSL}(2, 2)$, it has depth 2.

2.2 Dihedral subgroups

Proposition 2.1. *Let $q > 2$ and $\mathcal{D} < \mathrm{PSL}(2, q)$ be a dihedral subgroup of order not divisible by 4.*

(i) If $|\mathcal{D}| < 2(q + 1)$, then $\mathcal{D}$ has depth 3 in $\mathrm{PSL}(2, q)$.

(ii) If $|\mathcal{D}| = 2(2^f + 1)$, then $\mathcal{D}$ has depth 5 in $\mathrm{PSL}(2, 2^f)$.

Proof. The cyclic normal subgroup $\mathcal{C}$ of index 2 in $\mathcal{D}$ is a TI-subgroup of $\mathrm{PSL}(2, q)$. In consequence, $\mathcal{D}_G = \mathcal{D}$, $\mathcal{D}_G = \mathcal{C}$, $\mathcal{D}_G = \{E\}$ or $|\mathcal{D}_G| = 2$ for $G \in \mathrm{PSL}(2, q)$, whence $\mathcal{U}_1 = \mathcal{U}_\infty$ or $\mathcal{U}_1 = \mathcal{U}_\infty \setminus \{E\}$. Recall that $N(\mathcal{D}) = \mathcal{D}$, so $\mathcal{C} \not\subset \mathcal{U}_1$ if and only if $\mathcal{D} = N(\mathcal{C})$, i.e. $\mathcal{D}$ is a maximal dihedral subgroup of $\mathrm{PSL}(2, q)$. Otherwise, if $\mathcal{D}_{G_1,G_2} = \mathcal{C}$, we can find $G_3 \in \mathrm{PSL}(2, q)$, such that $\mathcal{D}_{G_3} = \mathcal{C}$ and $X^{G_1^{-1}} = X^{G_3^{-1}}$ for a generator X of $\mathcal{C}$: Take an element of order 2 of $N(\mathcal{C}) \setminus \mathcal{D}$ if $X^{G_1^{-1}} = X^{-1}$, or choose an element of order $\neq 2$ of $N(\mathcal{C}) \setminus \mathcal{D}$ if $X^{G_1^{-1}} = X$.
The involutions of $\mathcal{D}$ form a single conjugacy class. By Theorem 1.1 that means that the depth of $\mathcal{D}$ in $\mathrm{PSL}(2, q)$ is 3, if we find some $G \in \mathrm{PSL}(2, q)$, such that $\mathcal{D}_G = \{E\}$. In [4] it is shown that such elements can be found for each dihedral group except these of order $2(q + 1)$ for $q = 2^f$. The latter therefore have depth 5 in $\mathrm{PSL}(2, 2^f)$. □

We observe that for each dihedral subgroup $\mathcal{D} < \mathrm{PSL}(2, q)$, whose order is a multiple of 4, there exists a $G \in \mathrm{PSL}(2, q)$, such that $\mathcal{D}_G = \{E\}$ (see [4]). That is why we always get $\{E\} \in \mathcal{U}_1$ in the following.

Proposition 2.2. *Let q be odd and $\mathcal{D} < \mathrm{PSL}(2, q)$ be a dihedral group of order 4. Then, the depth of $\mathcal{D}$ in $\mathrm{PSL}(2, q)$ is 3 for $q > 3$ respectively 2 for $q = 3$.*

Proof. If $q \in \{5, 7, 9\}$, then $\mathcal{D}$ is a TI-subgroup of $\mathrm{PSL}(2, q)$: Suppose $X, Y \in \mathcal{D}$. For $q \in \{7, 9\}$ we have $\mathfrak{D}_4 \cong C(X) < N(\mathcal{D}) \cong \mathfrak{S}_4$, so if there is a $G \in \mathrm{PSL}(2, q)$ satisfying $X^G = Y$, then $G \in N(\mathcal{D})$ since $|C_{N(\mathcal{D})}(X)| = 8 = |C(X)|$. The argument for $q = 5$ is analog, here we have $\mathcal{D} = C(X)$ and $N(\mathcal{D}) \cong \mathfrak{A}_4$.

Assume $q \geq 11$. Certainly, $\mathcal{U}_1 = \mathcal{U}_2$, and it remains to determine, whether for all $G_1, G_2 \in \mathrm{PSL}(2,q)$ with $\mathcal{D}_{G_1,G_2} = \langle X \rangle$, $X \neq E$, there exists some G_3, such that $\mathcal{D}_{G_3} = \langle X \rangle$ and $X^{G_1^{-1}} = X^{G_3^{-1}}$. This is clear in case $G_1 \notin N(\mathcal{D})$. Moreover, $|C_{N(\mathcal{D})}(X)| < |C(X)|$, whence we can always find such a G_3.
Theorem 1.1 now implies that $\mathcal{D}$ has depth 3 in $\mathrm{PSL}(2,q)$ for $q > 3$. Furthermore, we have $\mathcal{D} \lhd \mathrm{PSL}(2,3)$ on the one hand, and $C(\mathcal{D}) = \mathcal{D}$ on the other hand, so the depth of $\mathcal{D}$ in $\mathrm{PSL}(2,3)$ is 3. □

Proposition 2.3. *Let $\mathcal{D} < \mathrm{PSL}(2,q)$ be a dihedral subgroup of order $2z$ divisible by 4, where $4 < 2z < (q \pm 1)/2$. Then, $\mathcal{D}$ has depth 4 in $\mathrm{PSL}(2,q)$.*

Proof. Any proper nontrivial subgroup, which occurs as the intersection of $\mathcal{D}$ and one of its conjugates, is either the cyclic subgroup $\mathcal{C}$ of index 2 in $\mathcal{D}$ or has order 2 resp. 4 by Theorem 1.2. Thus, $\mathcal{C}$ as well as each of these subgroups of order 4 is already contained in $\mathcal{U}_1$, if it lies in $\mathcal{U}_\infty$. Let $Z(\mathcal{D})$ denote the center of $\mathcal{D}$. Assume $E \neq X \in Z(\mathcal{D})$ and $Y \in \mathcal{D} \setminus Z(\mathcal{D})$ are elements of order 2. Then, we find some $G \in C(Y)$ of order ≥ 5 since $4 < 2z < (q \pm 1)/2$. Hence, G lies neither in $\mathcal{D}$ nor in $N(\langle X, Y \rangle) \cong \mathfrak{S}_4$ resp. $\cong \mathfrak{A}_4$, and we conclude $\mathcal{D}_G = \langle Y \rangle$.
Moreover, we can also show $\langle X \rangle \in \mathcal{U}_1$. We take some $G \in \mathrm{PSL}(2,q)$ such that $Y^G = X$. This yields $X \in \mathcal{D}^G \setminus Z(\mathcal{D}^G)$, so $\mathcal{D}_G = \langle X \rangle$ or $\mathcal{D}_G \cong \mathfrak{C}_2 \times \mathfrak{C}_2$. Assume $\langle X \rangle < \mathcal{D}_G$, then $\mathcal{D}_G$ is a subgroup of $C(Y_1)$ for some $Y_1 \in \mathcal{D}$. Since $2z < (q \pm 1)/2$, there is an $H \in C(X)$ such that $Y_2 = Y_1^H \notin \mathcal{D}$. Thus, $X \in \mathcal{D}^{GH} \subset C(Y_2)$ and we get $\mathcal{D}_{GH} = \langle X \rangle$.
We have just shown $\mathcal{U}_1 = \mathcal{U}_\infty$. However, $\mathcal{D}$ is subgroup of $C(X)$, whence there is no $G \in C(X)$ such that $\mathcal{D}_G = \langle X \rangle$. This implies $d_c(\mathcal{D}, \mathrm{PSL}(2,q)) = 4$. □

Proposition 2.4. *Let $\mathcal{D} < \mathrm{PSL}(2,q)$ be a dihedral subgroup of order $4 < 2z \in \{(q \pm 1)/2, q \pm 1\}$ divisible by 4. Then, $\mathcal{D}$ has depth 6 in $\mathrm{PSL}(2,q)$.*

Proof. As in the proof of Proposition 2.3, we obtain that subgroups of order 4 respectively subgroups generated by an element $Y \in \mathcal{D} \setminus Z(\mathcal{D})$ of order 2 lie in $\mathcal{U}_1$, if they are in $\mathcal{U}_\infty$ for $q \geq 11$. It is easy to show that these statements also hold for $q \in \{7, 9\}$. Set $E \neq X \in Z(\mathcal{D})$. Now, $\mathcal{D} \trianglelefteq C(X)$, implying that the cyclic subgroup $\mathcal{C}$ of index 2 in $\mathcal{D}$ is not contained in $\mathcal{U}_\infty$.
It remains to determine the smallest i, which satisfies $\langle X \rangle \in \mathcal{U}_i$. If $2z = q \pm 1$, then $X \in \mathcal{D}^G \setminus Z(\mathcal{D}^G)$ yields $XX^G = X^G X$, i.e. $X^G \in \mathcal{D}$. Assume $2z = (q \pm 1)/2$. Besides $\mathcal{D}$ there exists a further subgroup $\widetilde{\mathcal{D}} < C(X)$ which is isomorphic to $\mathcal{D}$. This subgroup is not a conjugate of $\mathcal{D}$ since $\mathcal{D} \lhd C(X)$. If there was a $G \in \mathrm{PSL}(2,q)$ with $\mathcal{D}_G = \langle X \rangle$, then $X^G = W$ for some $X \neq W \in \widetilde{\mathcal{D}}$ of order 2. Otherwise, there is a $\widetilde{G} \in N(\langle X, W \rangle) \setminus \widetilde{\mathcal{D}}$ such that $\widetilde{\mathcal{D}}^{\widetilde{G}} < C(W)$. This would yield $\mathcal{D}^G = \widetilde{\mathcal{D}}^{\widetilde{G}}$ in contradiction to the fact that $\widetilde{\mathcal{D}}$ is not a conjugate of $\mathcal{D}$. Hence $\langle X \rangle \notin \mathcal{U}_1$.
Let $Y, \widetilde{Y} \in \mathcal{D} \setminus \mathcal{C}$ such that $Y\widetilde{Y} \neq \widetilde{Y}Y$ and take $G \in N(\langle X, Y \rangle) \setminus \mathcal{D}$ as well as $\widetilde{G} \in N(\langle X, \widetilde{Y} \rangle) \setminus \mathcal{D}$. Then, $\mathcal{D}_G = \langle X, Y \rangle$, $\mathcal{D}_{\widetilde{G}} = \langle X, \widetilde{Y} \rangle$ and $\langle X, Y \rangle \neq \langle X, \widetilde{Y} \rangle$, that means $\mathcal{D}_{G,\widetilde{G}} = \langle X \rangle$. Since $\mathcal{D} \trianglelefteq C(X)$ and $\langle X \rangle \notin \mathcal{U}_1$, we get $X^{G_1^{-1}} \neq X$ whenever $\mathcal{D}_{G_1,G_2} = \langle X \rangle$ for $G_1, G_2 \in \mathrm{PSL}(2,q)$. Otherwise, $\mathcal{D}_{E,G,\widetilde{G}} = \langle X \rangle$ and $X^E = X$, whence $\mathcal{D}$ has depth 6 in $\mathrm{PSL}(2,q)$. □

2.3 Permutation subgroups

The major part of the determination of $d_c(\mathcal{A}, \mathrm{PSL}(2,q))$ for a subgroup $\mathcal{A} \cong \mathfrak{A}_4$ of $\mathrm{PSL}(2,q)$ has already been done in [4]. There, it was shown that there is some $G \in \mathrm{PSL}(2,q)$, such that $\mathcal{A}_G = \{E\}$, if and only if $q \geq 7$. The remaining calculations can be done fairly quickly.

Proposition 2.5. *Suppose q is odd and $\mathcal{A} \cong \mathfrak{A}_4$ is a subgroup of $\mathrm{PSL}(2,q)$. Then, the depth of $\mathcal{A}$ in $\mathrm{PSL}(2,q)$ is 3 if $q \geq 7$, and it is 5 in case $q = 5$.*

Proof. Due to Theorem 1.2, $\mathcal{A}$ is contained in the normalizer of a subgroup $\mathcal{N} \cong \mathfrak{C}_2 \times \mathfrak{C}_2$ of $\mathrm{PSL}(2,q)$. Thus, $\mathcal{A}_G \neq \mathcal{N}$ for any $G \in \mathrm{PSL}(2,q)$. All the other proper subgroups of $\mathcal{A}$ have order ≤ 3, so $\mathcal{U}_1 = \mathcal{U}_\infty$ if $q \geq 7$.

