On vertices of completely splittable modules for symmetric groups and simple modules labelled by two part partitions

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Abstract

We prove that the vertices of a completely splittable module D for a finite symmetric group over an algebraically closed field of characteristic $p > 0$ are precisely the defect groups of the block containing D. Moreover, we also determine the vertices of simple modules for the symmetric groups in characteristic $p > 2$ which are labelled by two part partitions $(n-m, m)$ such that $m < p(p+1)/2$.

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1 Introduction

In 1959 A. J. Green [9] introduced the notion of *vertices* of indecomposable modules over group algebras. Given a finite group G and an algebraically closed field F of positive characteristic p, by a vertex of an indecomposable FG-module M we understand a subgroup P of G which is minimal subject to the condition that M is isomorphic to a direct summand of $\mathrm{Ind}_P^G(\mathrm{Res}_P^G(M))$. It is well-known that the vertices of M are p-subgroups of G and that they are determined up to G-conjugacy. The concept of vertices thus enables us to relate indecomposable and, in particular, simple FG-modules to modules for p-subgroups of G and normalizers of these subgroups which is one of the major concerns in modular representation theory. However, there are only very few general results on vertices of simple modules for arbitrary finite groups available in the literature. For this reason, it is worthwhile to focus on certain classes of finite groups and their simple modules first.

In this article we aim to determine the vertices of two classes of simple modules for the symmetric group $\mathfrak{S}_n$ of degree $n \in \mathbb{N}$ over an algebraically closed field F of characteristic $p > 0$. As usual, we denote the simple $F\mathfrak{S}_n$-module parametrized by the p-regular partition λ of n by D^λ. The first class of modules under consideration consists of the so-called *completely splittable* $F\mathfrak{S}_n$-modules, introduced by Kleshchev in [14]. Following Kleshchev, a simple $F\mathfrak{S}_n$-module D is completely splittable, provided the restriction of D to $\mathfrak{S}_l \leq \mathfrak{S}_n$ is semisimple, for all $l < n$. The class of completely splittable $F\mathfrak{S}_n$-modules in particular contains the *Fibonacci modules*, studied by Ryba [22]. In Section 3 we will prove that the vertices of a completely splittable $F\mathfrak{S}_n$-module are the defect groups of its block.

Besides the completely splittable modules, we will focus on simple $F\mathfrak{S}_n$-modules in characteristic $p > 2$, labelled by two part partitions. These had been studied before by B. Fotsing [7] and R. Zimmermann [23]. Zimmermann determined the vertices of simple $F\mathfrak{S}_n$-modules in odd characteristic whose labelling partitions are of the form $(n-m, m)$, for $m \leq 3$. Later, Fotsing extended those results to the cases where $m < 2p$, or $m = 2p$ and $p > 3$. In Section 4, we will generalize to the case where $m < p(p+1)/2$, and will show that the vertices of the simple $F\mathfrak{S}_n$-module $D^{(n-m,m)}$ are precisely the defect groups of the block containing $D^{(n-m,m)}$, unless $D^{(n-m,m)} \cong S^{(n-m,m)}$ where $S^{(n-m,m)}$ is the $F\mathfrak{S}_n$-Specht module corresponding to $(n-m,m)$. In the latter case, the vertices of $D^{(n-m,m)}$ are known, by Grabmeier's Theorem [8]. Our method of proof basically builds on extensive use of modular branching rules and a result due to K. Erdmann [5]. Erdmann's Theorem enables us to determine the dimensions of the simple modules in question, without computing any decomposition numbers. In combination with our results on completely splittable $F\mathfrak{S}_n$-modules, we are able to determine the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions, provided $n \leq 2p^2-1$ and $p>3$, or $p=3$ and $n \leq 26$.

It shall be said at this point that the situation in characteristic 2 is more complicated. For $p=2$, in [18] J. Müller and R. Zimmermann have shown that the simple $F\mathfrak{S}_n$-module $D^{(n-1,1)}$ has the defect groups of its block as its vertices, unless $n=4$. Namely, $D^{(3,1)}$ has the unique Sylow 2-subgroup of the alternating group $\mathfrak{A}_4$ as its vertex. But, already for $m=\lfloor\frac{p+1}{2}\rfloor p-1=2$, the information on the corresponding module $D^{(n-m,m)}=D^{(n-2,2)}$ is far too scarce to determine its vertices in a similar manner. Currently, Müller/Zimmermann's result seems to be the only general result on vertices of simple $F\mathfrak{S}_n$-modules corresponding to two part partitions, for $p=2$. Nevertheless, we can provide some computational data in characteristic 2. In [2] the vertices of the simple $F\mathfrak{S}_n$-modules $D^{(n-2,2)}$ have been computed, for $n \leq 47$. The results will also appear in [3], and they suggest that, in analogy to the case of odd characteristic, $D^{(n-2,2)}$ has the Sylow 2-subgroups of $\mathfrak{S}_n$ as vertices, unless $S^{(n-2,2)} \cong D^{(n-2,2)}$ or $n=5$. In the latter case the Sylow 2-subgroup of $\mathfrak{A}_4$ is a vertex of $D^{(n-2,2)}=D^{(3,2)}$. If $S^{(n-2,2)} \cong D^{(n-2,2)}$ then the Sylow 2-subgroups of $\mathfrak{S}_{n-4}\times\mathfrak{S}_2\times\mathfrak{S}_2$ are known to be vertices of $D^{(n-2,2)}$, by Grabmeier's Theorem.

The present paper is organized as follows: we begin by collecting together some general notation and known results on vertices of simple modules over group algebras. Furthermore, we recall the facts about $F\mathfrak{S}_n$-modules which will be necessary throughout the subsequent sections, including Erdmann's Theorem mentioned above. In Sections 3, 4 and 5 we present our results on vertices of completely splittable $F\mathfrak{S}_n$-modules and simple $F\mathfrak{S}_n$-modules parametrized by two part partitions. In what follows, F will always denote an algebraically closed field of characteristic $p>0$. Moreover, given a finite group G, any FG-module is understood to be a finitely generated left FG-module. Given any integer z such that z is divisible by p^l but not by p^{l+1}, we write $\nu_p(z)=l$.

Detailed information concerning the representation theory of the symmetric groups is given in [12] and [13] where we also take most of our notation from. Our computations in Section 5 have been carried out using the computer algebra system MAGMA [1]; descriptions of the algorithms actually used to perform these computations can be found in [2] and [23], and will also appear in [4].

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2 Preliminaries and notation

2.1 Vertices of simple modules

We begin by fixing some notation. Given a finite group G and FG-modules M and N such that N is isomorphic to a direct summand of M, we write $N|M$. Suppose that H is a subgroup of G such that $M|\operatorname{Ind}_H^G(\operatorname{Res}_H^G(M))$. Then M is said to be *relatively H-projective.* Provided M is indecomposable, a *vertex* of M is a subgroup P of G which is minimal with respect to the condition that M is relatively P-projective. A vertex of M is a p-subgroup of G, and is determined up to G-conjugacy. Moreover, for any vertex P of M, there are a Sylow p-subgroup R of G and a defect group Δ of the block containing M such that $P \leq \Delta \leq R$. The following facts about vertices of indecomposable modules are well-known, and will be used repeatedly throughout the next sections:

Lemma 2.1 ([20], L. 4.3.4, Thm. 4.7.5)**.** *Let M be an indecomposable FG-module with vertex $P \leq R$, for a Sylow p-subgroup R of G. Furthermore, let H be a subgroup of G, and let N be an indecomposable FH-module with vertex Q. Then*

(i) $|R:P| \,\big|\, \dim(M)$.

(ii) If $N|\operatorname{Res}_H^G(M)$ *then* $Q \leq_G P$.

(iii) If $M|\operatorname{Ind}_H^G(N)$ *then* $P \leq_G Q$.

Theorem 2.2 (Knörr [16], Thm. 3.3)**.** *Let D be a simple FG-module belonging to the block B of FG. Let further P be a vertex of D. Then there exists a block b of $F[PC_G(P)]$ such that $b^G = B$, and P is a defect group of b. Hence there is a defect group Δ of B such that $C_\Delta(P) \leq P \leq \Delta$.*

In the case of the symmetric groups even more is true. Therefore we now suppose that $G = \mathfrak{S}_n$, for some $n \in \mathbb{N}$, and that the block B has p-weight w. By [13], Thm. 6.2.45, the defect groups of B are exactly the $\mathfrak{S}_n$-conjugates of the Sylow p-subgroups of $\mathfrak{S}_{pw}$. Then the following lemma is a direct consequence of Knörr's Theorem and a result by Olsson (cf. [21], Prop. 1.4).

Lemma 2.3. *Let D be a simple $F\mathfrak{S}_n$-module belonging to a block B of p-weight w. Moreover, let $P \leq \mathfrak{S}_{pw}$ be a vertex of D. Then $C_{\mathfrak{S}_{pw}}(P) = Z(P)$. In particular, D is not relatively $\mathfrak{S}_{pw-1}$-projective.*

We finally mention what the Sylow p-subgroups of the symmetric group $\mathfrak{S}_n$ look like. For this, let $n = \sum_{i=0}^r \alpha_i p^i$ be the p-adic expansion of n. Furthermore, set $P_1 := \{1\}$, $P_p := \langle(1,\ldots,p)\rangle$, and $P_{p^i} := P_{p^{i-1}} \wr P_p$, for $i \geq 2$. When considered as a subgroup of $\mathfrak{S}_{p^i}$ in the obvious way, P_{p^i} is a Sylow p-subgroup of $\mathfrak{S}_{p^i}$, for all $i \in \mathbb{N}_0$. In general the Sylow p-subgroups of $\mathfrak{S}_n$ are precisely the $\mathfrak{S}_n$-conjugates of

$$P_n := P_{p^r}^{\alpha_r} \times \cdots \times P_p^{\alpha_1} \leq \mathfrak{S}_{p^r}^{\alpha_r} \times \cdots \times \mathfrak{S}_p^{\alpha_1} \leq \mathfrak{S}_n,$$

with different direct factors acting on disjoint subsets of $\{1,\ldots,n\}$. A proof for this is given in [13], 4.1.22, 4.1.24. We will keep the above notation, for the remainder of this article.

