The Brauer group of character rings

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Abstract

Let G be a finite group and $R(G)$ be its character ring. Since $R(G)$ is a finite $\mathbb{Z}$-algebra, it is known that the Brauer group of the character ring is a finite sum of copies of $\mathbb{Z}/2\mathbb{Z}$. In this note, we determine the Brauer group of the character ring $R(G)$, on showing how it is related to the conjugacy classes of G and the values of the irreducible characters on these classes.

1 Introduction

The Brauer group was first introduced as an invariant of an algebraic number field. Auslander and Goldman generalized this by defining the Brauer group $\mathrm{Br}(R)$ of a commutative ring R [1]. The elements of $\mathrm{Br}(R)$ are equivalence classes of central separable R-algebras, i.e. Azumaya algebras, where two Azumaya algebras A_1, A_2 are said to be equivalent, if there are finitely generated projective faithful R-modules M_1, M_2, such that $A_1 \otimes_R \mathrm{End}_R(M_1) \cong A_2 \otimes_R \mathrm{End}_R(M_2)$. The group structure is then induced by the tensor product of algebras. Overviews of the properties of the Brauer group can be found in [6, 8].

This short article is motivated by [10], where it is shown that the Brauer group of the Burnside ring of a finite group G vanishes. Here, we determine the Brauer group $\mathrm{Br}(R(G))$ of the character ring. The set of $\mathbb{Z}$-linear combinations of complex characters of G is called the character ring $R(G)$ of G, where addition is defined in the canonical way and multiplication is induced by the tensor product of characters. Another formulation is that $R(G)$ is the Grothendieck ring of the semiring of complex characters of G equipped with the above-mentioned addition and multiplication.

The determination of $\mathrm{Br}(R(G))$ is done in Section 4. Prior to this, we take a short look at the maximal order of $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$ in Section 2, and at $\mathbb{Q}$-classes of G in Section 3.

2 The maximal order $R(G)'$

It follows immediately from the definition in Section 1 that $R(G)$ is a free $\mathbb{Z}$-module with basis $\mathrm{Irr}(G)$, the irreducible complex characters of G. Consequently, $R(G)$ is a $\mathbb{Z}$-order in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$. A survey of the theory of orders is given in [4]. It is known that there exists a unique maximal order $R(G)'$ in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$ containing $R(G)$, namely the integral closure of $\mathbb{Z}$ in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$. In other words, the order $R(G)'$ consists of those class functions of $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$, whose values on the elements of G are algebraic integers.

The gap between $R(G)$ and the maximal order $R(G)'$ is not very large. In particular, we have the following:

Lemma 2.1. *Let G, $R(G)$ and $R(G)'$ be as above. Then $|G| \cdot R(G)' \subseteq R(G)$.*

Proof. Let ε be a primitive $|G|$th root of unity and $\psi \in (\mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} R(G))'$ be a class function. It suffices to show $|G| \cdot \vartheta_g \in \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} R(G)$ for an arbitrary element $g \in G$, where

$$\vartheta_g(x) = \begin{cases} 1, & x \text{ is } G\text{-conjugate to } g, \\ 0, & \text{else} \end{cases}$$

since ψ can be written as a $\mathbb{Z}[\varepsilon]$-linear sum of various ϑ_h $(h \in G)$. The orthogonality relations yield

$$\vartheta_g = \frac{1}{|C_G(g)|} \sum_{\chi \in \mathrm{Irr(G)}} \chi(g^{-1})\chi \,.$$

Therefore $|G| \cdot \vartheta_g \in \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} R(G)$, whence $|G|(\mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} R(G))' \subseteq \mathbb{Z}[\varepsilon] \otimes_{\mathbb{Z}} R(G)$. Certainly, the last assertion implies $|G|R(G)' \subseteq R(G)$. □

3 $\mathbb{Q}$-classes of G

The idea of K-classes is due to Berman [2, 3]. Let K be a field of characteristic prime to the order of G, and ε be a primitive nth root of unity, where n is the exponent of G. We define two elements $g_1, g_2 \in G$ to be K-conjugate, if g_1 is G-conjugate to some g_2^h, where $\sigma(\varepsilon) = \varepsilon^h$ for a $\sigma \in \mathrm{Gal}(K(\varepsilon)/K)$, such that $\gcd(h, n) = 1$. A full set of K-conjugate elements of G is then called a K-class of G. In particular, the $\mathbb{Q}$-classes of G correspond to the conjugacy classes of cyclic subgroups of G.
We need some more definitions in the following. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the $\mathbb{Q}$-classes of G, and $C_1, \ldots, C_l$ be the conjugacy classes of G, such that $C_i \subseteq \mathcal{C}_i$. For an arbitrary element $x_i \in \mathcal{C}_i$, we set $\mathbb{Q}(\mathcal{C}_i) := \mathbb{Q}(\chi(x_i) : \chi \in \mathrm{Irr}(G))$. Clearly, this definition does not depend on the choice of x_i, and we have $\mathbb{Q}(\mathcal{C}_i) = \mathbb{Q}(C_i)$, if we define $\mathbb{Q}(C_i) := \mathbb{Q}(\chi(x_i) : \chi \in \mathrm{Irr}(G))$ with $x_i \in C_i$. Then, $\bigoplus_{i=1}^k \mathbb{Q}(\mathcal{C}_i)$ is a $\mathbb{Q}$-algebra in the natural way, and its maximal order is $S := \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{Q}(\mathcal{C}_i)}$, where $\mathcal{O}_{\mathbb{Q}(\mathcal{C}_i)}$ denotes the ring of algebraic integers in $\mathbb{Q}(\mathcal{C}_i)$.