Assume $\mathcal{A}_{G_1,G_2} = \langle X\rangle$ and $X^{G_1^{-1}} = Y$ for $E \neq X \in \mathcal{A}$ and $Y, G_1, G_2 \in \mathrm{PSL}(2,q)$. If $G_1 \notin N(\mathcal{A})$, we already get $\mathcal{A}_{G_1} = \langle X\rangle$. Otherwise, we obtain $\mathcal{A}_{CG_1} = \langle X\rangle$ and $X^{(CG_1)^{-1}} = Y$ for a suitable $C \in C(Y) \setminus N(\mathcal{A})$ if $q \geq 9$ (note that $|\mathcal{A}_{G_1,G_2}| \neq 2$ for $q = 9$). Furthermore, $\mathcal{A}$ is a TI-subgroup of $\mathrm{PSL}(2,7)$. Consequently, the depth of $\mathcal{A}$ in $\mathrm{PSL}(2,q)$ is 3 for $q \geq 7$.
The trivial subgroup $\{E\}$ is the only element of $\mathcal{U}_\infty \setminus \mathcal{U}_1$ in case $q = 5$ implying $d_c(\mathcal{A}, \mathrm{PSL}(2,5)) = 5$. □

Proposition 2.6. *Suppose q is even and $\mathcal{A} \cong \mathfrak{A}_4$ is a subgroup of $\mathrm{PSL}(2,q)$. Then, the depth of $\mathcal{A}$ in $\mathrm{PSL}(2,q)$ is 3 if $q > 4$, and it is 5 in case $q = 4$.*

Proof. Since $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$, the last assertion was just established in the preceding proposition. That is why we assume $q \geq 16$. Let $X, Y \in \mathcal{A}$ be (not necessarily different) elements of order 2 and $G \in \mathrm{PSL}(2,q)$ be such that $X^G = Y$. Then, there exist a $C \in C(X)$ and an $A \in \mathcal{A}$ satisfying $G = CA$, since X and Y are conjugate in $\mathcal{A}$. However, C commutes with the elements of the Sylow 2-subgroup of $\mathcal{A}$, whence the intersection of $\mathcal{A}$ and some of its conjugates cannot have order 2. This yields $\mathcal{U}_1 = \mathcal{U}_\infty$.
Let $\mathcal{A}_{G_1,G_2}$ be of order 3 resp. 4 and $G_1 \in N(\mathcal{A})$. For $X, Y \in \mathcal{A}_{G_1,G_2}$, the element $Y^{G_1^{-1}}$ is determined by $X^{G_1^{-1}}$, because $C(X) = C(Y)$. That is why it suffices to find a $G \in \mathrm{PSL}(2,q)$, such that $\mathcal{A}_G = \mathcal{A}_{G_1,G_2}$ and $X^{G^{-1}} = X^{G_1^{-1}}$. If $|\mathcal{A}_{G_1,G_2}| = 4$, the nontrivial elements of $\mathcal{A}_{G_1,G_2}$ are conjugate in $N(\mathcal{A})$. It follows that there is an $A \in N(\mathcal{A})$, which satisfies $\mathcal{A}_{AG_2} = \mathcal{A}_{G_1,G_2}$ and $X^{(AG_2)^{-1}} = X^{G_1^{-1}}$. In case $\mathcal{A}_{G_1,G_2} = \langle X\rangle$ has order 3, there is an $A \in \mathcal{A}$, such that either $X^{(AG_2)^{-1}} = X^{G_1^{-1}}$ or $X^{(AG_2)^{-1}} = \big(X^{G_1^{-1}}\big)^{-1}$. In the latter case, we take an element $B \in N\big(\langle X^{(AG_2)^{-1}}\rangle\big)$ of order 2 and obtain $X^{(BAG_2)^{-1}} = X^{G_1^{-1}}$. Suppose $BAG_2 \in N(\mathcal{A})$, then $\mathcal{A}^{(AG_2)^{-1}} = \mathcal{A}^B$. Since $N\big(\langle X^{(AG_2)^{-1}}\rangle\big)$ contains a $\widetilde{B}$ of order 2 with $|\langle \widetilde{B}B^{-1}\rangle| > 3$ (i.e. $\widetilde{B}B^{-1} \notin N(\mathcal{A})$), we get $\widetilde{B}AG_2 \notin N(\mathcal{A})$. Finally, this leads to $\mathcal{A}_{\widetilde{B}AG_2} = \mathcal{A}_{G_1,G_2}$ and $X^{(\widetilde{B}AG_2)^{-1}} = X^{G_1^{-1}}$. Obviously, $\mathcal{A}_{G_1} = \mathcal{A}_{G_1,G_2}$ if $G_1 \notin N(\mathcal{A})$. That means, we always find some G satisfying $\mathcal{A}_G = \mathcal{A}_{G_1,G_2}$ as well as $X^{G^{-1}} = X^{G_1^{-1}}$ and the assertion is proved. □

Lemma 2.7. *Let $q \geq 17$ and $\mathcal{S} \cong \mathfrak{S}_4$ be a subgroup of $\mathrm{PSL}(2,q)$. Then, there is no subgroup of order 1, 3, 6 or 8 in $\mathcal{U}_\infty \setminus \mathcal{U}_1$.*

Proof. From [4] we know that there is some $G \in \mathrm{PSL}(2,q)$ such that $\mathcal{S}_G = \{E\}$. Moreover, the subgroups of order 6 and 8 are maximal subgroups of $\mathcal{S}$. Hence, they are already contained in $\mathcal{U}_1$, if they lie in $\mathcal{U}_\infty$. Suppose $X \in \mathcal{S}$ has order 3. Then, for $C \in C(X)$ of order ≥ 5 the intersection $\mathcal{S}_C$ contains at most six elements. The groups of order 6 are certainly nonnormal subgroups of $N(X)$, whence there exists a $\widetilde{C} \in C(X)$ such that $|\mathcal{S}_{\widetilde{C}}| \leq 6$ and $\mathcal{S}_C \neq \mathcal{S}_{\widetilde{C}}$. However, $N_{\mathcal{S}}(\langle X\rangle)$ is the only subgroup of order 6 in $\mathcal{S}$ which contains X. That is why at least one of the groups $\mathcal{S}_C$ and $\mathcal{S}_{\widetilde{C}}$ coincides with $\langle X\rangle$. □

Lemma 2.8. *Let $q \geq 17$ and $\mathcal{S} \cong \mathfrak{S}_4$ be a subgroup of $\mathrm{PSL}(2,q)$. Furthermore, let $\mathcal{N}$ be the normal subgroup of order 4 of $\mathcal{S}$. Then, we have:*

(i) Each nonnormal subgroup of $\mathcal{S}$, which is isomorphic to $\mathfrak{C}_2 \times \mathfrak{C}_2$, lies in $\mathcal{U}_1$, whereas $\mathcal{N} \notin \mathcal{U}_\infty$.

(ii) For $q \geq 23$, the cyclic subgroups of order 4 of $\mathcal{S}$ lie in $\mathcal{U}_1$. If $q = 17$, then the Sylow 2-subgroups of $\mathcal{S}$ are contained in $\mathcal{U}_1$ in contrast to the cyclic subgroups of order 4 which do not lie in $\mathcal{U}_\infty$.

(iii) If $Y \in \mathcal{S} \setminus \mathcal{N}$ is an element of order 2, then $\langle Y\rangle \in \mathcal{U}_1$.

(iv) If $X \in \mathcal{N}$ is an element of order 2, then $\langle X\rangle \in \mathcal{U}_1$ for $q \geq 23$, and $\langle X\rangle \in \mathcal{U}_2 \setminus \mathcal{U}_1$ for $q = 17$.

Proof. (i) It is clear that $\mathcal{N} \notin \mathcal{U}_\infty$ since $\mathcal{S} = N(\mathcal{N})$. Assume $X \in \mathcal{N}$ and $Y \in \mathcal{S} \setminus \mathcal{N}$ are elements of order 2, such that $XY = YX$. Then, there is some $G \in N(\langle X, Y\rangle)$ which satisfies $X^G = Y$, $Y^G = XY$ and $(XY)^G = X$. This implies $X \notin \mathcal{N}^G$ and $Y \in \mathcal{N}^G$, so $\mathcal{S}_G = \langle X, Y\rangle$ as desired.

(ii) Assume X and Y are as before and $V \in \mathcal{S}$ is such that $V^2 = X$. For $G \in N(\langle V\rangle) \setminus \mathcal{S}$, we get $\langle V\rangle \leq \mathcal{S}_G < \mathcal{S}$. We want G to satisfy $\mathcal{N}^G \notin \langle X, Y\rangle$, so that $\mathcal{S}_G = \langle V\rangle$. Such a G exists if the length of the conjugacy class of $\mathcal{N}$ in $N(\langle V\rangle)$ is at least 3. This is the case for $q \geq 23$, since $N(\langle V\rangle)$ is a dihedral group of order $q \pm 1$. Hence, the cyclic groups of order 4 lie in $\mathcal{U}_1$ if $q \geq 23$.

Now, suppose $q = 17$. Then, the index of $\langle V, Y\rangle$ in $N(\langle V\rangle)$ is 2, so $\langle V, Y\rangle \lhd N(\langle V\rangle)$. Thus, the elements of order 2 of $\mathcal{S}$, which are $\neq X$ and commute with X, form a single conjugacy class in $C(X) = N(\langle V\rangle)$. Therefore, the conjugates of $\mathcal{N}$ in $N(\langle V\rangle)$ are $\mathcal{N}$ itself and $\langle X, Y\rangle$. This gives $\mathcal{N} \notin \mathcal{U}_\infty$, but the Sylow 2-subgroups of $\mathcal{S}$ lie in $\mathcal{U}_1$.

(iii) Let $E \neq X \in \mathcal{N}$ be the element which commutes with Y. Since $q \geq 17$, there is a $C \in C(Y)$ of order ≥ 8, so that $\langle X, Y\rangle^C \neq \langle X, Y\rangle$. Then, X^C commutes with Y, but not with X, so no nontrivial element of $\mathcal{N}$ lies in $\mathcal{S}^C$. Thus, it only remains to be shown that we can choose C, such that $\mathcal{S}_C$ does not contain elements of order 3.

There are two subgroups of order 6 in $\mathcal{S}$ containing Y, say $\langle A, Y\rangle$ and $\langle B, Y\rangle$ where $|\langle A\rangle| = |\langle B\rangle| = 3$. Since C does not lie in $C(A)$ resp. $C(B)$, because $AY \neq YA$ resp. $BY \neq YB$, and the order of C is ≥ 8, we obtain $A^C, A^{C^2} \notin \langle A\rangle$ and $B^C, B^{C^2} \notin \langle B\rangle$. Assume $A^C = B$. This certainly implies $A^{C^2} \notin \langle B\rangle$. Moreover, $B^{C^2} = A^{C^3} \notin \langle A\rangle$, again due to the order of C. From $\langle Y\rangle < \mathcal{S}_C$ we therefore deduce $\langle Y\rangle = \mathcal{S}_{C^2}$, so $\langle Y\rangle \in \mathcal{U}_1$.

(iv) Let $\widetilde{X} \in \mathcal{N}$ and $Y, \widetilde{Y} \in \mathcal{S} \setminus \mathcal{N}$ be elements of order 2, such that $XY = YX$, $\widetilde{X}\widetilde{Y} = \widetilde{Y}\widetilde{X}$ and $X \neq \widetilde{X}$. Then $X\widetilde{Y} \neq \widetilde{Y}X$. We take $G \in \mathrm{PSL}(2,q)$, such that $\widetilde{Y}^G = X$. This leads to $X \in \mathcal{S}_G$ and $X \notin \mathcal{N}^G$. There is exactly one subgroup of $\mathcal{S}^G$ that contains X and is isomorphic to $\mathfrak{C}_2 \times \mathfrak{C}_2$, namely $\langle X, \widetilde{X}^G\rangle$. In $C(X)$, a conjugacy class of elements of order 2, which does not contain X, has length $(q \pm 1)/4$. Otherwise, in $\mathcal{S}$ there are four elements of order 2 which commute with X. Thus, we find a $C \in C(X)$ that satisfies $\widetilde{X}^{GC} \notin \mathcal{S}$ for $q \geq 23$. This yields that there are no elements of order 2, which commute with X, and no elements of order 4 lying in $\mathcal{S}_{GC}$. Furthermore, $\mathcal{S}$ is generated by X and an arbitrary element of order 2 that does not commute with X, so there are no elements $\neq X$ of order 2 in $\mathcal{S}_{GC}$. Finally, if there was an element A of order 3 in $\mathcal{S}_{GC}$, then $\langle X, A\rangle \cong \mathfrak{A}_4$ implying $\mathcal{S}_{GC} = \mathcal{S}$, i.e. $GC \in N(\mathcal{S}) = \mathcal{S}$. Consequently, we get $\langle X\rangle \in \mathcal{U}_1$ for $q \geq 23$.