2.2 Modules for the symmetric groups

In this subsection, fix $n \in \mathbb{N}$. The set of partitions of n will be denoted by $\mathcal{P}_n$, and the subset of p-regular partitions of n by $\mathcal{P}_{n,p}$. Given $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathcal{P}_n$, we denote the corresponding $F\mathfrak{S}_n$-Specht module by S^λ. Moreover, we consider the *Young subgroup*

$$\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_s}$$

of $\mathfrak{S}_n$. Here $\mathfrak{S}_{\lambda_i}$ is understood to be acting on the set $\{\lambda_1 + \cdots + \lambda_{i-1} + 1, \dots, \lambda_1 + \cdots + \lambda_i\}$, for all $i = 1, \dots, s$. Then there is an indecomposable $F\mathfrak{S}_n$-module Y^λ which is unique up to isomorphism satisfying the following two conditions:

$$(1)\ Y^\lambda \mid M^\lambda \quad \text{and} \quad (2)\ S^\lambda \subseteq Y^\lambda,$$

where $M^\lambda := \operatorname{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(F)$ is the $F\mathfrak{S}_n$-permutation module corresponding to λ. This module Y^λ is called the *Young module* corresponding to the partition λ. Further details concerning Young modules can, for example, be found in [8] and in [17], Sec. 4.6. The following theorem, due to J. Grabmeier, determines the vertices of Y^λ.

Theorem 2.4 (Grabmeier [8], S. 7.8). *Let $n \in \mathbb{N}$, and let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of n such that $\lambda_s > 0$. Additionally, set $\lambda_{s+1} := 0$. Then the Sylow p-subgroups of the Young subgroup*

$$\prod_{i=1}^{s} (\mathfrak{S}_{\lambda_i - \lambda_{i+1}})^i$$

of $\mathfrak{S}_n$ are vertices of the $F\mathfrak{S}_n$-Young module Y^λ.

Remark 2.5. Note that in case $\lambda \in \mathcal{P}_{n,p}$ and $S^\lambda \cong D^\lambda$, we also have $S^\lambda \cong Y^\lambda$. This follows immediately from the fact that the module Y^λ has a Specht filtration

$$0 \subseteq S^\lambda = Y_k \subseteq Y_{k-1} \subseteq \ldots \subseteq Y_0 = Y^\lambda,$$

where $Y_j/Y_{j+1} \cong S^{\mu_j}$, for some $\mu_j \rhd \lambda$ and $j = 0, \dots, k-1$. Since Y^λ is, by construction, indecomposable and selfdual, this implies $Y^\lambda \cong D^\lambda \cong S^\lambda$. With this in mind, we will apply Grabmeier's Theorem several times in Sections 4 and 5. In order to determine when $S^\lambda \cong D^\lambda$ holds, we will make use of Carter's Criterion a proof of which can, for instance, be found in [13], Thm. 7.3.23:

Theorem 2.6 (Carter's Criterion). *Let λ be a p-regular partition of n. Then the corresponding $F\mathfrak{S}_n$-Specht module S^λ is simple if and only if the highest p-power dividing the hook length h_{xz} equals the highest p-power dividing the hook length h_{yz}, for all nodes $(x, z), (y, z)$ of the Young diagram $[\lambda]$.*

For determining the vertices of completely splittable modules and simple $F\mathfrak{S}_n$-modules labelled by certain two part partitions, we will extensively use modular branching rules, due to A. Kleshchev. A detailed introduction to this subject is given in [15]. Here, we just recall the facts needed in the course of the next sections.

Remark 2.7. Consider a partition λ of n with Young diagram $[\lambda]$. Moreover, let $i \in \{0, \dots, p-1\}$. Each i-removable node $(x, y) \in [\lambda]$, i.e. $y - x \equiv i \pmod p$, is marked by a "$-$". Analogously, any i-addable node $(u,v) \notin [\lambda]$, i.e. $v-u \equiv i \pmod p$, is marked by a "$+$". Going along the rim of $[\lambda]$ from bottom left to top right, we then obtain a sequence of pluses and minuses which is called the *i-signature* of λ. Successively cancelling all terms of the form "$-+$", we deduce the *reduced i-signature* of λ which has, by construction, the form "$+\cdots+-\cdots-$".

An i-removable node of $[\lambda]$ corresponding to a "$-$" in the reduced i-signature of $[\lambda]$ is called *i-normal*, and the total number of these is denoted by $\varepsilon_i(\lambda)$. An i-addable node of $[\lambda]$ corresponding to a "$+$" in the reduced i-signature is called *i-conormal*, and the total number of these is denoted by $\varphi_i(\lambda)$.

Keeping this notation, the following result will be substantial:

Theorem 2.8 (Kleshchev [15], Thms. 11.2.10, 11.2.11). *Let $\lambda \in \mathcal{P}_{n,p}$, and $i \in \{0,\ldots,p-1\}$. Let $\mu \in \mathcal{P}_{n-\varepsilon_i(\lambda),p}$ be the p-regular partition obtained after removing all i-normal nodes from $[\lambda]$. Furthermore, let $\nu \in \mathcal{P}_{n+\varphi_i(\lambda),p}$ be the p-regular partition obtained after adding all i-conormal nodes to $[\lambda]$. Then*

$$(\bigoplus_{\varepsilon_i(\lambda)!} D^\mu) \big| \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-\varepsilon_i(\lambda)}}(D^\lambda) \quad \textit{and} \quad (\bigoplus_{\varphi_i(\lambda)!} D^\nu) \big| \operatorname{Ind}^{\mathfrak{S}_{n+\varphi_i(\lambda)}}_{\mathfrak{S}_n}(D^\lambda).$$

Remark 2.9. In the situation of the previous theorem, any vertex of D^λ contains vertices of both D^μ and D^ν, by Lemma 2.1. If also $D^\lambda \big| \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-\varepsilon_i(\lambda)}}(D^\mu)$ then D^λ and D^μ have common vertices, by Lemma 2.1. We indicate this by writing $D^\lambda \longleftrightarrow_i D^\mu$. Analogously, if $D^\lambda \big| \operatorname{Res}^{\mathfrak{S}_{n+\varphi_i(\lambda)}}_{\mathfrak{S}_n}(D^\nu)$ then D^λ and D^ν have common vertices. In this case we write $D^\nu \longleftrightarrow_i D^\lambda$.

Remark 2.10. As mentioned in the introduction, the basic ingredients for the proof of our main theorem in Section 4 will be the modular branching rules and a result by K. Erdmann [5]. For this reason, we recall the necessary facts from [5].

Let $(n-m,m)$ be a p-regular partition of n, and let $s := n-2m$. Consider the p-adic expansion $s+1 = \sum_{i=1}^{l} d_i p^{t_i}$ where $0 \leq t_1 < \ldots < t_l =: t$ and $1 \leq d_i \leq p-1$, for $i=1,\ldots,l$. Moreover, set $\delta := (p-d_1)p^{t_1}$, and

$$\Gamma_s(q) := \prod_{i=2}^{l} \frac{(1-q^{(p-d_i)p^{t_i}})}{(1-q^{p^{t_{i-1}+1}})} = \prod_{i=2}^{l}(1+q^{p^{t_{i-1}+1}}+q^{2p^{t_{i-1}+1}}+\cdots+q^{e_i p^{t_{i-1}+1}}),$$

where $e_i := (p-d_i)p^{t_i-t_{i-1}-1}-1$. Writing $\Gamma_s(q) = \sum_{i=0}^{\infty} c_i q^i$, the following holds:

Theorem 2.11 (Erdmann [5], L. 5.4). *The dimension of $D^{(n-m,m)}$ is equal to*

$$\sum_{i=0}^{\infty} c_i \Big(\sum_{j=0}^{\lfloor m/p^{t+1} \rfloor} L_{i,j} - M_{i,j} \Big),$$

where

$$L_{i,j} := \binom{n}{m-i-jp^{t+1}} - \binom{n}{m-i-jp^{t+1}-1} \quad \textit{and}$$
$$M_{i,j} := \binom{n}{m-i-jp^{t+1}-\delta} - \binom{n}{m-i-jp^{t+1}-\delta-1}.$$

Note that, in the previous theorem, it suffices to determine the coefficients c_i, for $i \leq m$. For $i > m$, the respective summands $L_{i,j} - M_{i,j}$ in the dimension formula clearly vanish.

3 Completely splittable modules

In this section we will determine the vertices of the completely splittable modules for symmetric groups as introduced by Kleshchev in [14]. We briefly recall the notation needed throughout. Following [14], a simple $F\mathfrak{S}_n$-module D^λ is called *completely splittable* if its restriction to any Young subgroup $\mathfrak{S}_\mu$ of $\mathfrak{S}_n$ remains semisimple.

Given $\lambda \in \mathcal{P}_n$ with $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\lambda_s > 0$, we set $h(\lambda) := s$ and $\chi(\lambda) := \lambda_1 - \lambda_s + s$. With this definition, the following holds:

Theorem 3.1 (Kleshchev [14], Thm. 2.1, L. 1.7)**.** *Let D^λ be a simple $F\mathfrak{S}_n$-module. Then the following assertions are equivalent:*

(i) *D^λ is completely splittable.*

(ii) $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_l}(D^\lambda)$ *is semisimple, for all $l < n$.*

(iii) $\chi(\lambda) \leq p$.