4 The Brauer group of $R(G)$

In [10], the Brauer group of the Burnside ring $A(G)$ of G is not determined directly, but it is shown to be isomorphic to the Brauer group of the maximal order in $\mathbb{Q} \otimes_{\mathbb{Z}} A(G)$, i.e. the ghost ring, whose Brauer group is known. We will obtain the Brauer group of the character ring $R(G)$ by the same approach.
At first, we give another description of the maximal order $R(G)'$ in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$. Let S be as in Section 3.

Lemma 4.1. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the $\mathbb{Q}$-classes of G and $\{x_1, \ldots, x_k\}$ be a set of representatives of these $\mathbb{Q}$-classes where $x_i \in \mathcal{C}_i$ for $i = 1, \ldots, k$.*

(i) We have an isomorphism of $\mathbb{Q}$-algebras

$$\mathbb{Q} \otimes_{\mathbb{Z}} R(G) \to \bigoplus_{i=1}^k \mathbb{Q}(\mathcal{C}_i)\,, \quad a \otimes \psi \mapsto (a\psi(x_1), \ldots, a\psi(x_k))\,.$$

(ii) The map in (i) yields an isomorphism $R(G)' \cong S$.

Proof. (i) Obviously, this map is a homomorphism of $\mathbb{Q}$-algebras. Moreover, for $x \in \mathcal{C}_i$ we have $\psi(x) = \sigma(\psi(x_i))$ for a suitable $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mathcal{C}_i)/\mathbb{Q})$. Hence, the image of $a \otimes \psi$ is $(0, \ldots, 0)$ if and only if $a = 0$ resp. $\psi(x_i) = 0$ for all $i = 1, \ldots, k$, so the map is injective.

We will now show the surjectivity of this map. Let $\mathcal{C}$ be a union of conjugacy classes of G, such that $z^h \in \mathcal{C}$ for $\gcd(h, |G|) = 1$ whenever $z \in \mathcal{C}$. Set

$$\mathrm{Ch}(\mathcal{C}) := \left\{ \psi = \sum_{j=1}^{|\mathrm{Irr}(G)|} a_j \chi_j : a_j \in \mathbb{Z},\ \chi_j \in \mathrm{Irr}(G),\ \psi(z) = 0 \text{ for } z \notin \mathcal{C} \right\}.$$

Then, the $\mathbb{Z}$-rank of $\mathrm{Ch}(\mathcal{C})$ coincides with the number of conjugacy classes contained in $\mathcal{C}$ [9]. That is why the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ch}(\mathcal{C}_i)$ is the number of conjugacy classes which lie in $\mathcal{C}_i$.

Let $x_i \in C_i$ be as above and ζ_m be a primitive mth root of unity for $m \in \mathbb{N}$. Then, there is a canonical isomorphism between $\mathrm{Gal}(\mathbb{Q}(\zeta_{|\langle x_i \rangle|})/\mathbb{Q}(C_i))$ and $N_G(\langle x_i \rangle)/C_G(x_i)$, where $N_G(\langle x_i \rangle)$ is the normalizer of $\langle x_i \rangle$, and $C_G(x_i)$ is the centralizer of x_i in G [7]. Thus, we can write the equation $[\mathbb{Q}(\zeta_{|\langle x_i \rangle|}) : \mathbb{Q}] = [\mathbb{Q}(\zeta_{|\langle x_i \rangle|}) : \mathbb{Q}(C_i)] \cdot [\mathbb{Q}(C_i) : \mathbb{Q}]$ in the form $\varphi(|\langle x_i \rangle|) = |N_G(\langle x_i \rangle) : C_G(x_i)| \cdot [\mathbb{Q}(C_i) : \mathbb{Q}]$, where φ denotes Euler's phi function. Now, $\varphi(|\langle x_i \rangle|)$ corresponds to the number of elements of $\langle x_i \rangle$ which have the same order as x_i, and $|N_G(\langle x_i \rangle) : C_G(x_i)|$ equals the number of elements of $\langle x_i \rangle$ which are conjugate to x_i in G. Hence, $[\mathbb{Q}(\mathcal{C}_i) : \mathbb{Q}]$ is the number of conjugacy classes in $\mathcal{C}_i$, so $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ch}(\mathcal{C}_i)$ has dimension $[\mathbb{Q}(\mathcal{C}_i) : \mathbb{Q}]$.