Now, let $q = 17$. We show that apart from $\widetilde{Y}^G = X \in \mathcal{S}$, we also have $\widetilde{X}^G \in \mathcal{S}$. By the proof of part (ii), the elements $\neq \widetilde{X}$ of order 2 of $\mathcal{S}$, which commute with $\widetilde{X}$, form a single conjugacy class in $C(\widetilde{X})$. Therefore, there exists some $C_1 \in C(\widetilde{X})$ with $\widetilde{Y}^{C_1} = X$ and we can write $G = C_1C_2$ with a $C_2 \in C(X)$. That is why we get $\widetilde{X}^G = \widetilde{X}^{C_2}$, and this element is also contained in $\mathcal{S}$ by the same argumentation for X as for $\widetilde{X}$ before. Thus, $\langle X\rangle$ cannot lie in $\mathcal{U}_1$. However, the Sylow 2-subgroups of $\mathcal{S}$ lie in $\mathcal{U}_1$ by part (ii) as well as the subgroup $\langle X, Y\rangle$ by part (i). The intersection of the latter with a Sylow 2-subgroup not containing Y is $\langle X\rangle$, so $\langle X\rangle \in \mathcal{U}_2 \setminus \mathcal{U}_1$ as claimed. □

Proposition 2.9. *Let $\mathcal{S} \cong \mathfrak{S}_4$ be a subgroup of* $\mathrm{PSL}(2,q)$. *Then, the depth of $\mathcal{S}$ in* $\mathrm{PSL}(2,q)$ *is* 4 *for $q \geq 23$, it is* 6 *for $q = 17$, and it is* 5 *if $q \in \{7, 9\}$.*

Proof. Due to Lemmas 2.7 and 2.8, we have $\mathcal{U}_0 \neq \mathcal{U}_1 = \mathcal{U}_\infty$ for $q \geq 23$. If $X \neq E$ is an element of the normal subgroup of $\mathcal{S}$ of order 4 and $X^G = X$ for some $G \in \mathrm{PSL}(2,q)$, then $\mathcal{S}_G$ contains elements of order 4. On the other hand, there exists a $\widetilde{G} \in \mathrm{PSL}(2,q)$, such that $\mathcal{S}_{\widetilde{G}} = \langle X\rangle$ by Lemma 2.8. Thus, $\mathcal{S}_{E,\widetilde{G}} = \langle X\rangle$ and $X^E = X$, so $\mathcal{S}$ has depth 4 in $\mathrm{PSL}(2,q)$.
For $q = 17$, we get $\mathcal{U}_1 \neq \mathcal{U}_2 = \mathcal{U}_\infty$, and for X as above, $\langle X\rangle$ is the only element of $\mathcal{U}_2 \setminus \mathcal{U}_1$ by Lemmas 2.7 and 2.8. Assume $\mathcal{S}_{G_1,G_2} = \langle X\rangle$ for $G_1, G_2 \in \mathrm{PSL}(2,q)$. Then $X^{G_1} \neq X$, since otherwise, by the proof of (ii) of Lemma 2.8, $\mathcal{S}_{G_1}$ would contain both subgroups of $\mathcal{S}$, which are isomorphic to $\mathfrak{C}_2 \times \mathfrak{C}_2$ and contain X. Certainly, we have $\mathcal{S}_{E,G_1,G_2} = \langle X\rangle$ and $X^E = X$, so $\mathcal{S}$ has depth 6 in $\mathrm{PSL}(2,17)$.
It is not difficult to show $\mathcal{U}_2 = \mathcal{U}_1 \cup \{E\}$ for $q = 7$ resp. $q = 9$, and we deduce immediately that the depth of $\mathcal{S}$ in $\mathrm{PSL}(2,q)$ is 5 in both cases. □

The calculation of the depth of a subgroup isomorphic to $\mathfrak{A}_5$ in $\mathrm{PSL}(2,q)$ is done similarly. Again, for each subgroup, we determine the least i, for which this subgroup lies in $\mathcal{U}_i$. In [4] it was shown that for $q \geq 29$ odd and $q \not\equiv 0 \pmod 5$, a subgroup $\mathcal{A} \cong \mathfrak{A}_5$ of $\mathrm{PSL}(2,q)$ has a conjugate that intersects $\mathcal{A}$ trivially. The same assertion is valid for $q > 16$ even resp. $q > 25$ divisible by 5. This can be proved by similar arguments. For various other subgroups, the following lemmas are helpful.

Lemma 2.10. *Let $\mathcal{H} = \mathcal{Q} \rtimes \mathcal{C} < \mathrm{PSL}(2,p^f)$ with a subgroup $\mathcal{Q}$ of order p^m, $m \leq f$, and a cyclic subgroup $\mathcal{C}$ of order $\gcd(p^m-1,(p^f-1)/k)$ where $k = \gcd(p^f-1,2)$. Then $N(\mathcal{H}) = \mathcal{H}$.*

Proof. Suppose there is an element $G \in N(\mathcal{H}) \setminus \mathcal{H}$ of order prime to p. Then, $\mathcal{Q}$ would be a normal Sylow p-subgroup of a subgroup of $\mathrm{PSL}(2,p^f)$, whose order is bigger than $|\mathcal{H}|$. But this is impossible, as we can easily verify by a look at the list of all subgroups of $\mathrm{PSL}(2,p^f)$. Hence, there is also no element P of order p in $N(\mathcal{H}) \setminus \mathcal{H}$. Otherwise, PC would lie in $\mathcal{H}$ for $C \in \mathcal{C}$, but this contradicts $P \notin \mathcal{H}$. Thus, $N(\mathcal{H}) = \mathcal{H}$ as desired. □

Lemma 2.11. *Suppose $\mathcal{Q} < \mathrm{PSL}(2,p^f)$ is a subgroup of order p^m, which is contained in a further subgroup $\mathcal{Q} \rtimes \mathcal{C}$ of order $p^m(p^m-1)$ resp. $p^m(p^m-1)/2$. Then, $\mathcal{Q}$ is a TI-subgroup of $\mathrm{PSL}(2,p^f)$.*

Proof. Let $\mathcal{P}$ be the Sylow p-subgroup of $\mathrm{PSL}(2,p^f)$ which contains $\mathcal{Q}$. Then, $\mathcal{N} := N(\mathcal{Q}) = N_{N(\mathcal{P})}(\mathcal{Q})$ has order $p^f(p^m-1)$ resp. $p^f(p^m-1)/2$. The size of the centralizer of an element of order p is p^f, so any $G \in \mathrm{PSL}(2,p^f)$, which satisfies $P_1^G = P_2$ for $P_1, P_2 \in \mathcal{Q}$, already lies in $\mathcal{N}$ if $|\mathcal{N}| = p^f(p^m-1)$. Thus, in this case $\mathcal{Q}$ is a TI-subgroup.
Assume $|\mathcal{N}| = p^f(p^m-1)/2$ now. If $N_1, N_2 \in \mathcal{N}$ are conjugate, then $P^{N_1} = P^{N_2}$ for all $P \in \mathcal{P}$. (We may assume that $\mathcal{P}$ consists of lower triangular matrices with all diagonal entries 1 to confirm this fact.) The elements of $\mathcal{Q}$ split into two conjugacy classes $\hat{\mathcal{Q}}_1$, $\hat{\mathcal{Q}}_2$ of length $(p^m-1)/2$ in $\mathcal{Q} \rtimes \mathcal{C}$ as well as in $\mathcal{N}$. Suppose $P_1, P_2 \in \mathcal{Q}$ such that $P_1^G = P_2$ for some $G \in \mathrm{PSL}(2,p^f)$, and w.l.o.g. $P_2 \in \hat{\mathcal{Q}}_1$. Take $N \in \mathcal{Q} \rtimes \mathcal{C}$ of order $(p^m-1)/2$, and recall that $\mathcal{P}$ is a TI-subgroup of $\mathrm{PSL}(2,p^f)$, so $N^G \in \mathcal{N}$. This leads to $\{(P_2)^{N^j} : 1 \leq j \leq (p^m-1)/2\} = \hat{\mathcal{Q}}_1$, which also holds if N is replaced by N^G. Hence, not only P_2 but the whole class $\hat{\mathcal{Q}}_1$ lies in $(\mathcal{Q} \rtimes \mathcal{C})_G$. Obviously $(p^m-1)/2 > p^{m-1}-1$, so $\hat{\mathcal{Q}}_1$ generates a p-group of order $> p^{m-1}$, i.e. $\langle \hat{\mathcal{Q}}_1 \rangle = \mathcal{Q}$. This yields $\mathcal{Q} \leq (\mathcal{Q} \rtimes \mathcal{C})_G$, so $\mathcal{Q}^G = \mathcal{Q}$. Therefore, $\mathcal{Q}$ is a TI-subgroup of $\mathrm{PSL}(2,p^f)$ as desired. □

Proposition 2.12. *Let q be even and $\mathcal{A} \cong \mathfrak{A}_5$ be a subgroup of $\mathrm{PSL}(2,q)$. Then, the depth of $\mathcal{A}$ in $\mathrm{PSL}(2,q)$ is 3 for $q > 16$ resp. 5 for $q = 16$.*

Proof. The Sylow 2-subgroups of $\mathcal{A}$ satisfy the assumptions of Lemma 2.11, so $|\mathcal{A}_G|$ is either odd or divisible by 4 for any $G \in \mathrm{PSL}(2,q)$. Moreover, a subgroup of order 12 of $\mathcal{A}$ is isomorphic to $(\mathfrak{C}_2 \times \mathfrak{C}_2) \rtimes \mathfrak{C}_3$ and therefore coincides with its normalizer in $\mathrm{PSL}(2,q)$ by Lemma 2.10. Additionally, the subgroups of order 12 of $\mathcal{A}$ form a single conjugacy class in $\mathcal{A}$, that means they cannot lie in $\mathcal{U}_\infty$. This implies that proper nontrivial subgroups of $\mathcal{A}$ lying in $\mathcal{U}_\infty$ have order 3, 4 or 5 and therefore are not contained in $\mathcal{U}_\infty \setminus \mathcal{U}_1$. It follows $\mathcal{U}_0 \neq \mathcal{U}_1 = \mathcal{U}_\infty$ if $q > 16$, and $\mathcal{U}_2 = \mathcal{U}_1 \uplus \{E\}$ for $q = 16$, i.e. $\mathcal{A}$ has depth 5 in $\mathrm{PSL}(2,16)$.
Let $q > 16$. Assume $\mathcal{A}_{G_1,G_2} = \langle X \rangle$ for $G_1, G_2 \in \mathrm{PSL}(2,q)$ and X is of order 3. Since there is only one conjugacy class of elements of order 3 in $\mathcal{A}$, we find some $A \in \mathcal{A}$, such that $X^{G_1^{-1}} = X^{(AG_2)^{-1}}$. Thus, we have $\mathcal{A}_{G_1} = \mathcal{A}_{G_1,G_2}$, or $\mathcal{A}_{AG_2} = \mathcal{A}_{G_1,G_2}$ and $Y^{G_1^{-1}} = Y^{(AG_2)^{-1}}$ for $Y \in \mathcal{A}_{G_1,G_2}$. A similar assertion holds if X has order 5, since two elements of order 5 are conjugate in $\mathcal{A}$ if and only if they are conjugate in $\mathrm{PSL}(2,q)$. Finally, suppose $\mathcal{A}_{G_1,G_2} = \langle X, Y \rangle$ is a subgroup of order 4. The only interesting case is $\mathcal{A}_{G_1} \neq \langle X, Y \rangle$, that means $G_1 \in \mathcal{A}$. Set $\widetilde{X} = X^{G_1^{-1}}$ and $\widetilde{Y} = Y^{G_1^{-1}}$. Because of $N(\mathcal{A}) \cap C(\widetilde{X}) = \mathcal{A} \cap C(\widetilde{X}) = \langle \widetilde{X}, \widetilde{Y} \rangle$, we get $C(\widetilde{X}) \setminus N(\mathcal{A}) = C(\widetilde{X}) \setminus \langle \widetilde{X}, \widetilde{Y} \rangle$, so there is a $C \in C(\widetilde{X})$, such that $\mathcal{A}_C = \langle \widetilde{X}, \widetilde{Y} \rangle$. This yields $\mathcal{A}_{CG_1} = \langle X, Y \rangle$ and $X^{(CG_1)^{-1}} = \widetilde{X}$ as well as $Y^{(CG_1)^{-1}} = \widetilde{Y}$. In consequence, $\mathcal{A}$ has depth 3 in $\mathrm{PSL}(2,q)$. □

Lemma 2.13. *Let $q \geq 29$ be odd and $\mathcal{A} \cong \mathfrak{A}_5$ be a subgroup of $\mathrm{PSL}(2,q)$. Then $\mathcal{U}_0 \neq \mathcal{U}_1 = \mathcal{U}_\infty$.*