Definition 3.2. For each $n \in \mathbb{N}$, let $\Phi_n := \{D^n_1, \ldots, D^n_{z(n)}\}$ be a set of simple $F\mathfrak{S}_n$-modules satisfying the following conditions:

(1) Provided $n \geq 2$ and $j \in \{1, \ldots, z(n)\}$, we have $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^n_j) \cong a_1 D^{n-1}_1 \oplus \cdots \oplus a_{z(n-1)} D^{n-1}_{z(n-1)}$, for appropriate $a_1, \ldots, a_{z(n-1)} \in \mathbb{N}_0$.

(2) For each $n \geq 2$ and $i \in \{1, \ldots, z(n-1)\}$, there is some $j \in \{1, \ldots, z(n)\}$ such that $D^{n-1}_i \mid \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^n_j)$.

We then define $\Phi := \bigcup_{n \in \mathbb{N}} \Phi_n$, and call Φ a *semisimple inductive system.*

Remark 3.3. (a) By Theorem 3.1, all simple modules belonging to such a semisimple inductive system are completely splittable. In [14], Kleshchev investigates special classes of semisimple inductive systems which are constructed as follows: let $n \in \mathbb{N}$ and $s \in \{1, \ldots, p-1\}$. Furthermore, let

$$\xi^s_n := \{D^\mu | \mu \in \mathcal{P}_{n,p},\ h(\mu) = s,\ \chi(\mu) \leq p\} \ \text{ and } \ \omega^s_n := \{D^\mu | \mu \in \mathcal{P}_{n,p},\ h(\mu) < s,\ \mu_1 \leq p - s\}.$$

Now set $\Phi_n(s) := \xi^s_n \cup \omega^s_n$ and $\Phi(s) := \bigcup_{n \in \mathbb{N}} \Phi_n(s)$. By [14], Thm. 2.8 (i), $\Phi(s)$ is a semisimple inductive system, for each $s \in \{1, \ldots, p-1\}$.

(b) For $s \in \{1, \dots, p-1\}$, the semisimple inductive system $\Phi(s)$ in particular contains all completely splittable modules parametrized by p-regular partitions with exactly s non-zero parts. Conversely, each completely splittable module is parametrized by some p-regular partition with exactly s non-zero parts, where $s \in \{1, \dots, p-1\}$. This follows from Theorem 3.1. Thus each completely splittable module has to be contained in $\Phi(s)$, for an appropriate $s \in \{1, \dots, p-1\}$.

(c) Given a p-regular partition $\lambda := (n-m, m)$ of n such that $m > 0$, the corresponding simple $F\mathfrak{S}_n$-module D^λ is completely splittable if and only if $n - 2m + 2 = \chi(\lambda) \leq p$, i.e. if and only if $n - 2m \leq p - 2$, by Theorem 3.1 (iii). In consequence, in characteristic 2 there are no completely splittable modules parametrized by partitions with exactly two non-zero parts. Assuming $p > 2$ and $n > p - 2$, there are exactly $\frac{p-1}{2}$ completely splittable $F\mathfrak{S}_n$-modules labelled by partitions with exactly two non-zero parts. If $n = 2r$ then the corresponding partitions have the form $(r+i, r-i)$, for $0 \leq i \leq \frac{p-1}{2} - 1$. If $n = 2r+1$ then the corresponding partitions have the form $(r+1+i, r-i)$, for $0 \leq i \leq \frac{p-1}{2} - 1$. In particular, for $p = 3$ and $n \geq 2$, there is only one completely splittable $F\mathfrak{S}_n$-module labelled by a partition with exactly two non-zero parts, namely the alternating module.
For $p = 5$ and $n \geq 4$, there are precisely two completely splittable $F\mathfrak{S}_n$-modules labelled by partitions with exactly two non-zero parts. These are the so-called *Fibonacci modules*, introduced by Ryba in [22], whose dimensions equal the nth and $(n-1)$st Fibonacci number, respectively.

We are now prepared to determine the vertices of the completely splittable modules:

Theorem 3.4. *Let $n \in \mathbb{N}$, and let D^λ be a completely splittable $F\mathfrak{S}_n$-module. Then the vertices of D^λ are precisely the defect groups of the block of $F\mathfrak{S}_n$ containing D^λ.*

Proof. We will show that, given $s \in \{1, \dots, p-1\}$, each $F\mathfrak{S}_n$-module belonging to $\Phi_n(s)$ has the defect groups of its block as its vertices. By Knörr's Theorem we may assume that $n \geq p^2 > (p-s)(s-1)$. Then, in particular, $\Phi_n(s) = \xi_n^s$, i.e. each module in $\Phi_n(s)$ is labelled by a partition with exactly s non-zero parts. In case that $n \equiv 0 \pmod p$ exactly one of the $F\mathfrak{S}_n$-modules belonging to $\Phi_n(s)$ is contained in the principal block of $F\mathfrak{S}_n$, by [10], Thm. 3.3, L. 3.5. Moreover, the dimension of this module is not divisible by p, by [11], Thm. 5.7.1, so that its vertices are the Sylow p-subgroups of $F\mathfrak{S}_n$.
Let now D be any of the remaining $F\mathfrak{S}_n$-modules belonging to $\Phi_n(s)$. Then D is relatively $\mathfrak{S}_{n-1}$-projective, by [10], Thm. 3.3., and is thus contained in a block of p-weight $w < n/p$. If $w < p$ then the vertices of D are clearly the defect groups of its block, by Knörr's Theorem. Hence we may assume that $w \geq p$. Then we have $pw \geq p^2 > (p-s)(s-1)$ and thus $\Phi_{pw}(s) = \xi_{pw}^s$. Since D is completely splittable, $\mathrm{Res}_{\mathfrak{S}_{pw}}^{\mathfrak{S}_n}(D)$ is semisimple, and its simple direct summands belong to $\Phi_{pw}(s)$. One of those has a vertex in common with D. By [10], Thm. 3.3, L. 3.5. and [11], Thm. 5.7.1., the only module in $\Phi_{pw}(s)$ which is not relatively $\mathfrak{S}_{pw-1}$-projective belongs to the principal block of $F\mathfrak{S}_{pw}$, has a dimension coprime to p and therefore vertex P_{pw}. Consequently, D has vertex P_{pw} as well.

Let now $n \not\equiv 0 \pmod p$, and let D be a simple $F\mathfrak{S}_n$-module in $\Phi_n(s)$ belonging to a block of weight w. In view of Knörr's Theorem, we may again assume that $w \geq p$. As above, we then deduce that there is up to isomorphism only one not relatively $\mathfrak{S}_{pw-1}$-projective simple direct summand of $\mathrm{Res}_{\mathfrak{S}_{pw}}^{\mathfrak{S}_n}(D)$. That summand belongs to the principal block of $F\mathfrak{S}_{pw}$ and has vertex P_{pw}. Hence also D has vertex P_{pw}, and the assertion follows. $\square$

4 Simple $F\mathfrak{S}_n$-modules labelled by two part partitions

In this section, let p be odd. For the sake of simplicity, a *two part partition* of n is understood to be a partition having either one or two non-zero parts. We will determine the vertices of simple $F\mathfrak{S}_n$-modules labelled by certain two part partitions of n. This work was already begun in [23] where R. Zimmermann determined the vertices of simple $F\mathfrak{S}_n$-modules corresponding to the partitions $(n-m, m)$ with $m \leq 3$. Later, in [7] B. Fotsing determined the vertices of simple $F\mathfrak{S}_n$-modules corresponding to the partitions $(n-m, m)$ where $m \leq 2p-1$, or $m = 2p$ and $p > 3$. We will extend these results and show that the following holds:

Theorem 4.1. *Let $p > 2$, $m \in \{0, \dots, \frac{p+1}{2}p - 1\}$ and $n \geq 2m$. Furthermore, let $\lambda := (n-m, m)$. Then exactly one of the following cases occurs:*

(1) The vertices of D^λ are precisely the defect groups of its block.

(2) $m = kp + r$, for some $k \geq 1$ and some $r \in \{0, \dots, p-1\}$, such that $n \equiv bp + 2r - 1 \pmod{p^2}$, for some $b \in \{1, \dots, k-1, 2k\}$. In this case we have $D^\lambda \cong S^\lambda \cong Y^\lambda$ with vertex $P := P_{n-2m} \times (P_m)^2$. Moreover, P is not a defect group of the block of $F\mathfrak{S}_n$ containing D^λ.

Remark 4.2. In order to prove the theorem we will proceed as follows:

- We determine when $D^\lambda \cong S^\lambda$ holds true, and when $D^\lambda \cong S^\lambda$ has vertex P which is not a defect group of its block.
- We prove the assertion of the theorem, by induction on m. For this, we will basically distinguish between two cases, depending on whether or not m is divisible by p.

For the remainder of this section, we write $m = kp + r$, for appropriate $k \in \{0, \dots, \frac{p-1}{2}\}$ and $r \in \{0, \dots, p-1\}$. Moreover, for any integer z, let $\bar{z}$ be its residue class modulo p. We will often make use of both the p-residue diagram and the hook diagram of λ. These are of the following shapes:

$[\lambda]_p=$

$\bar{0}$	$\cdots$	$\overline{r-2}$	$\overline{r-1}$	$\cdots$	$\overline{n-r-1}$	$\overline{n-r}$
$\overline{p-1}$	$\cdots$	$\overline{r-3}$	$\overline{r-2}$	$\overline{r-1}$		
$\overline{p-2}$						

$[\lambda]_{\text{hook}} =$

$n-kp-r+1$	$\cdots$	$n-kp-2r+1$	$\cdots$	$n-(2k-1)p-2r+1$	$\cdots$		$\cdots$
$kp+r$	$\cdots$	kp	$\cdots$	p	$\cdots$	1	

We will now break up the entire proof of Theorem 4.1 into a series of propositions. As a first step in the proof of Theorem 4.1, we will determine when $D^{(n-m,m)} \cong S^{(n-m,m)}$ holds, and will show that the assertion of Theorem 4.1 holds, for $m < p$. The first three propositions are consequences of Theorem 2.8 and Theorem 2.6, and do in fact also hold for $k > \frac{p-1}{2}$.