Otherwise, $\mathbb{Q}(\mathcal{C}_i)$ is also a $\mathbb{Q}$-vector space of dimension $[\mathbb{Q}(\mathcal{C}_i) : \mathbb{Q}]$. Since we have already seen that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Ch}(\mathcal{C}_i)$ injects into $0 \oplus \cdots \oplus 0 \oplus \mathbb{Q}(\mathcal{C}_i) \oplus 0 \oplus \cdots \oplus 0$, the injection is, in fact, a bijection.

(ii) This is an immediate consequence of (i), since $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$ and $\bigoplus_{i=1}^k \mathbb{Q}(\mathcal{C}_i)$ are commutative $\mathbb{Q}$-algebras, whose maximal orders are $R(G)'$ resp. S. □

Theorem 4.2. *Let G be a finite group, $R(G)$ be its character ring and $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the $\mathbb{Q}$-classes of G. Moreover, we denote the number of real places of $\mathbb{Q}(\mathcal{C}_i)$ by r_i. Then*

$$\mathrm{Br}(R(G)) \cong \bigoplus_{i=1}^{k} \bigoplus_{m=1}^{r_i - 1} \mathbb{Z}/2\mathbb{Z}\,.$$

Proof. Identifying $R(G)$ with its image in S, we obtain $|G|S \subseteq R(G)$ from Lemmas 2.1 and 4.1 (ii). Therefore, we can write $R(G)$ as a pullback

$$\begin{array}{ccc} R(G) & \longrightarrow & S \\ \downarrow & & \downarrow \\ R(G)/|G|S & \longrightarrow & S/|G|S \end{array}$$

with a surjective map $S \to S/|G|S$. It is clear that $S/|G|S$ is a finite commutative ring of dimension 0. Thus, we get an exact sequence

$$\mathrm{Pic}(S/|G|S) \to \mathrm{Br}(R(G)) \to \mathrm{Br}(R(G)/|G|S) \oplus \mathrm{Br}(S) \to \mathrm{Br}(S/|G|S)$$

by [5, Thm 3.1], with $\mathrm{Pic}(R)$ denoting the Picard group of a commutative ring R. Recall that the Picard group and the Brauer group of a finite commutative ring are trivial [10], so we deduce $\mathrm{Br}(R(G)) \cong \mathrm{Br}(S)$. From [5, proof of Thm 3.5] we know that the Brauer group of a direct product of domains is isomorphic to the direct product of the single Brauer groups, that means $\mathrm{Br}(S) \cong \bigoplus_{i=1}^k \mathrm{Br}(\mathcal{O}_{\mathbb{Q}(\mathcal{C}_i)})$. Now, it only remains to determine the Brauer group of the ring of algebraic integers of an algebraic number field. This is done in [8, Thm 6.36], from which we obtain $\mathrm{Br}(\mathcal{O}_{\mathbb{Q}(\mathcal{C}_i)}) \cong \bigoplus_{m=1}^{r_i-1} \mathbb{Z}/2\mathbb{Z}$, where r_i denotes the number of real places of $\mathbb{Q}(\mathcal{C}_i)$. (This coincides with the number of inequivalent real archimedean valuations of $\mathbb{Q}(\mathcal{C}_i)$.) Hence, we have

$$\mathrm{Br}(R(G)) \cong \bigoplus_{i=1}^{k} \bigoplus_{m=1}^{r_i - 1} \mathbb{Z}/2\mathbb{Z}$$

as desired. □

In particular, the Brauer group of the character ring of a symmetric group $\mathfrak{S}$ is trivial, since every conjugacy class of $\mathfrak{S}$ is a $\mathbb{Q}$-class (i.e. is a rational class). Furthermore, if $\mathcal{C}$ is a conjugacy class of an abelian group, then $\mathbb{Q}(\mathcal{C})$ is either $\mathbb{Q}$ or a cyclotomic field. Thus, $\mathbb{Q}(\mathcal{C})$ has at most one real place, and we deduce that the Brauer group of the character ring of an abelian group is trivial as well. This result can also be obtained from [5], where the Brauer group of the group ring of an abelian group is shown to be trivial.
However, there are examples of character rings with nontrivial Brauer groups. For instance, in the dihedral group $\mathfrak{D}_8$ of order 16, the elements of order 8 form a $\mathbb{Q}$-class $\mathcal{C}$ with $\mathbb{Q}(\mathcal{C}) = \mathbb{Q}(\sqrt{2})$. Since $\mathbb{Q}(\sqrt{2})$ has two real places, we get $\mathrm{Br}(\mathcal{O}_{\mathbb{Q}(\sqrt{2})}) \cong \mathbb{Z}/2\mathbb{Z}$ (and therefore $\mathrm{Br}(\mathfrak{D}_8) \cong \mathbb{Z}/2\mathbb{Z}$). In [6, III, Rmk 6.5], a nontrivial Azumaya algebra over $\mathbb{Z}[\sqrt{2}] = \mathcal{O}_{\mathbb{Q}(\sqrt{2})}$ is constructed explicitely. This gives rise to the nontrivial element of $\mathrm{Br}(R(\mathfrak{D}_8))$.

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