Proof. We have already mentioned $\{E\} \in \mathcal{U}_1$. Obviously, the subgroups of order 10 and 12 are maximal subgroups of $\mathcal{A}$, so they cannot lie in $\mathcal{U}_\infty \setminus \mathcal{U}_1$. The subgroups of order 6 are maximal subgroups of $\mathcal{A}$ as well, since all subgroups of $\mathcal{A}$ of order 12 are isomorphic to $\mathfrak{A}_4$. Suppose $\mathcal{V}$ is a subgroup of $\mathcal{A}$ of order 4, then $N_\mathcal{A}(\mathcal{V}) \cong \mathfrak{A}_4$. In $\mathrm{PSL}(2,q)$, there is only one subgroup isomorphic to $\mathfrak{A}_4$ which contains $\mathcal{V}$. Hence, if $\mathcal{V}$ is contained in the intersection of $\mathcal{A}$ and some of its conjugates, then $N_\mathcal{A}(\mathcal{V})$ also lies in that intersection. This implies $\mathcal{V} \notin \mathcal{U}_\infty$. Now, it remains to examine the subgroups of order 2, 3 and 5.
At first, let $q \not\equiv 0 \pmod 5$ and $X \in \mathcal{A}$ be an element of order 5. An element $C \in C(X)$ of order > 10 cannot lie in $N(\mathcal{A}) = \mathcal{A}$, whence $5 \leq |\mathcal{A}_C| \leq 10$. On the one hand $|N_\mathcal{A}(\langle X\rangle)| = 10$, but on the other hand $C \notin N(N_\mathcal{A}(\langle X\rangle))$ due to its order. This yields $\mathcal{A}_C = \langle X\rangle$, so $\langle X\rangle \in \mathcal{U}_1$.
Now, let $q \equiv 0 \pmod 5$ and $X \in \mathcal{A}$ be an element of order 5. The normalizer of $\mathcal{A}$ possesses at most 120 elements. Since $|C(X)| > 25$, there is a $C \in C(X)$, such that $\mathcal{A}_C < \mathcal{A}$. For $q = 5^f$ with f odd, we have $N(\mathcal{H}) = \mathcal{H}$ for $\mathcal{H} \cong \mathfrak{C}_5 \rtimes \mathfrak{C}_2$ by Lemma 2.10. Otherwise, for f even, we get $|N(\mathcal{H})| = 20$ by the same lemma. In this case, the normalizer of $\mathcal{A}$ is isomorphic to $\mathrm{PGL}(2,5)$, so $|N(N_\mathcal{A}(\mathcal{H}))| = 20$. This leads to $N(\mathcal{H}) < N(\mathcal{A})$, independent of the choice of f. It follows $\mathcal{A}_C = \langle X\rangle$, that means $\langle X\rangle \in \mathcal{U}_1$.
In the next step, we consider subgroups of order 3. Let $q \not\equiv 0 \pmod 3$ and $X \in \mathcal{A}$ be an element of order 3. Clearly, an element $C \in C(X)$ of order > 12 cannot lie in $N(\mathcal{A})$ or $N(N_\mathcal{A}(\langle X\rangle))$, so $\mathcal{A}_C = \langle X\rangle$ or $|\mathcal{A}_C| = 12$. Assume the latter holds true. There are exactly two subgroups $\mathcal{A}_1, \mathcal{A}_2 < \mathcal{A}$ isomorphic to $\mathfrak{A}_4$ which contain X. Let $\mathcal{A}_1 = \mathcal{A}_C$ w.l.o.g. and set $\mathcal{A}_3 = \mathcal{A}_1^C$. Since $C \notin N(\mathcal{A}_1)$, we get $\mathcal{A}_2^C = \mathcal{A}_1$ and therefore $\mathcal{A}_{C^{-1}} = \mathcal{A}_2$. If $\mathcal{A}_3$ is also contained in $\mathcal{A}_{C^{-1}}$, then $\mathcal{A}_3^{C^3} = \mathcal{A}_2^{C^2} = \mathcal{A}_1^C = \mathcal{A}_3$ would follow. However, this is impossible, since the order of C^3 is > 4. This yields $|\mathcal{A}^C \cap \mathcal{A}^{C^{-1}}| < 12$, whence $3 \leq |\mathcal{A}_{C^2}| < 12$. We have already mentioned $C \notin N(N_\mathcal{A}(\langle X\rangle))$, so $\mathcal{A}_{C^2} = \langle X\rangle$ and $\langle X\rangle \in \mathcal{U}_1$.
For $q \equiv 0 \pmod 3$, the centralizer of an element $X \in \mathcal{A}$ of order 3 is an elementary-abelian group of order q, thus $|C(X) \cap \mathcal{A}| = 3$. Since X lies in exactly two subgroups $\mathcal{A}_1, \mathcal{A}_2$ of order 12 of $\mathcal{A}$, there are at most nine elements $C \in C(X)$, such that $|\mathcal{A}_C| \geq 12$. The condition $q \geq 29$ now guarantees that there is some $G \in C(X)$ satisfying $|\mathcal{A}_G| < 12$. Due to Lemma 2.10, a subgroup isomorphic to $\mathfrak{C}_3 \rtimes \mathfrak{C}_2$ is self-normalizing in $\mathrm{PSL}(2,q)$. This implies $G \notin N(N_\mathcal{A}(\langle X\rangle))$, that means $|\mathcal{A}_G| = 3$ and $\langle X\rangle \in \mathcal{U}_1$.
The argument for subgroups of order 2 is analogous. Let $X \in \mathcal{A}$ be of order 2 and $C \in C(X)$ be of order > 12 and $\neq 16$. Then, $\mathcal{A}_C$ consists of two, six or ten elements. Suppose $|\mathcal{A}_C| = 6$. There are exactly two subgroups of $\mathcal{A}$ of order 6 containing X, say $\langle X, Y_1\rangle$ and $\langle X, Y_2\rangle$ with elements Y_1, Y_2 of order 3, such that $Y_1^C = Y_2$. A similar argument as above shows $|\mathcal{A}_{C^2}| \neq 6$ because of the order of C. Hence, if $\mathcal{A}_{C^2} \neq \langle X\rangle$, then $|\mathcal{A}_{C^2}| = 10$. There are exactly two subgroups of $\mathcal{A}$ of order 10 containing X, so $|\mathcal{A}_{C^4}| \neq 10$ again by the order of C. Furthermore, $|\mathcal{A}_{C^4}| \neq 6$. Otherwise, we would have $Y_i^{C^4} \in \langle Y_j\rangle$ for some $i, j \in \{1,2\}$. But $Y_i^{C^4} \in \langle Y_i\rangle$ would imply $C^4 \in N(\langle Y_i\rangle)$, for $Y_1^{C^4} \in \langle Y_2\rangle$ we would need $C^3 \in N(\langle Y_2\rangle)$, and $Y_2^{C^4} \in \langle Y_1\rangle$ would give rise to $C^5 \in N(\langle Y_1\rangle)$. Since C has an even order > 12, none of these assertions applies. This gives $|\mathcal{A}_{C^4}| \notin \{6, 10\}$ if $|\mathcal{A}_C| = 6$ and $|\mathcal{A}_{C^2}| = 10$. Moreover, C^4 cannot lie in the normalizer of a subgroup isomorphic to $\mathfrak{A}_4$, since the order of C is $\neq 16$, and that is why $|\mathcal{A}_{C^4}| \neq 12$. This implies $\mathcal{A}_{C^4} = \langle X\rangle$. The same argument works for $|\mathcal{A}_C| = 10$ and $|\mathcal{A}_{C^2}| = 6$. Except for $q = 31$, we always find an element C with a required order, whence $\langle X\rangle \in \mathcal{U}_1$ in these cases. A slight modification of the above shows that the assertion is also true in case $q = 31$. We take $C \in C(X)$ of order 16, so $|\mathcal{A}_C| \in \{2, 6, 10\}$ again. Here, we do not consider C^2 and C^4, but C^3 and C^6, and deduce that at least one of the subgroups $\mathcal{A}_C$, $\mathcal{A}_{C^3}$ and $\mathcal{A}_{C^6}$ coincides with $\langle X\rangle$, so $\langle X\rangle \in \mathcal{U}_1$. □

Proposition 2.14. *Let q be odd and $\mathcal{A} \cong \mathfrak{A}_5$ be a subgroup of* $\mathrm{PSL}(2,q)$.

(i) For $q \in \{11, 19, 25\}$, the depth of $\mathcal{A}$ in $\mathrm{PSL}(2,q)$ is 5. Moreover, $d_c(\mathcal{A}, \mathrm{PSL}(2,9)) = 7$.

(ii) If $q \geq 29$ satisfies $q^2 \equiv 1 \pmod{16}$ and $q \not\equiv 0 \pmod 5$, then $\mathcal{A}$ has depth 4 in $\mathrm{PSL}(2,q)$.

(iii) If $q \geq 29$ satisfies $q^2 \not\equiv 1 \pmod{16}$ or $q \equiv 0 \pmod 5$, then $\mathcal{A}$ has depth 3 in $\mathrm{PSL}(2,q)$.

Proof. (i) For $q \in \{11, 19, 25\}$, it is not difficult to show that $\mathcal{U}_2 = \mathcal{U}_\infty$, and besides $\{E\}$, there can only arise cyclic subgroups in $\mathcal{U}_2 \setminus \mathcal{U}_1$. Two elements of these cyclic subgroups are conjugate in $\mathrm{PSL}(2,q)$ if and only if they are conjugate in $\mathcal{A}$, whence $\mathcal{A}$ has depth 5 in $\mathrm{PSL}(2,q)$. Since $\mathrm{PSL}(2,9) \cong \mathfrak{A}_6$, we obtain $d_c(\mathcal{A}, \mathrm{PSL}(2,9)) = 7$ from [1], where the depth of $\mathfrak{A}_n$ in $\mathfrak{A}_{n+1}$ is determined for all $n \geq 2$.

(ii) For $q^2 \equiv 1 \pmod{16}$, we find subgroups isomorphic to $\mathfrak{S}_4$ in $\mathrm{PSL}(2,q)$. Furthermore, in $N(\mathcal{A})$ there is no element of order 4 if $q \not\equiv 0 \pmod 5$. Hence, we find some $G \in \mathrm{PSL}(2,q)$, such that $\mathcal{H} := \mathcal{A}_G \cong \mathfrak{A}_4$. Obviously, $\mathcal{A}_{E,G} = \mathcal{H}$. However, any element $\widetilde{G} \in \mathrm{PSL}(2,q)$ satisfying $\mathcal{A}_{\widetilde{G}} = \mathcal{H}$ cannot lie in the centralizer of the subgroup of order 4 of $\mathcal{H}$, since $\mathcal{H}$ is self-centralizing in $\mathrm{PSL}(2,q)$. Together with Lemma 2.13, this implies that $\mathcal{A}$ has depth 4 in $\mathrm{PSL}(2,q)$.

(iii) Each element of $\mathrm{PSL}(2,q)$, which maps a subgroup isomorphic to $\mathfrak{A}_4$ of $\mathcal{A}$ to a (not necessarily different) subgroup of $\mathcal{A}$, is already contained in $N(\mathcal{A})$. Thus, the subgroups of order 12 of $\mathcal{A}$ do not lie in $\mathcal{U}_\infty$. For G_1, G_2, such that $|\mathcal{A}_{G_1,G_2}| \in \{2,3,5\}$, we always find a $G \in \mathrm{PSL}(2,q)$ satisfying $\mathcal{A}_G = \mathcal{A}_{G_1,G_2}$ and $X^{G^{-1}} = X^{G_1^{-1}}$ for $X \in \mathcal{A}_G$. This can be verified by mimicking the last part of the proof of Proposition 2.12. Suppose $\mathcal{A}_{G_1,G_2} = \langle X, Y \rangle$ is a subgroup of order 6 and $|\mathcal{A}_{G_1}| \neq 6$. This gives $|\mathcal{A}_{G_2}| = 6$, since $\langle X, Y \rangle$ is a maximal subgroup of $\mathcal{A}$. Let X be of order 3 w.l.o.g. Since there is only one conjugacy class of elements of order 3 in $\mathcal{A}$, there exists an $A \in \mathcal{A}$, such that $X^{(AG_2)^{-1}} = X^{G_1^{-1}}$. Moreover, X acts transitive on the elements of order 2 of $\langle X, Y \rangle$ by conjugation. That is why there is an $i \in \{0,1,2\}$, such that $Y^{(X^i AG_2)^{-1}} = Y^{G_1^{-1}}$, and we conclude $\mathcal{A}_{X^i AG_2} = \mathcal{A}_{G_1,G_2}$ as well as $Z^{(X^i AG_2)^{-1}} = Z^{G_1^{-1}}$ for all $Z \in \langle X, Y \rangle$. The same argument applies in case $|\mathcal{A}_{G_1,G_2}| = 10$, since two elements of order 5 of $\mathcal{A}$ are conjugate in $\mathrm{PSL}(2,q)$ if and only if they are conjugate in $\mathcal{A}$. Consequently, the depth of $\mathcal{A}$ in $\mathrm{PSL}(2,q)$ is 3 as claimed. □

2.4 Projective linear subgroups

It could be expected that the determination of the depth of subgroups isomorphic to $\mathrm{PSL}(2,p^m)$ resp. $\mathrm{PGL}(2,p^m)$ of $\mathrm{PSL}(2,p^f)$ would be more difficult than for the other subgroups, due to the large number of subgroups such projective linear subgroups possess. In fact, most of these subgroups cannot be described as an intersection of conjugates of projective linear subgroups.