Proposition 4.3. *Let $k > 0$ and $r = 0$. Furthermore, let $-1 \not\equiv n \not\equiv 0 \pmod{p}$. Then D^λ and $D^{(n-kp-1,kp)}$ have common vertices. Moreover, $D^\lambda \not\cong S^\lambda$, and $D^{(n-kp-1,kp)} \not\cong S^{(n-kp-1,kp)}$.*

Proof. Again we denote the residue class of $n-1$ modulo p by $\overline{n-1} =: i$. In direct consequence of Theorem 2.6, $D^\lambda \not\cong S^\lambda$ and $D^{(n-kp-1,kp)} \not\cong S^{(n-kp-1,kp)}$. Furthermore, λ has reduced i-signature "$-$", and $(n-kp-1,kp)$ has reduced i-signature "$+$". Thus $D^\lambda \longleftrightarrow_i D^{(n-kp-1,kp)}$, and the assertion follows. □

Proposition 4.4. *Let $k \geq 0$, and let $r \geq 1$.*

(i) If $2r-1 \not\equiv n \not\equiv 2r-2 \pmod p$ then D^λ and $D^{(n-kp-r,kp+r-1)}$ have common vertices.

(ii) If $n \equiv 2r-1 \pmod p$ then D^λ and $D^{(n-kp-r-1,kp+r-1)}$ have common vertices.

(iii) If $r > 1$ and $n \equiv 2r-2 \pmod p$ then D^λ and $D^{(n-kp-r-1,kp+r-1)}$ have common vertices.

Proof. (i) In case that $2r-1 \not\equiv n \not\equiv 2r-2 \pmod p$ we have $r-2 \not\equiv n-r-1 \not\equiv r-3 \pmod p$. Thus, in this case, λ has reduced $(\overline{r-2})$-signature "$-$", and $(n-kp-r, kp+r-1)$ has reduced $(\overline{r-2})$-signature "$+$". Therefore, we have $D^\lambda \longleftrightarrow_{\overline{r-2}} D^{(n-kp-r,kp+r-1)}$, and (i) follows.

(ii) In case that $n \equiv 2r-1 \pmod p$, we have $n-r-1 \equiv r-2 \pmod p$ so that λ has reduced $(\overline{r-2})$-signature "$--$", and $(n-kp-r-1, kp+r-1)$ has reduced $(\overline{r-2})$-signature "$++$". Consequently, $D^\lambda \longleftrightarrow_{\overline{r-2}} D^{(n-kp-r-1,kp+r-1)}$, and we thus obtain (ii).

(iii) In case that $n \equiv 2r-2 \pmod p$ we have $n-r-1 \equiv r-3 \pmod p$ so that λ has reduced $(\overline{r-3})$-signature "$-$", and reduced $(\overline{r-2})$-signature $\emptyset$. Thus $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^\lambda) \cong D^{(n-kp-r-1,kp+r)}$, by Theorem 2.8. Furthermore, $r-1 \not\equiv r-3 \not\equiv p-2 \pmod p$, since $r > 1$. Hence $(n-kp-r-1, kp+r)$ has reduced $(\overline{r-3})$-signature "$+$" so that $D^\lambda \longleftrightarrow_{\overline{r-3}} D^{(n-kp-r-1,kp+r)}$. Since $n-1 \equiv 2r-3 \pmod p$ and $2r-1 \not\equiv 2r-3 \not\equiv 2r-2 \pmod p$, from (i) we deduce that D^λ and $D^{(n-kp-r-1,kp+r-1)}$ have common vertices. □

Proposition 4.5. *Let $k \geq 0$ and $r \geq 1$.*

(i) If $2r-1 \not\equiv n \not\equiv 2r-2 \pmod p$ then $D^\lambda \cong S^\lambda$ if and only if $D^{(n-kp-r,kp+r-1)} \cong S^{(n-kp-r,kp+r-1)}$. In particular, $D^{(n-kp-r,kp+r-1)} \not\cong S^{(n-kp-r,kp+r-1)}$, for $k > 0$.

(ii) If $n \equiv 2r-1 \pmod p$ then $D^\lambda \cong S^\lambda$ if and only if

$$D^{(n-kp-r-1,kp+r-1)} \cong S^{(n-kp-r-1,kp+r-1)}.$$

(iii) If $r > 1$ and $n \equiv 2r-2 \pmod p$ then $D^\lambda \not\cong S^\lambda$, and

$$D^{(n-kp-r-1,kp+r-1)} \not\cong S^{(n-kp-r-1,kp+r-1)}.$$

Proof. The assertion follows immediately from Theorem 2.6. □

Proposition 4.6. *With the notation of Remark 4.2, the following holds:*

(i) If $k \geq 1$ then $S^\lambda \cong D^\lambda$ if and only if $n \equiv bp + 2r - 1 \pmod{p^2}$, for some $b \in \{0, \ldots, k-1, 2k, \ldots, p-1\}$.

(ii) If $k = 0$ and $r \geq 1$ then $S^\lambda \cong D^\lambda$ if and only if $n \not\equiv r + l - 1 \pmod p$, for all $l \in \{0, \ldots, r-1\}$.

Proof. Let $k \geq 1$. By Theorem 2.6, we then have that $S^\lambda \cong D^\lambda$ if and only if $n-(2r-1) \equiv 0 \pmod{p}$ and $\nu_p(n-kp-lp-2r+1) = \nu_p((k-l)p) = 1$, for all $l = 0, \ldots, k-1$. The latter condition is equivalent to $n = ap^2 + bp + 2r - 1$ with $a \in \mathbb{N}_0$ and $b \in \{0, \ldots, p-1\}$ such that $\nu_p(ap^2 + (b-k-l)p) = \nu_p((k-l)p) = 1$, for all $l = 0, \ldots, k-1$. Consequently, $S^\lambda \cong D^\lambda$ if and only if $n = ap^2 + bp + 2r - 1$, for some $a \in \mathbb{N}_0$ and some $b \in \{0, \ldots, k-1, 2k, \ldots, p-1\}$. This proves (i).

Now let $k = 0$ and $r \geq 1$. Then we immediately get that $S^\lambda \cong D^\lambda$ if and only if $n-r-l+1 \not\equiv 0 \pmod{p}$, for all $l = 0, \ldots, r-1$, by Theorem 2.6. This proves (ii). □

Remark 4.7. Note that in the case where $k = 0 = r$ we have $\lambda = (n)$ and thus $D^\lambda \cong S^\lambda$, for all $n \in \mathbb{N}$.

Proposition 4.8. *With the notation of Remark 4.2, assume that $D^\lambda \cong S^\lambda$. Moreover, let P be a vertex of D^λ.*

(i) If $k = 0$ then P is a defect group of the block of $F\mathfrak{S}_n$ containing D^λ.

(ii) If $k \geq 1$ then P is a defect group of the block of $F\mathfrak{S}_n$ containing D^λ if and only if $n \equiv bp + 2r - 1 \pmod{p^2}$, for some $b \in \{0, 2k+1, \ldots, p-1\}$.

Proof. (i) By Proposition 4.6, we have $n \not\equiv r + l - 1 \pmod{p}$, for all $l \in \{0, \ldots, r-1\}$. Furthermore, D^λ has vertex P_{n-2r}, by Theorem 2.4. In order to show that D^λ belongs to a block with defect group P_{n-2r}, we assume that λ has p-core κ such that $|\kappa| < 2r$. Then $\kappa = (r-1, l)$, for some $l \in \{0, \ldots, r-1\}$. Consequently, $n - r + 1 - l = n - |\kappa| \equiv 0 \pmod{p}$ which leads to the contradiction $n \equiv r + l - 1 \pmod{p}$. Hence D^λ belongs to a block B of $F\mathfrak{S}_n$ with p-core κ such that $|\kappa| \geq 2r$, and the defect groups of B are conjugate to $P_{n-|\kappa|} = P_{n-2r}$. This proves (i).

(ii) Let now $k \geq 1$. By Proposition 4.6, we have $n = ap^2 + bp + 2r - 1$, for some $a \in \mathbb{N}_0$ and some $b \in \{0, \ldots, k-1, 2k, \ldots, p-1\}$. We distinguish between three cases.

Case 1: $1 \leq b \leq 2k$. Then, in particular, $a \geq 1$. Moreover, λ has p-core $(p+r-1, r)$ and p-weight $w := ap + b - 1$ so that $P_{pw} = P_{ap^2} \times (P_p)^{b-1}$. Furthermore, D^λ has vertex $P := P_{n-2kp-2r} \times (P_{kp+r})^2 =_{\mathfrak{S}_n} P_{n-2kp-2r} \times (P_p)^{2k}$, by Theorem 2.4. Since

$$n - 2kp - 2r = ap^2 + (b-2k)p - 1 = (a-1)p^2 + (p+b-2k-1)p + p - 1,$$

we obtain $P =_{\mathfrak{S}_n} P_{(a-1)p^2} \times (P_p)^{p+b-1} <_{\mathfrak{S}_n} P_{pw}$.

Case 2: $b = 0$. Then $a \geq 1$, and λ has p-core $(p+r-1, r)$ and p-weight $w := (a-1)p + p - 1$. Hence $P_{pw} = P_{(a-1)p^2} \times (P_p)^{p-1}$. Since, in this case,

$$n - 2kp - 2r = ap^2 - 2kp - 1 = (a-1)p^2 + (p-2k-1)p + p - 1,$$

we obtain that D^λ has vertex $P = P_{n-2kp-2r} \times (P_p)^{2k} =_{\mathfrak{S}_n} P_{(a-1)p^2} \times (P_p)^{p-1} = P_{pw}$.

Case 3: $b \geq 2k+1$. Then λ has p-core $(p+r-1, r)$ and p-weight $w := ap + b - 1$ so that $P_{pw} = P_{ap^2} \times (P_p)^{b-1}$. Since

$$n - 2kp - 2r = ap^2 + (b-2k)p - 1 = ap^2 + (b-2k-1)p + p - 1,$$

also in this case, D^λ has vertex $P := P_{n-2kp-2r} \times (P_p)^{2k} =_{\mathfrak{S}_n} P_{ap^2} \times (P_p)^{b-1} = P_{pw}$. This proves the proposition. ☐

Corollary 4.9. *The assertion of Theorem 4.1 holds, for $m \in \{0, \ldots, p-1\}$.*

Proof. The vertices of $D^{(n)}$ and $D^{(n-1,1)}$ are precisely the defect groups of the corresponding blocks of $F\mathfrak{S}_n$, by [18]. By Proposition 4.4, the vertices of $D^{(n-m,m)}$ are then also the defect groups of its block, for all $m = 2, \ldots, p-1$. ☐

Remark 4.10. We recall that we aim to prove Theorem 4.1 by induction on m. For this, Corollary 4.9 is the start of the induction argument. In order to do the induction step, we fix $m \in \{1, \ldots, \frac{p+1}{2}p - 1\}$, and assume that the statement of Theorem 4.1 holds for all two part partitions whose second part is stricty less than m. We need to show that the assertion of Theorem 4.1 then also holds for $\lambda = (n-m, m)$. It will be helpful to distinguish between the cases $m \equiv 0 \pmod p$ and $m \not\equiv 0 \pmod p$.

4.1 Induction step - the case $m \equiv 0 \pmod p$

Remark 4.11. In this subsection, we do the induction step, for $m \equiv 0 \pmod p$. In this case, $\lambda = (n-kp, kp)$, for some $k \in \{1, \ldots, \frac{p-1}{2}\}$, so that D^λ is contained in the principal block of $F\mathfrak{S}_n$. By Proposition 4.8 we already know that the vertices of D^λ differ from the Sylow p-subgroups of $\mathfrak{S}_n$, provided $n \equiv bp - 1 \pmod{p^2}$, for some $b \in \{1, \ldots, k-1, 2k\}$. In the following, we will show that in all other cases D^λ has precisely the Sylow p-subgroups of $\mathfrak{S}_n$ as its vertices. By Proposition 4.3, we also know that it suffices to consider the cases $n \equiv -1 \pmod p$, and $n \equiv 0 \pmod p$ explicitly.

Proposition 4.12. *Let $n \equiv -1 \pmod p$. Then either the vertices of D^λ are the Sylow p-subgroups of $\mathfrak{S}_n$, or $n \equiv bp - 1 \pmod{p^2}$, for some $b \in \{1, \ldots, k-1, 2k\}$. In the latter case, D^λ has vertex $P_{n-2kp} \times (P_p)^{2k} \notin \mathrm{Syl}_p(\mathfrak{S}_n)$.*

Proof. By Proposition 4.6, $D^\lambda \cong S^\lambda$ if and only if $n = ap^2 + bp - 1$, for appropriate $a \in \mathbb{N}_0$ and $b \in \{0, \ldots, k-1, 2k, \ldots, p-1\}$. By Proposition 4.8, $D^\lambda \cong S^\lambda$ has vertex P_n if and only if $b \in \{0, 2k+1, \ldots, p-1\}$.

It thus remains to show that D^λ has vertex P_n, for $n = ap^2 + bp - 1$ where $a \in \mathbb{N}_0$ and $b \in \{k, \ldots, 2k-1\}$. Therefore we write $n = \tilde{a}p^2 + \tilde{b}p + p - 1$, for $\tilde{a} \in \mathbb{N}_0$ and $\tilde{b} \in \{k-1, \ldots, 2k-2\}$. In addition, we may assume that $\tilde{a} > 0$, since otherwise $n < p^2$, and the assertion then holds, by Knörr's Theorem. We distinguish between two cases.

Case 1: $k = 1$. Then $\tilde{b} = 0$, i.e. $n = \tilde{a}p^2 + (p-1)$. With the notation of Remark 2.10, we obtain the following:

$$\begin{aligned}
s + 1 &= n - 2kp + 1 = \tilde{a}p^2 + p - 2p = (\tilde{a} - 1)p^2 + (p-1)p,\\
t_1 &= 1,\ d_1 = p - 1,\ \delta = p,\ e_2 = (p - d_2)p^{t_2 - 2} - 1,\ t \geq 1,\\
\left\lfloor \frac{kp}{p^{t+1}} \right\rfloor &= \left\lfloor \frac{p}{p^{t+1}} \right\rfloor = 0,\\
\sum_{i=0}^{\infty} c_i q^i &= \Gamma_s(q) = (1 + q^{p^2} + q^{2p^2} + \cdots + q^{e_2 p^2})(1 + q^{p^{t_2+1}} + q^{2p^{t_2+1}} + \cdots + q^{e_3 p^{t_2+1}}) \cdots .
\end{aligned}$$

Obviously, $c_0 = 1$ and $c_1 = \ldots = c_p = 0$. Hence we have

$$\dim(D^\lambda) = \binom{n}{p} - \binom{n}{p-1} - \binom{n}{p-\delta} + \binom{n}{p-\delta-1} = \binom{n}{p} - \binom{n}{p-1} - 1,$$

by Theorem 2.11. This shows that $\dim(D^\lambda) \equiv -2 \pmod{p}$.

Case 2: $k > 1$. Then $\tilde{b} \geq 1$. Using the notation of Remark 2.10 and Theorem 2.11, we now have

$$\begin{aligned}
s+1 = n - 2kp + 1 = (\tilde{a}-1)p^2 + (p + \tilde{b} - 2k + 1)p, \\
t_1 = 1,\ d_1 = p + \tilde{b} - 2k + 1,\ \delta = (2k - \tilde{b} - 1)p,\ t \geq 1, \\
\left\lfloor \frac{kp}{p^{t+1}} \right\rfloor = 0,\ e_2 = (p - d_2)p^{t_2 - 2} - 1,
\end{aligned}$$

$$\sum_{i=0}^{\infty} c_i q^i = \Gamma_s(q) = (1 + q^{p^2} + q^{2p^2} + \cdots + q^{e_2 p^2})(1 + q^{p^{t_2+1}} + q^{2p^{t_2+1}} + \cdots + q^{e_3 p^{t_2+1}}) \cdots .$$

Consequently, also here we have $c_0 = 1$ and $c_1 = \ldots = c_{kp} = 0$. Hence

$$\dim(D^\lambda) = \binom{n}{kp} - \binom{n}{kp-1} - \binom{n}{p(\tilde{b}-k+1)} + \binom{n}{p(\tilde{b}-k+1)-1}.$$

We subsequently show that $\dim(D^\lambda) \not\equiv 0 \pmod{p}$. Firstly, let $\tilde{b} = k-1$ so that

$$\dim(D^\lambda) = \binom{n}{kp} - \binom{n}{kp-1} - \binom{n}{0}.$$

An easy calculation shows that $\binom{n}{kp} \equiv 0 \pmod{p}$ and $\binom{n}{kp-1} \equiv 1 \pmod{p}$ so that $\dim(D^\lambda) \equiv -2 \pmod{p}$.

We now assume that $\tilde{b} \geq k$. Then we have

$$\dim(D^\lambda) = \frac{n-2kp+1}{(kp)!} \prod_{i=0}^{kp-2} (n-i) - \frac{n - 2(\tilde{b}-k+1)p + 1}{((\tilde{b}-k+1)p)!} \prod_{i=0}^{(\tilde{b}-k+1)p-2} (n-i) = \frac{z}{(kp)!},$$

where

$$z := (n-2kp+1) \prod_{i=0}^{kp-2} (n-i) - (n-2(\tilde{b}-k+1)p+1) \prod_{i=0}^{(\tilde{b}-k+1)p-2} (n-i)((\tilde{b}-k+1)p+1) \cdots kp.$$

Since $\nu_p((kp)!) = k$, we need to show that $z \not\equiv 0 \pmod{p^{k+1}}$. Firstly, we have:

$$\begin{aligned}
(n-2kp+1)n(n-1)\cdots(n-kp+2) &= (\tilde{a}p^2 + (\tilde{b}-2k+1)p) \underbrace{n(n-1)\cdots(n-kp+2)}_{\equiv 0 \pmod{p^{k-1}}} \\
&\equiv (\tilde{b}-2k+1)pn(n-1)\cdots(n-kp+2) \pmod{p^{k+1}} \\
&\equiv p^k[(\tilde{b}-2k+1)((p-1)!)^k \tilde{b} \cdots (\tilde{b}-(k-2))] \pmod{p^{k+1}}.
\end{aligned}$$