Lemma 2.15. *Let $p^m \geq 7$ be odd and $\mathcal{H} \cong \mathrm{PSL}(2,p^m)$ be a subgroup of $\mathrm{PSL}(2,p^f)$ where $1 \leq m < f$. Then, the intersection of $\mathcal{H}$ and one of its conjugates is $\{E\}$, $\mathcal{H}$, a cyclic subgroup of order prime to p, or a Sylow p-subgroup of $\mathcal{H}$.*

Proof. Let $X \neq E$ be an element of order $\neq p$ and $\mathcal{C}$ be the maximal cyclic subgroup of $\mathcal{H}$ which contains X. Recall that $|\mathcal{C}| = (p^m \pm 1)/2$ and $\mathcal{C}$ is a TI-subgroup of $\mathrm{PSL}(2,p^f)$. Thus, if X lies in the intersection of $\mathcal{H}$ and some conjugate of $\mathcal{H}$, then $\mathcal{C}$ also does. The dihedral subgroups as well as the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ contain maximal cyclic subgroups of order 2 which are clearly not maximal in $\mathrm{PSL}(2,p^f)$, so they cannot be isomorphic to $\mathcal{H}_G$ for any $G \in \mathrm{PSL}(2,p^f)$. Moreover, the cyclic subgroups of $\mathrm{PSL}(2,p^n)$ resp. $\mathrm{PGL}(2,p^n)$ which contain X have order $(p^n \pm 1)/2$ resp. $p^n - 1$. For $1 \leq n < m$ and $p^m \neq 9$, these numbers are smaller than $(p^m \pm 1)/2$, that means $\mathcal{C}$ does not lie in the corresponding subgroups. Since $\mathrm{PGL}(2,3) \cong \mathfrak{S}_4$, we already know that $\mathcal{H}_G \ncong \mathrm{PGL}(2,3)$ in case $p^m \neq 9$.
Suppose $\mathcal{H}_G = \mathcal{Q} \rtimes \mathcal{C} \cong \mathfrak{C}_p^r \rtimes \mathfrak{C}_t$ for $1 \leq r \leq m$ and $t > 1$ divides $p^r - 1$ as well as $(p^m - 1)/2$. The same argument as above leads to $t = (p^m - 1)/2$ and therefore $r = m$. It is clear that $N(\mathcal{Q} \rtimes \mathcal{C})$ is either $\mathcal{Q} \rtimes \mathcal{C}$ itself or has order $p^m(p^m - 1)$. In the first case, we have $N(\mathcal{Q} \rtimes \mathcal{C}) < \mathcal{H}$, so each $\widetilde{G} \in \mathrm{PSL}(2,p^f)$, which satisfies $\mathcal{Q} \rtimes \mathcal{C} \leq \mathcal{H}_{\widetilde{G}}$, lies already in $N(\mathcal{H})$. Hence, $\mathcal{H}_G = \mathcal{Q} \rtimes \mathcal{C}$ is impossible. In the second case, we find an element $N \in N(\mathcal{Q} \rtimes \mathcal{C})$ of order $p^m - 1$. There is a set $\mathcal{M}$ of $p^m + 1$ points of the projective line $\mathbb{P}^1(\mathbb{F}_{p^f})$, which are permuted by $\mathcal{H}$. Since N normalizes $\mathcal{Q} \rtimes \mathcal{C}$, it also permutes the points of $\mathcal{M}$. Moreover $N^2 \in \mathcal{Q} \rtimes \mathcal{C}$, that means N has two fixed points in $\mathcal{M}$ by Theorem 1.2. Consequently, besides $\mathcal{Q}$ there is a further Sylow

p-subgroup $\widetilde{\mathcal{Q}}$ of $\mathcal{H}$ which is normalized by N. However, $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ generate $\mathcal{H}$. This implies $N \in N(\mathcal{H})$, so $\mathcal{H}_G = \mathcal{Q} \rtimes \mathcal{C}$ can also not hold in this case.
Finally, the Sylow p-subgroups of $\mathcal{H}$ are the only p-groups, which can coincide with $\mathcal{H}_G$, by Lemma 2.11. □

Lemma 2.16. *Let $p^m \geq 7$ be odd and $\mathcal{H} \cong \mathrm{PSL}(2, p^m)$ be a subgroup of $\mathrm{PSL}(2, p^f)$ where $1 \leq m < f$. Then $\mathcal{U}_0 \neq \mathcal{U}_1 = \mathcal{U}_\infty$ for $f > 2m$, and $\mathcal{U}_2 = \mathcal{U}_1 \uplus \{E\}$ if $f = 2m$.*

Proof. From Lemmas 2.11 and 2.15 we conclude that all nontrivial subgroups of $\mathcal{H}$, which lie in $\mathcal{U}_\infty$, are already contained in $\mathcal{U}_1$. Thus, it only remains to determine whether $\{E\} \in \mathcal{U}_1$. We will do this by counting the elements $G \in \mathrm{PSL}(2, p^f)$ which satisfy $\mathcal{H}_G = \{E\}$.
At first, we consider the case $N(\mathcal{H}) = \mathcal{H}$, i.e. $|N(\mathcal{H})| = (p^{3m} - p^m)/2$. Then, the proof of Lemma 2.15 yields that the normalizer of a Sylow p-subgroup $\mathcal{Q}$ of $\mathcal{H}$ has order $p^f(p^m - 1)/2$. However, $p^m(p^m - 1)/2$ elements of $N(\mathcal{Q})$ lie in $\mathcal{H}$. Since $\mathcal{H}$ contains exactly $p^m + 1$ Sylow p-subgroups, we obtain

$$\left|\left\{G \in \mathrm{PSL}(2, p^f) : \mathcal{H}_G = \mathcal{Q}\right\}\right| = \left(p^f - p^m\right) \frac{p^m - 1}{2} \cdot (p^m + 1)$$

and therefore

$$\begin{aligned}\left|\left\{G \in \mathrm{PSL}(2, p^f) : |\mathcal{H}_G| = p^m\right\}\right| &= \left(p^f - p^m\right) \frac{p^m - 1}{2} \cdot (p^m + 1)^2 \\ &= \frac{p^{f+3m} + p^{f+2m} - p^{f+m} - p^f - p^{4m} - p^{3m} + p^{2m} + p^m}{2}.\end{aligned}$$

Moreover, $\mathcal{H}$ contains exactly $(p^{2m} + p^m)/2$ cyclic groups of order $(p^m - 1)/2$, and $(p^{2m} - p^m)/2$ cyclic groups of order $(p^m + 1)/2$. Let $\mathcal{C}_1 < \mathcal{H}$ be cyclic of order $(p^m - 1)/2$. Then, we get $|N(\mathcal{C}_1)| = p^f - 1$ as well as $|N_{\mathcal{H}}(\mathcal{C}_1)| = p^m - 1$, and this implies

$$\begin{aligned}\left|\left\{G \in \mathrm{PSL}(2, p^f) : |\mathcal{H}_G| = \frac{p^m - 1}{2}\right\}\right| &= \left(p^f - p^m\right) \cdot \left(\frac{p^{2m} + p^m}{2}\right)^2 \\ &= \frac{p^{f+4m} + 2p^{f+3m} + p^{f+2m} - p^{5m} - 2p^{4m} - p^{3m}}{4}.\end{aligned}$$

For a cyclic subgroup $\mathcal{C}_2$ of $\mathcal{H}$ of order $(p^m+1)/2$, we have $|N(\mathcal{C}_2)| = p^f+1$ and $|N_{\mathcal{H}}(\mathcal{C}_2)| = p^m+1$. This leads to

$$\begin{aligned}\left|\left\{G \in \mathrm{PSL}(2, p^f) : |\mathcal{H}_G| = \frac{p^m + 1}{2}\right\}\right| &= \left(p^f - p^m\right) \cdot \left(\frac{p^{2m} - p^m}{2}\right)^2 \\ &= \frac{p^{f+4m} - 2p^{f+3m} + p^{f+2m} - p^{5m} + 2p^{4m} - p^{3m}}{4}.\end{aligned}$$

Altogether, there are

$$\begin{aligned}&\frac{p^{3m} - p^m}{2} + \frac{p^{f+3m} + p^{f+2m} - p^{f+m} - p^f - p^{4m} - p^{3m} + p^{2m} + p^m}{2} \\ &+ \frac{p^{f+4m} + 2p^{f+3m} + p^{f+2m} - p^{5m} - 2p^{4m} - p^{3m}}{4} + \frac{p^{f+4m} - 2p^{f+3m} + p^{f+2m} - p^{5m} + 2p^{4m} - p^{3m}}{4} \\ &= \frac{p^{f+4m} + p^{f+3m} + 2p^{f+2m} - p^{f+m} - p^f - p^{5m} - p^{4m} - p^{3m} + p^{2m}}{2}\end{aligned}$$

different elements G of $\mathrm{PSL}(2, p^f)$, which provide a nontrivial intersection $\mathcal{H}_G$. The inequality

$$\begin{aligned}&\frac{p^{f+4m} + p^{f+3m} + 2p^{f+2m} - p^{f+m} - p^f - p^{5m} - p^{4m} - p^{3m} + p^{2m}}{2} < \frac{p^{3f} - p^f}{2} \\ \Longleftrightarrow\quad & p^{3f} - p^{f+4m} - p^{f+3m} - 2p^{f+2m} + p^{f+m} + p^{5m} + p^{4m} + p^{3m} - p^{2m} > 0\end{aligned}$$

obviously holds for $f \geq 3m$, that means there is some $\widetilde{G} \in \mathrm{PSL}(2, p^f)$, such that $\mathcal{H}_{\widetilde{G}} = \{E\}$. Otherwise, if $f = 2m$, there would be more elements providing a nontrivial intersection $\mathcal{H}_G$ than $\mathrm{PSL}(2, p^f)$ possesses. That is why $N(\mathcal{H}) > \mathcal{H}$ for $f = 2m$.
The case $|N(\mathcal{H})| = p^{3m} - p^m$ can be discussed analogously. Thereby, we get $|N(\mathcal{Q})| = p^f(p^m - 1)$, $|N_{N(\mathcal{H})}(\mathcal{Q})| = p^m(p^m - 1)$, $|N(\mathcal{C}_1)| = p^f - 1$, $|N_{N(\mathcal{H})}(\mathcal{C}_1)| = 2(p^m - 1)$, $|N(\mathcal{C}_2)| = p^f - 1$ and $|N_{N(\mathcal{H})}(\mathcal{C}_2)| = 2(p^m + 1)$. Thus, there are

$$\frac{p^{f+4m} + 2p^{f+3m} + 3p^{f+2m} - 2p^{f+m} - 2p^f - 2p^{5m} - 3p^{4m} + 2p^{3m} + p^{2m}}{2}$$

different elements $G \in \mathrm{PSL}(2, p^f)$, such that $\mathcal{H}_G \neq \{E\}$. For $f = 2m$, this number coincides with $(p^{3f} - p^f)/2$, so $\{E\} \notin \mathcal{U}_1$ in this case. However, if $f > 2m$, this number is clearly smaller than $(p^{3f} - p^f)/2$, that means $\{E\}$ lies in $\mathcal{U}_1$.
Since there are no other possibilities for $|N(\mathcal{H})|$, the proof is complete. □

Proposition 2.17. *Let $p^m \geq 5$ be odd and $\mathcal{H} \cong \mathrm{PSL}(2, p^m)$ be a subgroup of $\mathrm{PSL}(2, p^f)$ where $1 \leq m < f$. If $f > 2m$, then $d_c(\mathcal{H}, \mathrm{PSL}(2, p^f)) = 3$, and for $f = 2m$, the depth of $\mathcal{H}$ in $\mathrm{PSL}(2, p^f)$ is 5. Moreover, $d_c(\mathrm{PSL}(2, 3), \mathrm{PSL}(2, 3^f)) = 3$ for all $f \geq 2$.*

Proof. For $p^m \geq 7$ and $f > 2m$, we have just seen $\mathcal{U}_0 \neq \mathcal{U}_1 = \mathcal{U}_\infty$ in Lemma 2.16. Hence, $\mathcal{H}$ has depth 3 or 4 in $\mathrm{PSL}(2, p^f)$. Assume $\mathcal{H}_{G_1, G_2} = \mathcal{Q}$ for $G_1, G_2 \in \mathrm{PSL}(2, p^f)$ where $|\mathcal{Q}| = p^m$. The case $G_1 \notin N(\mathcal{H})$ is trivial, so let $G \in N(\mathcal{H})$. We can choose a $P \notin N(\mathcal{H})$ which lies in the Sylow p-subgroup of $\mathrm{PSL}(2, p^f)$ containing $\mathcal{Q}$. It is clear that P satisfies $\mathcal{H}_{G_1 P} = \mathcal{H}_{G_1, G_2}$ as well as $Q^{(G_1 P)^{-1}} = Q^{G_1^{-1}}$ for any $Q \in \mathcal{Q}$. By similar arguments, if $\mathcal{H}_{G_1, G_2} = \mathcal{C}$ where $|\mathcal{C}| = (p^m \pm 1)/2$, we find some $X \in N(\mathcal{C})$ of order $\geq (p^m \pm 1)/2$, such that $\mathcal{H}_{G_1, G_2} = \mathcal{H}_{G_1 X}$ and $C^{(G_1 X)^{-1}} = C^{G_1^{-1}}$ for any $C \in \mathcal{C}$. Due to Lemmas 2.11 and 2.15, there are no further cases to investigate, so the depth of $\mathcal{H}$ in $\mathrm{PSL}(2, p^f)$ is 3.
For $f = 2m$ and $p^m \geq 7$, we know that $\mathcal{U}_2 = \mathcal{U}_1 \uplus \{E\}$ by Lemma 2.16. That is why $\mathcal{H}$ has depth 5 in $\mathrm{PSL}(2, p^f)$.
Furthermore, we obtain $d_c(\mathrm{PSL}(2, 3), \mathrm{PSL}(2, 3^f)) = 3$ for $f \geq 2$ from Proposition 2.5, because $\mathrm{PSL}(2, 3) \cong \mathfrak{A}_4$. Finally, we have already seen $d_c(\mathrm{PSL}(2, 5), \mathrm{PSL}(2, 5^f)) = 3$ for $f \geq 3$ and $d_c(\mathrm{PSL}(2, 5), \mathrm{PSL}(2, 5^2)) = 5$ in Proposition 2.14, since $\mathrm{PSL}(2, 5) \cong \mathfrak{A}_5$. □

There are hardly any differences between the determination of the depth of a subgroup isomorphic to $\mathrm{PSL}(2, p^m)$ and a subgroup $\mathcal{H} \cong \mathrm{PGL}(2, p^m)$ in $\mathrm{PSL}(2, p^f)$. Let $p^m \geq 5$. Since each element of order prime to p lies in a cyclic subgroup of order $p^m + 1$ resp. $p^m - 1$ of $\mathcal{H}$, we conclude that for $G \in \mathrm{PSL}(2, p^f)$ the intersection $\mathcal{H}_G$ is $\{E\}$, $\mathcal{H}$, a cyclic group of order $p^m \pm 1$, or a Sylow p-subgroup of $\mathcal{H}$ as in Lemma 2.15. Thus, we have $\mathcal{U}_\infty = \mathcal{U}_1$ or $\mathcal{U}_\infty = \mathcal{U}_1 \uplus \{E\}$.
We derive from the same calculations as in Lemma 2.16 whether the trivial subgroup is contained in $\mathcal{U}_1$. It can be done in this way, because $\mathcal{H}$ possesses a subgroup $\widetilde{\mathcal{H}} \cong \mathrm{PSL}(2, p^m)$, which has the same Sylow p-subgroups and the same number of maximal cyclic subgroups as $\mathcal{H}$. Since $N(\widetilde{\mathcal{H}}) = \mathcal{H} = N(\mathcal{H})$, the normalizers of the single subgroups of $N(\widetilde{\mathcal{H}})$ and $N(\mathcal{H})$ also coincide. Moreover, for $f > 2m$, we can show $d_c(\mathcal{H}, \mathrm{PSL}(2, p^f)) = 3$ as in Proposition 2.17.
It is well known that $\mathrm{PGL}(2, 3) \cong \mathfrak{S}_4$. Hence, the depth of $\mathrm{PGL}(2, 3)$ in $\mathrm{PSL}(2, 3^f)$ can be taken from Proposition 2.9. In conclusion, we have:

Proposition 2.18. *Let $\mathcal{H} \cong \mathrm{PGL}(2, p^m)$ be a subgroup of $\mathrm{PSL}(2, p^f)$, $2 \leq 2m \leq f$. If $p^m \geq 5$, then $\mathcal{H}$ has depth 3 resp. 5 in $\mathrm{PSL}(2, p^f)$ if $f > 2m$ resp. $f = 2m$. Moreover, if $p^m = 3$, then the depth of $\mathcal{H}$ in $\mathrm{PSL}(2, 3^f)$ is 5 for $f = 2$ resp. 4 for $f \geq 4$.*

At last, we are looking for the results in the case that q is even. Naturally, we use similar arguments as for q odd.

Proposition 2.19. *Let $2 \leq m < f$ and $\mathcal{H} \cong \mathrm{PSL}(2, 2^m)$ be a subgroup of $\mathrm{PSL}(2, 2^f)$. Then, $\mathcal{H}$ has depth 3 if $f > 2m$ resp. 5 if $f = 2m$. Moreover, a subgroup isomorphic to $\mathrm{PSL}(2, 2)$ has depth 3 in $\mathrm{PSL}(2, 2^f)$ for $f \geq 2$.*

Proof. Since $\mathrm{PSL}(2,2) \cong \mathfrak{D}_3$ and $\mathrm{PSL}(2,4) \cong \mathfrak{A}_5$, the results in case $m = 2$ can be taken from Propositions 2.1 and 2.12. We have got $d_c(\mathrm{PSL}(2,2), \mathrm{PSL}(2,2^f)) = 3$ $(f > 1)$, $d_c(\mathrm{PSL}(2,2^2), \mathrm{PSL}(2,2^{2f})) = 3$ $(f > 2)$ and $d_c(\mathrm{PSL}(2,2^2), \mathrm{PSL}(2,2^4)) = 5$.
Now assume $m \geq 3$. As in Lemma 2.15 we conclude that $\mathcal{H}_G$ cannot be isomorphic to one of the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$, $\mathfrak{A}_4$ or $\mathrm{PSL}(2,2^n)$, $n < m$, for any $G \in \mathrm{PSL}(2,2^f)$, since the maximal cyclic subgroups of odd order of $\mathcal{H}$ contain elements of order $2^m \pm 1$. Likewise, $\mathcal{H}_G$ is not isomorphic to a semidirect product $\mathfrak{C}_2^n \rtimes \mathfrak{C}_t$, $1 \leq n \leq m$ and $t > 1$, by the same arguments as in the proof of Lemma 2.15. Moreover, the Sylow 2-subgroups of $\mathcal{H}$ are TI-subgroups in $\mathrm{PSL}(2,2^f)$. Hence, $\mathcal{H}_G$ is not a dihedral group, because the involutions of a dihedral group generate their own centralizers. This gives $\mathcal{U}_\infty = \mathcal{U}_1$ or $\mathcal{U}_\infty = \mathcal{U}_1 \uplus \{E\}$.
It remains to determine whether $\{E\}$ is in $\mathcal{U}_1$. We count the elements G, which give rise to a nontrivial intersection $\mathcal{H}_G$. This is done as in Lemma 2.16. For f/m is odd, this leads to a similar inequality as in case p odd, implying $\{E\} \in \mathcal{U}_1$. If f/m is even, we have $|N(\mathcal{C})| = 2 \cdot (2^f - 1)$ and $|N_{N(\mathcal{H})}(\mathcal{C})| = 2 \cdot (2^m \pm 1)$ for a cyclic subgroup $\mathcal{C} < \mathcal{H}$ of order $2^m \pm 1$. For this reason, we get an inequality, which is slightly different from the one for p odd, namely

$$\begin{aligned} & 2^{f+4m} + 2^{f+3m} + 2 \cdot 2^{f+2m} - 2^{f+m} - 2^f - 2^{5m} - 2 \cdot 2^{4m} + 2^{3m} < 2^{3f} - 2^f \\ \iff \quad & 2^{3f} - 2^{f+4m} - 2^{f+3m} - 2 \cdot 2^{f+2m} + 2^{f+m} + 2^{5m} + 2 \cdot 2^{4m} - 2^{3m} > 0. \end{aligned}$$

This inequality holds for $f > 2m$, whereas the left-hand side is 0 if $f = 2m$. Consequently, the depth of $\mathcal{H}$ in $\mathrm{PSL}(2,2^f)$ is 5 for $f = 2m$.
The same arguments as in Proposition 2.17 show that $\mathcal{H}$ has depth 3 in $\mathrm{PSL}(2,2^f)$ in case $f > 2m$. □

2.5 Abelian p-subgroups and semidirect products

In contrast to the calculation of the ordinary depths of the elementary-abelian p-subgroups, the determination of their combinatorial depths is anything but trivial. Solely, the depth of a subgroup of order p resp. p^f of $\mathrm{PSL}(2,p^f)$ can be given immediately. It is 3, since such a group is a nonnormal TI-subgroup of $\mathrm{PSL}(2,p^f)$.
Already by looking at the subgroups of order p^2 it becomes obvious why the calculation of the depths is difficult for the elementary-abelian subgroups. For $f > 2$ even, we find subgroups of the form $\mathfrak{C}_p^2 \rtimes \mathfrak{C}_{p^2-1}$ or $\mathfrak{C}_p^2 \rtimes \mathfrak{C}_{(p^2-1)/2}$. In Lemma 2.11 we showed that their Sylow p-subgroups are TI-subgroups of $\mathrm{PSL}(2,p^f)$, so $\mathrm{PSL}(2,p^f)$ possesses subgroups of order p^2 having depth 3. However, there also exist subgroups of order p^2, which are not TI-subgroups of $\mathrm{PSL}(2,p^f)$: Take a Sylow p-subgroup $\mathcal{P} < \mathrm{PSL}(2,p^f)$, $X \in \mathcal{P}$ and $G \in N(\mathcal{P})$ of order $> p^2 - 1$. Then, the orbit of X under G has length $> p^2 - 1$, so $\langle X, X^G \rangle$ is a subgroup of order p^2, which is certainly no TI-subgroup. Corollary 2.21 now yields that these subgroups have depth 4 in $\mathrm{PSL}(2,q)$. Thus, there are subgroups of order p^2 in $\mathrm{PSL}(2,p^f)$ having depth 3 (i.e. the TI-subgroups) as well as subgroups having depth 4 for $f > 2$ even. For a given subgroup $\mathcal{Q}$ of order p^2, it is just the question whether $\mathcal{Q}$ is a TI-subgroup in the normalizer of the Sylow p-subgroup containing $\mathcal{Q}$. Nevertheless, it does not seem to be easy to decide whether $\mathcal{Q}$ is a TI-subgroup in $\mathrm{PSL}(2,p^f)$ or not.
In the general case, we will only give bounds for the depths. Recall that if A and B are subgroups of a finite group G, then $|AB| = |A||B|/|A \cap B|$. Thus, if $\mathcal{P}$ is a Sylow p-subgroup of $\mathrm{PSL}(2,p^f)$ with $\mathcal{H}_1, \mathcal{H}_2 < \mathcal{P}$ of order p^{m_1} resp. p^{m_2}, then $|\mathcal{H}_1 \cap \mathcal{H}_2| = p^{m_1}p^{m_2}/|\mathcal{H}_1\mathcal{H}_2| \geq p^{m_1+m_2-f}$.
Let $\lceil x \rceil$ denote the least integer not smaller than x for $x \in \mathbb{R}$.

Theorem 2.20. *Let $\mathcal{H} < \mathrm{PSL}(2,p^f)$ be a subgroup of order p^m, $1 < m < f$, which is not a TI-subgroup. Moreover, let $|N(\mathcal{H})| \in \{p^f(p^r-1), p^f(p^r-1)/2\}$. Then,*

$$2\left\lceil \frac{f-r}{f-m} \right\rceil \leq d_c(\mathcal{H}, \mathrm{PSL}(2,p^f)) \leq 2\frac{m}{r}$$

and $d_c(\mathcal{H}, \mathrm{PSL}(2,p^f))$ is even.

Proof. At first, we show that the depth of $\mathcal{H}$ is even. If $\mathcal{U}_{i-2} \neq \mathcal{U}_{i-1} = \mathcal{U}_i$ for some $i \geq 2$ and $\mathcal{H}_{G_1,\dots,G_{i-1}} \notin \mathcal{U}_{i-2}$, then G_1 does not lie in the centralizer of any nontrivial element of $\mathcal{H}$. Hence, there are no elements $\widetilde{G}_1, \dots, \widetilde{G}_{i-1} \in \mathrm{PSL}(2,p^f)$ satisfying $\mathcal{H}_{\widetilde{G}_1,\dots,\widetilde{G}_{i-1}} = \mathcal{H}_{E,G_1,\dots,G_{i-1}}$ as well as $X^{E^{-1}} = X^{\widetilde{G}_1^{-1}}$ for $X \in \mathcal{H} \setminus \{E\}$.
In the following, we may assume $\mathcal{H} < \mathcal{P}$, where $\mathcal{P}$ is the Sylow p-subgroup of $\mathrm{PSL}(2,p^f)$, which consists of lower triangular matrices whose diagonal entries are 1. We have

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x+y & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}^{\lambda} = \begin{pmatrix} 1 & 0 \\ \lambda x & 1 \end{pmatrix} \quad \text{as well as}$$
$$\begin{pmatrix} a^{-1} & 0 \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^2 x & 1 \end{pmatrix},$$

where $x, y, b \in \mathbb{F}_q$, $a \in \mathbb{F}_q^{\times}$, and $\lambda \in \mathbb{F}_p$. That means that the lower left entries of $\mathcal{P}$ form an $\mathbb{F}_p$-vector space $\mathcal{V}$, and we have a natural isomorphism $\mathcal{P} \cong \mathcal{V}$. In this context, the lower left entries of the matrices of $\mathcal{H}$ can be seen as a subspace $\mathcal{W}$ of $\mathcal{V}$. Moreover, conjugation with an element $G := \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \in N(\mathcal{P})$ induces a linear map $\gamma : \mathcal{V} \to \mathcal{V}, x \mapsto a^2 x$, with $\gamma(\mathcal{W}) = \mathcal{W}$, i.e. $a^2 \in \mathcal{W}$, if and only if $G \in N(\mathcal{P})$. So, we may think of $\mathcal{W}$ as an $\mathbb{F}_p(a^2)$-vector space. The factor group $N(\mathcal{H})/\mathcal{P}$ is isomorphic to the subgroup of $N(\mathcal{H})$, which consists of the diagonal matrices. This implies that $\mathcal{W}$ is a vector space over

$$\mathbb{E} := \mathbb{F}_p\left(\left\{\alpha^2 : \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in N(\mathcal{H})\right\}\right).$$