Secondly, we have:

$$
\begin{aligned}
&(n-2(\tilde{b}-k+1)p+1)n(n-1)\cdots(n-(\tilde{b}-k+1)p+2)((\tilde{b}-k+1)p+1)\cdots kp \\
&\quad = (\tilde{a}p^2+(2k-\tilde{b}-1)p)\underbrace{n(n-1)\cdots(n-(\tilde{b}-k+1)p+2)}_{\equiv 0 \pmod{p^{\tilde{b}-k}}}\underbrace{((\tilde{b}-k+1)p+1)\cdots kp}_{\equiv 0 \pmod{p^{2k-\tilde{b}-1}}} \\
&\quad \equiv (2k-\tilde{b}-1)pn(n-1)\cdots(n-(\tilde{b}-k+1)p+2)((\tilde{b}-k+1)p+1)\cdots kp \pmod{p^{k+1}} \\
&\quad \equiv p^k[(2k-\tilde{b}-1)((p-1)!)^k\tilde{b}(\tilde{b}-1)\cdots(k+1)(\tilde{b}-(k-2))(\tilde{b}-(k-3))\cdots k] \pmod{p^{k+1}} \\
&\quad \equiv -p^k[(\tilde{b}-2k+1)((p-1)!)^k\tilde{b}(\tilde{b}-1)\cdots(\tilde{b}-(k-2))] \pmod{p^{k+1}}.
\end{aligned}
$$

Thus we finally deduce that

$$
\begin{aligned}
z &\equiv 2p^k((p-1)!)^k\tilde{b}(\tilde{b}-1)\cdots(\tilde{b}-(k-2))(\tilde{b}-2k+1) \pmod{p^{k+1}} \\
&\not\equiv 0 \pmod{p^{k+1}}.
\end{aligned}
$$

We have now shown that also in this case $\dim(D^\lambda)\not\equiv 0 \pmod p$. Consequently, D^λ has vertex P_n, for $n\equiv bp-1 \pmod{p^2}$ with $b\in\{k,\ldots,2k-1\}$, and the proposition is proved. ☐

Proposition 4.13. *Let $n\equiv 0 \pmod p$. Then $D^\lambda = D^{(n-kp,kp)}$ has vertex P_n.*

Proof. By Theorem 2.8, we get $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^\lambda)\cong D^{(n-kp-1,kp)}\oplus D^{(n-kp,kp-1)}$. Here

$$D^{(n-kp,kp-1)} = D^{(n-1-(k-1)p-(p-1),(k-1)p+p-1)}$$

belongs to the block of $F\mathfrak{S}_{n-1}$ with p-core $(p-2,1)$ and p-weight $\frac{n}{p}-1$. Furthermore, $n-1\equiv -1\not\equiv 2(p-1)-1 \pmod p$. Thus

$$D^{(n-1-(k-1)p-(p-1),(k-1)p+p-1)}\not\cong S^{(n-1-(k-1)p-(p-1),(k-1)p+p-1)},$$

by Proposition 4.6, and has vertex P_{n-p}, by our inductive hypothesis. In the case where $n\not\equiv 0 \pmod{p^2}$ we have $|P_n : P_{n-p}| = p$, and the assertion then immediately follows from Lemma 2.3.

For the remainder of the proof, we may thus assume that $n\equiv 0 \pmod{p^2}$. Therefore we write $n = ap^2$, for some $a\in\mathbb{N}$. With the notation of Remark 2.10 and Theorem 2.11, we then get:

$$
\begin{aligned}
&s+1 = n-2kp+1 = (a-1)p^2+\underbrace{(p-2k)}_{>0}p+1, \\
&\quad t_1=0,\ d_1=1,\ \delta=p-1,\ t_2=1,\ d_2=p-2k,\ t\geq 1, \\
&\left\lfloor\frac{kp}{p^{t+1}}\right\rfloor = 0,\ e_2 = 2k-1, \\
&\sum_{i=0}^{\infty} c_iq^i = \Gamma_s(q) = (1+q^p+q^{2p}+\cdots+q^{e_2p})(1+q^{p^2}+q^{2p^2}+\cdots+q^{e_3p^2})\cdots.
\end{aligned}
$$

Consequently, $c_i = 1$ for $i\in\{0,p,2p,\ldots,kp\}$, and $c_i=0$ for $i\in\{0,\ldots,kp\}\setminus\{0,p,2p,\ldots,kp\}$, and hence

$$\dim(D^\lambda) = \sum_{l=0}^{k}\binom{n}{lp}-\binom{n}{lp-1}-\binom{n}{lp-p+1}+\binom{n}{lp-p}.$$

We set

$$\sigma_j := \sum_{l=0}^{j} \binom{n}{lp} - \binom{n}{lp-1} - \binom{n}{lp-p+1} + \binom{n}{lp-p},$$

and show that $\sigma_j \not\equiv 0 \pmod p$, for all $0 \leq j \leq \frac{p-1}{2}$. We clearly have

$$\sigma_0 = \binom{n}{0} = 1 \quad \text{and} \quad \sigma_1 = 2 + \binom{n}{p} - \binom{n}{p-1} - n.$$

It is easily verified that $\binom{n}{p} \equiv 0 \equiv \binom{n}{p-1} \pmod p$ so that $\sigma_1 \equiv 2 \pmod p$. Thus we may now assume that $j > 1$, and argue by induction on j. Then

$$\sigma_j = \sigma_{j-1} + \binom{n}{jp} - \binom{n}{jp-1} - \binom{n}{(j-1)p+1} + \binom{n}{(j-1)p}.$$

A small calculation shows that

$$\binom{n}{jp} \equiv \binom{n}{jp-1} \equiv \binom{n}{(j-1)p+1} \equiv \binom{n}{(j-1)p} \equiv 0 \pmod p.$$

Consequently, $\sigma_j \equiv \sigma_{j-1} \equiv 2 \pmod p$.

Hence we have proved that $\dim(D^\lambda) \not\equiv 0 \pmod p$, provided $n \equiv 0 \pmod{p^2}$. Thus D^λ has vertex P_n in this case, and the proposition is proved. □

Proposition 4.14. *If $-1 \not\equiv n \not\equiv 0 \pmod p$ then $D^\lambda = D^{(n-kp,kp)}$ has vertex P_n.*

Proof. This follows immediately from Propositions 4.3, 4.12 and 4.13. □

Remark 4.15. In consequence of Propositions 4.12, 4.13 and 4.14, we have now completed the induction, in the case where $m \equiv 0 \pmod p$. Therefore we may now assume that $m \not\equiv 0 \pmod p$, and prove the assertion of Theorem 4.1 for this case. This will be done in the next subsection.

4.2 Induction step - the case $m \not\equiv 0 \pmod p$

Remark 4.16. Throughout this subsection, we assume that $m \not\equiv 0 \pmod p$. Again we write $m = kp + r$, for appropriate $k \in \{0, \ldots, \frac{p-1}{2}\}$ and $r \in \{1, \ldots, p-1\}$. By Proposition 4.4, we know that it suffices to determine the vertices of $D^{(n-kp-r,kp+r)}$, for $r = 1$ and $n \equiv 0 \pmod p$. In particular, we then have $S^{(n-kp-r,kp+r)} \not\cong D^{(n-kp-r,kp+r)}$, and $D^{(n-kp-r,kp+r)}$ is contained in the principal block of $F\mathfrak{S}_n$.

Proposition 4.17. *With the assumptions of the previous remark, let $r = 1$ and $n \equiv 0 \pmod p$. Then $D^\lambda = D^{(n-kp-r,kp+r)}$ has vertex P_n.*

Proof. By Theorem 2.8, we have $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^\lambda) \cong D^{(n-kp-2,kp+1)}$. Furthermore, $D^{(n-kp-2,kp+1)}$ is contained in the block of $F\mathfrak{S}_{n-1}$ with p-core $(p-2,1)$ and p-weight $\frac{n}{p} - 1$. Since $r = 1$ and $2r - 1 \not\equiv n - 1 \not\equiv 2r - 2 \pmod p$, the simple modules $D^{(n-kp-2,kp+1)}$ and $D^{(n-kp-2,kp)}$ have common vertices, by Proposition 4.4. By our inductive hypothesis and Proposition 4.6, the $F\mathfrak{S}_{n-2}$-module $D^{(n-kp-2,kp)}$ has vertex P_{n-p}. Hence $D^{(n-kp-2,kp+1)}$ has vertex P_{n-p} as well.

In the case where $n \not\equiv 0 \pmod{p^2}$, we have $|P_n : P_{n-p}| = p$ so that then D^λ has vertex P_n, by Lemma 2.3.

For this reason, we now assume that $n \equiv 0 \pmod{p^2}$ and write $n = ap^2$, for some $a \in \mathbb{N}$. In the following, we will show that $\dim(D^\lambda) \equiv -2 \pmod{p}$. For this we use the notation of Remark 2.10 and Theorem 2.11 again, and obtain

$$s+1 = ap^2 - 2kp - 1 = (a-1)p^2 + \underbrace{(p-2k-1)}_{\geq 0} p + p - 1.$$

We distinguish between three cases.

Case 1: $a = 1,\ k = \frac{p-1}{2}$. Then we get

$$s+1 = p-1,\ t_1 = 0,\ d_1 = p-1,\ \delta = 1,\ t = 0,$$
$$\sum_{i=0}^{\infty} c_i q^i = \Gamma_s(q) = 1,\ \left\lfloor \frac{kp+1}{p^{t+1}} \right\rfloor = \left\lfloor \frac{kp+1}{p} \right\rfloor = k.$$

Consequently, $c_0 = 1$ and $c_i = 0$, for $i \geq 1$ so that

$$\dim(D^\lambda) = \sum_{l=0}^{k} \left[\binom{n}{lp+1} - 2\binom{n}{lp} + \binom{n}{lp-1} \right].$$

Case 2: $k < \frac{p-1}{2}$, i.e. $p - 2k - 1 > 0$. Thus we then get

$$t_1 = 0,\ d_1 = p-1,\ \delta = 1,\ t_2 = 1,\ d_2 = p-2k-1,\ t \geq 1,$$
$$\left\lfloor \frac{kp+1}{p^{t+1}} \right\rfloor = 0,\ e_2 = 2k,$$
$$\sum_{i=0}^{\infty} c_i q^i = \Gamma_s(q) = (1 + q^p + q^{2p} + \cdots + q^{e_2 p})(1 + q^{p^2} + q^{2p^2} + \cdots + q^{e_3 p^2}) \cdots$$

This shows that $c_0 = c_p = c_{2p} = \ldots = c_{kp} = 1$ and $c_i = 0$, for $i \in \{0, \ldots, kp+1\} \setminus \{0, p, 2p, \ldots, kp\}$, and also here we get

$$\dim(D^\lambda) = \sum_{l=0}^{k} \left[\binom{n}{lp+1} - 2\binom{n}{lp} + \binom{n}{lp-1} \right].$$

Case 3: $k = \frac{p-1}{2}$ and $a \geq 2$. Then we have

$$s+1 = (a-1)p^2 + p - 1,\ t_1 = 0,\ d_1 = p-1,\ \delta = 1,\ t_2 \geq 2,\ t \geq 2,$$
$$\left\lfloor \frac{kp+1}{p^{t+1}} \right\rfloor = 0,\ e_2 = (p - d_2)p^{t_2 - 1} - 1 > k,$$
$$\sum_{i=0}^{\infty} c_i q^i = \Gamma_s(q) = (1 + q^p + q^{2p} + \cdots + q^{e_2 p})(1 + q^{p^{t_2+1}} + \cdots + q^{e_3 p^{t_2+1}}) \cdots,$$

Again we obtain $c_0 = c_p = \ldots = c_{kp} = 1$ and $c_i = 0$, for $i \in \{0, \ldots, kp+1\} \setminus \{0, p, 2p, \ldots, kp\}$, and

$$\dim(D^\lambda) = \sum_{l=0}^{k} \left[\binom{n}{lp+1} - 2\binom{n}{lp} + \binom{n}{lp-1} \right].$$

We now set

$$\sigma_j := \sum_{l=0}^{j} \left[\binom{n}{lp+1} - 2\binom{n}{lp} + \binom{n}{lp-1} \right],$$

and show that $\sigma_j \equiv -2 \pmod p$, for all $0 \leq j \leq \frac{p-1}{2}$. We argue by induction on j. If $j = 0$, then $\sigma_j = \sigma_0 = n - 2 \equiv -2 \pmod p$. For $j \geq 1$, we obviously have

$$\sigma_j = \sigma_{j-1} + \binom{n}{jp+1} - 2\binom{n}{jp} + \binom{n}{jp-1}.$$

An easy calculation shows that

$$\binom{n}{jp+1} \equiv \binom{n}{jp} \equiv \binom{n}{jp-1} \equiv 0 \pmod p,$$

so that $\sigma_j \equiv \sigma_{j-1} \equiv -2 \pmod p$, by our inductive hypothesis. Consequently, $\dim(D^\lambda) \equiv -2 \not\equiv 0 \pmod p$, for $n \equiv 0 \pmod{p^2}$, and D^λ has then vertex P_n. This finally proves the proposition. □

Proposition 4.18. *With the assumptions of Remark 4.16, let $r > 1$ or $n \not\equiv 0 \pmod p$. Then the assertion of Theorem 4.1 holds, for $\lambda = (n - m, m) = (n - kp - r, kp + r)$.*

Proof. By Proposition 4.4, there exists some $j \in \{0, 1\}$ such that D^λ and the $F\mathfrak{S}_{n-j-1}$-module $D^{(n-kp-r-j,kp+r-1)}$ have common vertices. Furthermore, Proposition 4.5 ensures that $S^\lambda \cong D^\lambda$ if and only if $S^{(n-kp-r-j,kp+r-1)} \cong D^{(n-kp-r-j,kp+r-1)}$. Arguing by induction on r, and applying Propositions 4.6 and 4.8, we deduce that the vertices of D^λ differ from the defect groups of its block if and only if condition (2) of Theorem 4.1 is satisfied. From this the proposition follows. □

Remark 4.19. To summarize, we have now completed the induction, also for the case $m \not\equiv 0 \pmod p$. This finally completes the proof of Theorem 4.1.

5 Further results

Let p be odd, and consider a simple $F\mathfrak{S}_n$-module $D^{(n-m,m)}$. Provided $m \leq \frac{p+1}{2}p - 1$, the vertices of $D^{(n-m,m)}$ are now known, by Theorem 4.1. On the other hand, as mentioned in Remark 3.3, $D^{(n-m,m)}$ is completely splittable, for $m \geq \frac{n-p+2}{2}$. As a direct consequence of these results, the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions are known, for $n \leq p^2 + 2p - 2$. Indeed, we can do better. In the following, we will show that the above methods enable us to determine the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions, for $n < 2p^2$. In case $p = 3$, we can even achieve $n < 27$, by carrying out some additional computer calculations.

Remark 5.1. (a) First of all we should emphasize that the dimension arguments used throughout the previous section generally fail, for $m \geq \frac{p(p+1)}{2}$. For instance, assuming that $k = \frac{p+1}{2}$, $n = 2p^2$, $m = kp$, and $\lambda = (n-m, m)$, we have

$$s+1 = 2p^2 - (p+1)p + 1 = (p-1)p + 1,$$
$$t_1 = 0,\ d_1 = 1,\ \delta = p-1,\ t = t_2 = 1,\ d_2 = p-1,$$
$$\left\lfloor \frac{m}{p^2} \right\rfloor = \left\lfloor \frac{kp}{p^2} \right\rfloor = 0,\ \sum_{i=0}^{\infty} c_i q^i = \Gamma_s(q) = 1,$$

in the notation of Theorem 2.11 and Remark 2.10. Consequently, $c_0 = 1$, $c_i = 0$ for $i \geq 1$, and hence

$$\dim(D^\lambda) = \binom{n}{kp} - \binom{n}{kp-1} - \binom{n}{kp-p+1} + \binom{n}{kp-p} \equiv 0 \pmod{p}.$$

(b) Note that Propositions 4.3 and 4.4 do also hold, for $k > \frac{p-1}{2}$. Consequently, in order to determine the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions $(n-m, m)$, it suffices to consider the cases where $m \equiv 0 \pmod{p}$ and $n \equiv 0 \pmod{p}$, $m \equiv 0 \pmod{p}$ and $n \equiv -1 \pmod{p}$, and the case where $m \equiv 1 \pmod{p}$ and $n \equiv 0 \pmod{p}$.

With this in mind, we obtain the following:

Proposition 5.2. *Let $k := \frac{p+1}{2}$, $r \in \{0, \ldots, p-1\}$, and $m = kp + r$. Furthermore, let $p^2 + p \leq 2m \leq n < 2p^2$ and $\lambda := (n-m, m)$. Then one of the following cases occurs:*

(i) The vertices of D^λ are the defect groups of the block of $F\mathfrak{S}_n$ containing D^λ.

(ii) $n \equiv bp + 2r - 1 \pmod{p^2}$, for some $b \in \{1, \ldots, \frac{p-1}{2}\}$. In this case, $D^\lambda \cong S^\lambda$ with vertex $P := P_{n-2kp-2r} \times (P_{kp})^2$, and P is not a defect group of the block containing D^λ.

Proof. The proof is similar to that of Theorem 4.1. Therefore we omit the details in the calculations. Firstly, by Theorem 2.6, we obtain that $S^\lambda \cong D^\lambda$ if and only if $n \equiv bp + 2r - 1 \pmod{p^2}$, for some $b \in \{1, \ldots, \frac{p-1}{2}\}$. In all these cases, D^λ has vertex $P := P_{n-2kp-2r} \times (P_{kp+r})^2 =_{\mathfrak{S}_n} P_{n-p^2-p-2r} \times (P_p)^{p+1}$, by Theorem 2.4. Moreover, D^λ is then contained in the block of $F\mathfrak{S}_n$ with p-core $(p+r-1, r)$ and p-weight $w = \frac{n-2r+1-p}{p}$. This shows that $P =_{\mathfrak{S}_n} P_{n-p^2-p-2r} \times (P_p)^{p+1} <_{\mathfrak{S}_n} P_{n-p-2r+1} = P_{pw}$ so that P is not a defect group of the block containing D^λ.

Next, we consider the case where $r = 0$ and $n \equiv bp - 1 \pmod{p^2}$, for some $b \in \{0, \frac{p+1}{2}, \ldots, p-1\}$. Then, we again apply Theorem 2.11, in order to determine $\dim(D^\lambda)$. If $n = ap^2 + \tilde{b}p + p - 1$, for some $a \in \mathbb{N}$ and some $\tilde{b} \in \{\frac{p-1}{2}, \ldots, p-1\}$, then Theorem 2.11 yields

$$\dim(D^\lambda) = \binom{n}{kp} - \binom{n}{kp-1} - \binom{n}{(k-p+\tilde{b})p} + \binom{n}{(k-p+\tilde{b})p-1}.$$

This shows that $\dim(D^\lambda) \equiv -2 \pmod{p}$, for $\tilde{b} = \frac{p-1}{2}$. Furthermore, for $\tilde{b} > \frac{p-1}{2}$, we have

$$\dim(D^\lambda) = \frac{n-2kp+1}{(kp)!} \prod_{i=0}^{kp-2} (n-i) - \frac{n-2(k-p+\tilde{b})p+1}{((k-p+\tilde{b})p)!} \prod_{i=0}^{(k-p+\tilde{b})p-2} (n-i) = \frac{z}{(kp)!},$$

where $z \equiv 2p^k((p-1)!)^k\tilde{b}^2(\tilde{b}-1)\cdots(\tilde{b}-(k-2)) \not\equiv 0 \pmod{p^{k+1}}$. Since $\nu_p((kp)!) = k$, this implies $\dim(D^\lambda) \not\equiv 0 \pmod{p}$. Hence D^λ has vertex P_n, in this case.
Now, let $r = 0$ and $n \equiv 0 \pmod{p}$. Then $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^\lambda) \cong D^{(n-kp-1,kp)} \oplus D^{(n-kp,kp-1)}$, by Theorem 2.8. Moreover, $D^{(n-kp,kp-1)}$ has vertex P_{n-p}, by Theorem 4.1. Since $n \not\equiv 0 \pmod{p^2}$, we have $|P_n : P_{n-p}| = p$, and D^λ has thus vertex P_n, by Lemma 2.3. If $r = 0$ and $-1 \not\equiv n \not\equiv 0 \pmod{p}$, then D^λ and $D^{(n-kp-1,kp)}$ have common vertices, and $D^\lambda \not\cong S^\lambda$ and $D^{(n-kp-1,kp)} \not\cong S^{(n-kp-1,kp)}$. This follows from Proposition 4.3. Consequently, in these cases D^λ has vertex P_n, by what we just proved.