Now, we choose a generator $\begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$ of $\left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in N(\mathcal{H})\right\}$ and obtain $\mathbb{E} = \mathbb{F}_p(\theta^2)$. The order of $N(\mathcal{H})$ yields that θ is a root of unity of order $2(p^r - 1)$, $p^r - 1$ or $(p^r - 1)/2$ in $\mathbb{F}_{p^f}$. Therefore, θ^2 has order $p^r - 1$, $(p^r - 1)/2$ or $(p^r - 1)/4$. In any case, we get $\mathbb{E} = \mathbb{F}_p(\theta^2) \cong \mathbb{F}_{p^r}$. Certainly, $\mathcal{V}$ is also an $\mathbb{E}$-vector space, since r divides f.
The intersection of $\mathcal{W}$ with its image under a linear transformation γ is again an $\mathbb{E}$-vector space. Thus, this intersection contains $(p^r)^j$ elements for some $j \in \{0, \dots, m/r\}$. This leads to $|\mathcal{H}_G| = p^{jr}$. So, any subgroup of $\mathcal{U}_\infty$ has an order which is a power of p^r. It follows $\mathcal{U}_{m/r} = \mathcal{U}_\infty$. Since $\{E\}$ is already in $\mathcal{U}_1$, we even get $\mathcal{U}_{m/r-1} = \mathcal{U}_{m/r}$, and the minimal combinatorial depth of $\mathcal{H}$ in $\mathrm{PSL}(2,p^f)$ is at most $2m/r$.
Now, we look at the lower bound of $d_c(\mathcal{H}, \mathrm{PSL}(2,p^f))$. Since $\mathcal{P}$ is a TI-subgroup, two elements of $\mathcal{P}$, which are conjugate in $\mathrm{PSL}(2,p^f)$, are already conjugate in $N(\mathcal{P})$. Thus, for p odd, there are only two conjugacy classes of elements of order p in $N(\mathcal{P})$. Both of these classes contain $(p^f - 1)/2 > p^{f-1}$ elements, so any proper nontrivial subgroup of $\mathcal{P}$ is nonnormal in $N(\mathcal{P})$. If $p = 2$, then all elements of order 2 are conjugate in $N(\mathcal{P})$, so $\{E\}$ and $\mathcal{P}$ are the only normal subgroups of $N(\mathcal{P})$ again. In conclusion, we have $\mathrm{Core}_{N(\mathcal{P})}(\mathcal{H}) = \{E\}$.
Moreover, we can show that there is a subgroup of order p^r in $\mathcal{U}_\infty$. Suppose Q has minimal order amongst the elements of $\mathcal{U}_\infty \setminus \{E\}$, that means $|\mathcal{Q}| = p^{kr}$ for some positive integer k. Then, $\mathcal{Q}$ is a TI-subgroup of $\mathrm{PSL}(2,p^f)$. The minimality of $\mathcal{Q}$ implies that the conjugates of $\mathcal{Q}$ either lie in $\mathcal{H}$ or intersect $\mathcal{H}$ trivially. Hence, $\mathcal{H}$ is a union of several conjugates of $\mathcal{Q}$. Now, the normalizers of $\mathcal{Q}$ and all of its conjugates in $N(\mathcal{P})$ coincide and contain elements of order $(p^{kr} - 1)/2$, since the conjugates of $\mathcal{Q}$ are TI-subgroups. That is why each element of $N(\mathcal{Q})$ also lies in $N(\mathcal{H})$, so $k = 1$ by the order of $N(\mathcal{H})$.
Otherwise, for $G \in N(\mathcal{P})$, we get $|\mathcal{H}_G| \geq p^{2m-f}$ by the remark above. The iteration of this leads to $|\mathcal{H}_{G_1,\dots,G_{j-1}}| \geq p^{jm-(j-1)f} = p^{f+j(m-f)}$ for $G_1, \dots, G_{j-1} \in N(\mathcal{P})$. So, if $\mathcal{H}_{G_1,\dots,G_{j-1}}$ is a subgroup of order p^r, then $f + j(m-f) \leq r$ and $j \geq (f-r)/(f-m)$ must hold. Furthermore, $\{E\} \in \mathcal{U}_1$ because $\mathcal{P}$ is not normal in $\mathrm{PSL}(2,p^f)$. Hence, $\mathcal{U}_{j-1} = \mathcal{U}_j$ can hold only for $j \geq (f-r)/(f-m)$, that means $2\lceil (f-r)/(f-m) \rceil - 1 \leq d_c(\mathcal{H}, \mathrm{PSL}(2,p^f))$, and since the depth is even, it is at least $2\lceil (f-r)/(f-m) \rceil$. □

Corollary 2.21. *(i) Let $\mathcal{Q} < \mathrm{PSL}(2,p^f)$ be a subgroup of order p^2, which is not a TI-subgroup. Then, $\mathcal{Q}$ has depth 4 in $\mathrm{PSL}(2,p^f)$.*

(ii) Let $\mathcal{H} < \mathrm{PSL}(2,p^f)$ be a subgroup of order p^{f-1} and $f \geq 3$. Then, the depth of $\mathcal{H}$ in $\mathrm{PSL}(2,p^f)$ is $2f-2$.

(iii) Let $\mathcal{H} < \mathrm{PSL}(2,p^f)$ be a subgroup of order p^{f-2} and $f \geq 6$. If $(p^2-1)/2$ divides $|N(\mathcal{H})|$, then the depth of $\mathcal{H}$ in $\mathrm{PSL}(2,p^f)$ is $f-2$.

With the above assertions, the depth of any p-subgroup of $\mathrm{PSL}(2,p^f)$ for $f \leq 4$ can be determined. However, the problem to determine the depth of an arbitrary p-subgroup of $\mathrm{PSL}(2,p^f)$ for $f \geq 5$ is still open.
The problems in the determination of the depths of semidirect products $\mathfrak{C}_p^m \rtimes \mathfrak{C}_t$ in $\mathrm{PSL}(2,p^f)$ are essentially the same as for the depths of elementary-abelian p-groups. In fact, we will show that the depth of such a semidirect product in $\mathrm{PSL}(2,p^f)$ is determined once the depth of its Sylow p-subgroup is known. We assume $t > 1$ in the following. The calculation of the depth of a subgroup $\mathcal{H} = \mathcal{Q} \rtimes \mathcal{C} \cong \mathfrak{C}_p^f \rtimes \mathfrak{C}_t$ is comparatively simple. For $G \in \mathrm{PSL}(2,p^f)$, the intersection $\mathcal{H}_G$ is not trivial, since the ordinary depth of $\mathcal{H}$ in $\mathrm{PSL}(2,p^f)$ is 5 [4]. Thus, $\mathcal{H}_G$ is either $\mathcal{H}$, $\mathcal{Q}$, or isomorphic to $\mathfrak{C}_t$, since $\mathcal{Q}$ and $\mathcal{C}$ are TI-subgroups in $\mathrm{PSL}(2,p^f)$. This implies $\mathcal{U}_2 = \mathcal{U}_1 \uplus \{E\}$, so the depth of $\mathcal{H}$ in $\mathrm{PSL}(2,p^f)$ is 5.
Another case, which can be handled easily, is the case of a subgroup $\mathcal{H} = \mathcal{Q} \rtimes \mathcal{C} \cong \mathfrak{C}_p^m \rtimes \mathfrak{C}_t$ with a TI-subgroup $\mathcal{Q}$ of order $m < f$. Let $\mathcal{P}$ be the Sylow p-subgroup of $\mathrm{PSL}(2,p^f)$ containing $\mathcal{Q}$. From [4] we obtain $\{E\} \in \mathcal{U}_1$, whence $\mathcal{U}_\infty = \mathcal{U}_1$. Now, the length of the conjugacy class of $\mathcal{C}$ in $N(\mathcal{P})$ is p^f, since no element of order p lies in $N(\mathcal{C})$. This implies that $N(\mathcal{Q}) \setminus N(\mathcal{H})$ is not empty. Otherwise, any element of $N(\mathcal{Q}) \setminus \mathcal{P}$ lies already in $N(\mathcal{H})$, since $N(\mathcal{Q})$ is a semidirect product of $\mathcal{Q}$ with a cyclic group. So, if $E \neq G_1 \in \mathcal{C}$ and $G_2 \in N(\mathcal{Q}) \setminus N(\mathcal{H})$, there is no $G \in \mathrm{PSL}(2,p^f)$, such that $\mathcal{H}_{G_1,G_2} = \mathcal{H}_G$ and $X^{G_1^{-1}} = X^{G^{-1}}$ for $E \neq X \in \mathcal{Q}$. Thus, $\mathcal{H}$ has depth 4 in $\mathrm{PSL}(2,p^f)$.
Now, we proceed to these semidirect products, whose Sylow p-subgroups are not TI-subgroups of $\mathrm{PSL}(2,p^f)$.

Theorem 2.22. *Let $\mathcal{H} = \mathcal{Q} \rtimes \mathcal{C} \cong \mathfrak{C}_p^m \rtimes \mathfrak{C}_t$ be a subgroup of $\mathrm{PSL}(2,p^f)$, where $\mathcal{Q}$ is not a TI-subgroup of $\mathrm{PSL}(2,p^f)$ and $t > 1$. Moreover, let $|N(\mathcal{Q})| \in \{p^f(p^r-1), p^f(p^r-1)/2\}$. Then*

$$d_c(\mathcal{H}, \mathrm{PSL}(2,p^f)) = \begin{cases} d_c(\mathcal{Q}, \mathrm{PSL}(2,p^f)) + 1, & \text{if } d_c(\mathcal{Q}, \mathrm{PSL}(2,p^f)) = 2(f-r)/(f-m), \\ d_c(\mathcal{Q}, \mathrm{PSL}(2,p^f)), & \text{else.} \end{cases}$$

Proof. Let $\mathcal{P}$ be the Sylow p-subgroup of $\mathrm{PSL}(2,p^f)$, which contains $\mathcal{Q}$. Since $N(\mathcal{P})$ is a Frobenius group, it is clear that we can find some $P \in \mathcal{P}$, such that $\mathcal{H}_P = \mathcal{Q}$. Therefore, we conclude $\mathrm{Core}_{N(\mathcal{P})}(\mathcal{H}) = \{E\}$ as in the proof of Theorem 2.20. Moreover, $\mathcal{C}$ is a TI-subgroup, whence the intersection of $\mathcal{H}$ and some of its conjugates has order p^s resp. $p^s t$ for an $s \in \mathbb{N}_0$.
We may assume that $\mathcal{P}$ consists of the lower triangular matrices with diagonal entries being 1. The set of lower left entries of these matrices forms the field $\mathbb{F}_{p^f}$, which we will consider as $\mathbb{F}_p$-vector space $\mathcal{V}$ in the following. This yields an isomorphism $\varphi : \mathcal{P} \to \mathcal{V}$, $XY \mapsto x+y$ for $X, Y \in \mathcal{P}$ and $x, y \in \mathbb{F}_{p^f}$, such that $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ resp. $Y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$. Moreover, the conjugation by an element $G := \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \in N(\mathcal{P})$ induces the multiplication by a^2, i.e. $\gamma(x) := \varphi(X^G) = a^2 x$. Obviously, the map γ corresponding to G is linear.
At first, let $\widetilde{\mathcal{C}} < \mathcal{H}$ be a conjugate of $\mathcal{C}$ and $\{E\} < \mathcal{R} = \mathcal{Q}_{G_1,\ldots,G_i}$ where $G_1 = \begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \ldots,$ $G_i = \begin{pmatrix} a_i & 0 \\ b_i & a_i^{-1} \end{pmatrix}$. Then, $\mathcal{R}$ does not depend on the entries $b_1, \ldots, b_i$. Moreover, we find

$\tilde{b}_1, \ldots, \tilde{b}_i$, such that $\widetilde{G}_1 := \begin{pmatrix} a_1 & 0 \\ \tilde{b}_1 & a_1^{-1} \end{pmatrix}, \ldots, \widetilde{G}_i := \begin{pmatrix} a_i & 0 \\ \tilde{b}_i & a_i^{-1} \end{pmatrix}$ lie in the same maximal cyclic subgroup of $N(\mathcal{P})$ as $\widetilde{C}$. That is why $\mathcal{H}_{\widetilde{G}_1,\ldots,\widetilde{G}_i} = \mathcal{R} \rtimes \widetilde{\mathcal{C}}$, so $\mathcal{R} \rtimes \widetilde{\mathcal{C}}$ lies in $\mathcal{U}_i(\mathcal{H}, \mathrm{PSL}(2,p^f))$ if $\mathcal{R} \in \mathcal{U}_i(\mathcal{Q}, \mathrm{PSL}(2,p^f))$. Otherwise, since $\mathcal{Q} \in \mathcal{U}_1(\mathcal{H}, \mathrm{PSL}(2,p^f))$, it is clear that each p-subgroup of $\mathcal{H}$, which lies in $\mathcal{U}_i(\mathcal{Q}, \mathrm{PSL}(2,p^f))$, is contained in $\mathcal{U}_{i+1}(\mathcal{H}, \mathrm{PSL}(2,p^f))$. Hence, we get $\mathcal{U}_{i+1}(\mathcal{H}, \mathrm{PSL}(2,p^f)) = \mathcal{U}_\infty(\mathcal{H}, \mathrm{PSL}(2,p^f))$ if $\mathcal{U}_i(\mathcal{Q}, \mathrm{PSL}(2,p^f)) = \mathcal{U}_\infty(\mathcal{Q}, \mathrm{PSL}(2,p^f))$.
Let j denote the least i, such that $\mathcal{U}_i(\mathcal{H}, \mathrm{PSL}(2,p^f)) = \mathcal{U}_\infty(\mathcal{H}, \mathrm{PSL}(2,p^f))$, and let k be the least i, such that $\mathcal{U}_i(\mathcal{Q}, \mathrm{PSL}(2,p^f)) = \mathcal{U}_\infty(\mathcal{Q}, \mathrm{PSL}(2,p^f))$. If j and k coincide, then the same argument as in Theorem 2.20 yields $d_c(\mathcal{H}, \mathrm{PSL}(2,p^f)) = d_c(\mathcal{Q}, \mathrm{PSL}(2,p^f))$.
On the other hand, if $j = k+1$, then there are only p-subgroups in $\mathcal{U}_j(\mathcal{H}, \mathrm{PSL}(2,p^f)) \setminus \mathcal{U}_{j-1}(\mathcal{H}, \mathrm{PSL}(2,p^f))$. Take $G_1, \ldots, G_{j+1}$, such that $\mathcal{H}_{G_1,\ldots,G_{j+1}} \in \mathcal{U}_j(\mathcal{H}, \mathrm{PSL}(2,p^f)) \setminus \mathcal{U}_{j-1}(\mathcal{H}, \mathrm{PSL}(2,p^f))$. The case $G_1 \notin N(\mathcal{H})$ is trivial, so we may assume $G_1 \in N(\mathcal{H})$, whence $G_1 = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$. Since $\mathcal{Q} < \mathcal{P}$, we find some $\tilde{b} \in \mathbb{F}_{p^f}$, such that $\widetilde{G}_1 = \begin{pmatrix} a & 0 \\ \tilde{b} & a^{-1} \end{pmatrix} \notin N(\mathcal{H})$. This gives $X^{G_1^{-1}} = X^{\widetilde{G}_1^{-1}}$ for any $X \in \mathcal{H}_{G_1,\ldots,G_{j+1}}$ and $\mathcal{H}_{\widetilde{G}_1} < \mathcal{H}$. However, $\mathcal{Q}^{G_1} = \mathcal{Q}^{\widetilde{G}_1}$, because the maps γ_1 and $\tilde{\gamma}_1$ are the same on $\mathcal{V}$, so $\mathcal{H}_{\widetilde{G}_1} = \mathcal{Q}$. Furthermore, since $\mathcal{U}_\infty(\mathcal{Q}, \mathrm{PSL}(2,p^f)) = \mathcal{U}_{j-1}(\mathcal{Q}, \mathrm{PSL}(2,p^f))$, we find $\widetilde{G}_2, \ldots, \widetilde{G}_j$, such that $\mathcal{H}_{G_1,\ldots,G_{j+1}} = \mathcal{Q}_{\widetilde{G}_2,\ldots,\widetilde{G}_j}$. Therefore, $\mathcal{H}_{\widetilde{G}_1,\ldots,\widetilde{G}_j} = \mathcal{H}_{G_1,\ldots,G_{j+1}}$, and we conclude that $\mathcal{H}$ has depth $2j+1$ in $\mathrm{PSL}(2,p^f)$. From Theorem 2.20 we know that the depth of $\mathcal{Q}$ in $\mathrm{PSL}(2,p^f)$ is $2j$, that means $d_c(\mathcal{H}, \mathrm{PSL}(2,p^f)) = d_c(\mathcal{Q}, \mathrm{PSL}(2,p^f)) + 1$.
It remains to decide whether $j = k$. To do this, we consider affine subspaces of $\mathcal{V}$ as follows: Suppose $H = \begin{pmatrix} c & 0 \\ d & c^{-1} \end{pmatrix} \in \mathcal{H}$ with $c \neq 1$ and take $G = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$. Then

$$(H\mathcal{Q})^G = \left\{ \begin{pmatrix} c & 0 \\ a^2 d + ab(c^{-1} - c) + a^2 c^{-1} x & c^{-1} \end{pmatrix} : \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \mathcal{Q} \right\}.$$

Thus, the lower left entries of $(H\mathcal{Q})^G$ form an affine subspace $\mathcal{W}$ of $\mathcal{V}$ of dimension m.
Now, let $\mathcal{Q}_{G_1,\ldots,G_k} \in \mathcal{U}_k(\mathcal{Q}, \mathrm{PSL}(2,p^f)) \setminus \mathcal{U}_{k-1}(\mathcal{Q}, \mathrm{PSL}(2,p^f))$. Suppose, there exists an $i \in \{1, \ldots, k\}$, such that

$$\frac{|\mathcal{Q}_{G_1,\ldots,G_{i-1}}|}{|\mathcal{Q}_{G_1,\ldots,G_i}|} < p^{f-m}$$

(if $i = 1$, the numerator is $|\mathcal{Q}|$), and choose i to be minimal with respect to this property. The corresponding affine subspace of $(H\mathcal{Q})_{G_1,\ldots,G_{i-1}}$ is called $\mathcal{W}_1$ and, similarly, $\mathcal{W}_2$ corresponds to $(H\mathcal{Q})^{G_i}$. By our assumption on i, we get $\dim \mathcal{W}_1 = im - (i-1)f$ and $\dim(\mathcal{W}_1 \cap \mathcal{W}_2) > (i+1)m - if$. Hence,

$$\dim(\mathcal{W}_1 + \mathcal{W}_2) = \dim \mathcal{W}_1 + \dim \mathcal{W}_2 - \dim(\mathcal{W}_1 \cap \mathcal{W}_2) < im - (i-1)f + m - (i+1)m + if = f.$$

Moreover, the lower left entry b_i of G_i has no influence on $\mathcal{Q}^{G_i}$. However, the support vector of $\mathcal{W}_2$ depends on b_i. Since the dimension of $\mathcal{W}_1 + \mathcal{W}_2$ is smaller than the dimension of $\mathcal{V}$, there is a $\tilde{b}_i \in \mathbb{F}_{p^f}$, such that $\mathcal{W}_1$ and $\widetilde{\mathcal{W}}_2$ do not intersect. Thereby, $\widetilde{\mathcal{W}}_2$ is the affine subspace corresponding to $(H\mathcal{Q})^{\widetilde{G}_i}$ with $\widetilde{G}_i = \begin{pmatrix} a & 0 \\ \tilde{b}_i & a^{-1} \end{pmatrix}$. That means $(H\mathcal{Q})_{G_1,\ldots,G_{i-1}} \cap (H\mathcal{Q})^{\widetilde{G}_i} = \emptyset$. But two matrices of $N(\mathcal{P})$ are conjugate in $\mathrm{PSL}(2,p^f)$ if and only if their diagonal entries coincide. Thus, we even get $(H\mathcal{Q})_{G_1,\ldots,G_{i-1}} \cap \mathcal{H}^{\widetilde{G}_i} = \emptyset$. Since the subgroups of order t of $\mathcal{H}$ are TI-subgroups in $\mathrm{PSL}(2,p^f)$, and each of these subgroups contains an element of $H\mathcal{Q}$, we conclude $t \nmid |\mathcal{H}_{G_1,\ldots,G_{i-1},\widetilde{G}_i}|$. Hence, $\mathcal{Q}_{G_1,\ldots,G_k} = \mathcal{H}_{G_1,\ldots,G_{i-1},\widetilde{G}_i,G_{i+1},\ldots,G_k} \in \mathcal{U}_k(\mathcal{H}, \mathrm{PSL}(2,p^f))$.
Otherwise, assume $(|\mathcal{Q}_{G_1,\ldots,G_{i-1}}|)/(|\mathcal{Q}_{G_1,\ldots,G_i}|) = p^{f-m}$ for all $i = 1, \ldots, k$. Then, for each i, the dimension of the sum of the affine subspaces $\mathcal{W}_1$and $\mathcal{W}_2$ corresponding to $(H\mathcal{Q})_{G_1,\ldots,G_{i-1}}$ resp. $(H\mathcal{Q})^{G_i}$ is f. Thus, $\mathcal{W}_1$ and $\mathcal{W}_2$ intersect nontrivially, independently of the lower left entry of

G_i. That means $(H\mathcal{Q})_{G_1,\ldots,G_{i-1}} \cap (H\mathcal{Q})^{\widetilde{G}_i} \neq \emptyset$, so t divides $|\mathcal{H}_{G_1,\ldots,G_i}|$. Since this holds for all $i = 1, \ldots, k$, we obtain $\mathcal{Q}_{G_1,\ldots,G_k} \notin \mathcal{U}_k(\mathcal{H}, \mathrm{PSL}(2, p^f))$.
From Theorem 2.20 we know that the subgroups of $\mathcal{Q}$ of order p^r lie in $\mathcal{U}_k(\mathcal{Q}, \mathrm{PSL}(2, p^f))$, and there are no smaller nontrivial subgroups in $\mathcal{U}_\infty(\mathcal{Q}, \mathrm{PSL}(2, p^f))$. Therefore, $j \neq k$ if and only if $m-(k-1)(f-m) = r$, i.e. $(f-r)/(f-m) = k \in \mathbb{N}$, and $\mathcal{U}_k(\mathcal{Q}, \mathrm{PSL}(2, p^f)) = \mathcal{U}_\infty(\mathcal{Q}, \mathrm{PSL}(2, p^f))$. But this is just another description of $d_c(\mathcal{Q}, \mathrm{PSL}(2, p^f)) = 2(f-r)/(f-m)$, since the depth of $\mathcal{Q}$ is even. □

Corollary 2.23. *(i) Let $\mathcal{H} \cong \mathfrak{C}_p^2 \rtimes \mathfrak{C}_t$ be a subgroup of $\mathrm{PSL}(2, p^f)$, whose Sylow p-subgroup is not a TI-subgroup of $\mathrm{PSL}(2, q)$. Then, $\mathcal{H}$ has depth 4 in $\mathrm{PSL}(2, p^f)$.*

(ii) Let $\mathcal{H} \cong \mathfrak{C}_p^{f-1} \rtimes \mathfrak{C}_t$ be a subgroup of $\mathrm{PSL}(2, p^f)$ with $f \geq 3$ and $t > 1$. Then, the depth of $\mathcal{H}$ in $\mathrm{PSL}(2, p^f)$ is $2f-1$.

(iii) Let $\mathcal{H} \cong \mathfrak{C}_p^{f-2} \rtimes \mathfrak{C}_t$ be a subgroup of $\mathrm{PSL}(2, p^f)$ with $f \geq 6$ and $t > 1$. If $(p^2-1)/2$ divides $|N(\mathcal{H})|$, then the depth of $\mathcal{H}$ in $\mathrm{PSL}(2, p^f)$ is $f-1$.

2.6 Conclusion

While the depths of subgroups considered in Sections 2.1–2.4 are comparatively small, we get interesting results in section 2.5. Combining these results, we conclude that every positive integer occurs as the minimal combinatorial depth of a subgroup $\mathcal{H}$ in $\mathrm{PSL}(2, q)$ for suitable q and $\mathcal{H}$. In particular, Corollary 2.21 shows, how to choose q and $\mathcal{H}$, such that a given even integer ≥ 4 equals $d_c(\mathcal{H}, \mathrm{PSL}(2, q))$. Up to now, only the similar result for the positive odd integers has been known [1].
Moreover, we also can answer the following question: Suppose, we have subgroups $H < K$ of a given group G. If we have $d_0(H, G) < d_0(K, G)$ for the ordinary depths of H and K in G, can we bound $d_c(H, G)$ by $d_c(K, G)$ or at least a multiple of $d_c(K, G)$? The results for the ordinary and minimal combinatorial depths of semidirect products in $\mathrm{PSL}(2, q)$ give a negative answer to this question: Let $q = p^f$ with $f > 3$, $\mathcal{P}$ be a Sylow p-subgroup of $\mathrm{PSL}(2, p^f)$ and $\mathcal{Q} < \mathcal{P}$ be of order p^{f-1}. From [4] we obtain $d_0(N(\mathcal{Q}), \mathrm{PSL}(2, q)) = 3 < 5 = d_0(N(\mathcal{P}), \mathrm{PSL}(2, q))$. However, the minimal combinatorial depth of $N(\mathcal{P})$ in $\mathrm{PSL}(2, q)$ is also 5 by the paragraph previous to Theorem 2.22, whereas $N(\mathcal{Q})$ has depth $2f-1$ in $\mathrm{PSL}(2, p^f)$ by Corollary 2.23. This gets in line with the questions in [1, 5.2], which are answered negatively as well.
The problem of the determination of the depth of an arbitrary elementary-abelian subgroup of $\mathrm{PSL}(2, p^f)$ is not solved for $f \geq 5$. I conjecture that the lower bound in Theorem 2.20 coincides with the minimal combinatorial depth, that means, the depth of an elementary-abelian subgroup $\mathcal{H} < \mathrm{PSL}(2, p^f)$ of order p^m $(1 < m < f)$, whose normalizer is of order $p^f(p^r-1)$ resp. $p^f(p^r-1)/2$, would be $2\lceil (f-r)/(f-m) \rceil$. Unfortunately, I have not been able to prove this yet.

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References

[1] R. Boltje, S. Danz, B. Külshammer, *On the depth of subgroups and group algebra extensions*, J. Algebra **335** (2011) 258–281.

[2] S. Burciu, L. Kadison, B. Külshammer, *On subgroup depth*, Int. Electron. J. Algebra **9** (2011) 133–166.

[3] T. Fritzsche, *Die Tiefe von Untergruppen von* $\mathrm{PSL}(2,q)$, Diploma thesis, Friedrich Schiller University Jena, unpublished.

[4] T. Fritzsche, *The depth of subgroups of* $\mathrm{PSL}(2,q)$, J. Algebra **349** (2012) 217–233.

[5] F. Goodman, P. de la Harpe, V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer, New York, 1989.

[6] B. Huppert, *Endliche Gruppen I*, Springer, Berlin 1967.

[7] L. Kadison, *Finite depth and Jacobson-Bourbaki correspondence*, J. Pure Appl. Algebra **212** (2008) 1822–1839.