We may now assume that $r = 1$. Assume further that $n \equiv 0 \pmod{p}$. Then $D^{(n-kp-2,kp)} \mid \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-2}}(D^\lambda)$, by Theorem 2.8. Since $n-2 \not\equiv -1 \pmod{p}$, we already know that $D^{(n-kp-2,kp)}$ has vertex P_{n-p}. Since $n \not\equiv 0 \pmod{p^2}$, we again have $|P_n : P_{n-p}| = p$, and D^λ has vertex P_n, by Lemma 2.3. If $n \equiv 1 \pmod{p}$ then D^λ and $D^{(n-kp-2,kp)}$ have common vertices, by Proposition 4.4, and $D^\lambda \cong S^\lambda$ if and only if $D^{(n-kp-2,kp)} \cong S^{(n-kp-2,kp)}$, by Proposition 4.5. Thus the vertices of D^λ are as claimed, by what we have proved above. If $0 \not\equiv n \not\equiv 1 \pmod{p}$ then D^λ and $D^{(n-kp-1,kp)}$ have common vertices, by Proposition 4.4, and $D^\lambda \cong S^\lambda$ if and only if $D^{(n-kp-1,kp)} \cong S^{(n-kp-1,kp)}$, by Proposition 4.5. By our previous considerations, this yields the claimed assertion on the vertices of D^λ.

Finally, assume that $r > 1$. Then D^λ and $D^\mu := D^{(n-kp-r-j,kp+r-1)}$ have common vertices, for some $j \in \{0,1\}$, by Proposition 4.4. Furthermore, $D^\lambda \cong S^\lambda$ if and only if $D^\mu \cong S^\mu$, by Proposition 4.5. We now argue by induction on r, and finally obtain the vertices of D^λ as claimed. This completes the proof of the proposition. □

Proposition 5.3. *Let $1 \leq n \leq 2p^2-1$. Then the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions are known. If $m \in \{0,\ldots,\lfloor \frac{n}{2} \rfloor\}$ then $D^{(n-m,m)}$ has the defect groups of its block as its vertices, unless $S^{(n-m,m)} \cong D^{(n-m,m)}$. In the latter case $D^{(n-m,m)}$ has vertex $P_{n-2m} \times (P_m)^2$.*

Proof. For $p = 3$, the assertion follows immediately from Theorem 4.1 and Proposition 5.2. Therefore, we may now assume that $p > 3$. By Theorem 4.1 and Proposition 5.2, the vertices of $D^{(n-m,m)}$ are known to be as claimed, for $m \leq \frac{p+1}{2}p + p - 1$. Moreover, for $m \geq \frac{n-p+2}{2}$, we have $n - 2m \leq p - 2$. Consequently, $D^{(n-m,m)}$ is then completely splittable, and the defect groups of its block are vertices of $D^{(n-m,m)}$, by Theorem 3.4. In particular, the assertion of the proposition follows, for $n \leq p^2 + 4p - 2$.

It thus remains to consider the cases where $p^2+4p-1 \leq n \leq 2p^2-1$ and $m \in \{\frac{p+3}{2}p,\ldots,\lfloor\frac{n-p+1}{2}\rfloor\}$. For this, we write $m = kp + r$, for appropriate $k \in \{\frac{p+3}{2},\ldots,p-1\}$ and $r \in \{0,\ldots,p-1\}$. In analogy to our proof of Theorem 4.1, we will argue by induction on m, and will distinguish between three cases.

Case 1: $m \equiv 0 \pmod{p}$ and $n \equiv -1 \pmod{p}$. Then $m = kp$, for some $k \in \{\frac{p+3}{2},\ldots,p-1\}$, and $n = p^2 + xp - 1$, for some $x \in \{4,\ldots,p\}$. In the notation of Theorem 2.11, we thus have

$$s+1 = p(p+x-2k),\ t = t_1 = 1,\ d_1 = p+x-2k,\ \delta = (2k-x)p,$$
$$\left\lfloor \frac{m}{p^{t+1}} \right\rfloor = 0,\ \Gamma_s(q) = 1.$$

Consequently,

$$\dim(D^{(n-m,m)}) = \binom{n}{kp} - \binom{n}{kp-1} - \binom{n}{p(x-k)} + \binom{n}{p(x-k)-1},$$

and, in particular, $D^{(n-m,m)} \cong S^{(n-m,m)}$ if and only if $k > x$. Hence, for $k > x$, the module $D^{(n-m,m)}$ has vertex $P_{n-2m} \times (P_m)^2$, by Theorem 2.4. Now suppose that $k < x$. Then we obtain

$$\dim(D^{(n-m,m)}) = \frac{n-2kp+1}{(kp)!} \prod_{i=0}^{kp-2} (n-i) - \frac{n-2(x-k)p+1}{(p(x-k))!} \prod_{i=0}^{p(x-k)-2} (n-i) = \frac{z}{(xp)!},$$

where

$$z \equiv \begin{cases} 2p^x(x-2k)((p-1)!)^x(x-1)\cdots(x-(k-1))(k+1)\cdots x \pmod{p^{x+1}}, & \text{for } x<p \\ -2p^{x+1}2k((p-1)!)^x(p-1)\cdots(p-(k-1))(k+1)\cdots(p-1) \pmod{p^{x+2}}, & \text{for } x=p. \end{cases}$$

Since $\nu_p((xp)!) = x$ for $x < p$, and $\nu_p((xp)!)) = x+1$ for $x = p$, this shows that $\dim(D^{(n-m,m)}) \not\equiv 0 \pmod p$. Finally, suppose that $k = x < p$. Then we have

$$\dim(D^{(n-m,m)}) = \binom{n}{xp} - \binom{n}{xp-1} - 1 \equiv -2 \pmod p.$$

Hence, also in this case, $\dim(D^{(n-m,m)}) \not\equiv 0 \pmod p$. Consequently, $D^{(n-m,m)}$ has vertex P_n, provided $n = p^2 + xp - 1$, for some $x \in \{4, \ldots, p\}$, and $m = kp$, for some $k \in \{\frac{p+3}{2}, \ldots, x\}$ with $k \leq p-1$. This completes case 1.

Case 2: $m \equiv 0 \pmod p$ and $n \not\equiv -1 \pmod p$. If $n \equiv 0 \pmod p$ then $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^{(n-m,m)}) \cong D^{(n-m,m-1)} \oplus D^{(n-m-1,m)}$, by Theorem 2.8. The simple $F\mathfrak{S}_{n-1}$-module $D^{(n-m,m-1)}$ belongs to the block of $F\mathfrak{S}_{n-1}$ with p-core $(p-2,1)$ and p-weight $\frac{n-p}{p}$. Moreover, $D^{(n-m,m-1)} \not\cong S^{(n-m,m-1)}$, by Theorem 2.6. Thus, by induction, $D^{(n-m,m-1)}$ has vertex P_{n-p}. Since $n \not\equiv 0 \pmod{p^2}$, this implies $|P_n : P_{n-p}| = p$ so that $D^{(n-m,m)}$ has then vertex P_n, by Lemma 2.3.

If $n \not\equiv 0 \pmod p$ then $D^{(n-m,m)}$ and $D^{(n-m-1,m)}$ have common vertices, by Proposition 4.3. Moreover, $D^{(n-m,m)} \not\cong S^{(n-m,m)}$ and $D^{(n-m-1,m)} \not\cong S^{(n-m-1,m)}$, by Proposition 4.3. Therefore we may then argue by induction on n, and obtain that $D^{(n-m,m)}$ has vertex P_n, by our previous considerations.

Case 3: $m \not\equiv 0 \pmod p$. Then $m = kp + r$, for some $k \in \{\frac{p+3}{2}, \ldots, p-1\}$ and some $r \in \{1, \ldots, p-1\}$. If $r = 1$ and $n \equiv 0 \pmod p$ then $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(D^{(n-m,m)}) \cong D^{(n-m-1,m)} = D^{(n-kp-2,kp+1)}$, by Theorem 2.8. Since $2r-1 \not\equiv n-1 \not\equiv 2r-2 \pmod p$, the modules $D^{(n-kp-2,kp+1)}$ and $D^{(n-kp-2,kp)}$ have common vertices, by Proposition 4.4. By Theorem 2.6,

$D^{(n-kp-2,kp)} \not\cong S^{(n-kp-2,kp)}$ so that $D^{(n-kp-2,kp)}$ has vertex P_{n-p}, by our inductive hypothesis. Since $n \not\equiv 0 \pmod{p^2}$, we have $|P_n : P_{n-p}| = p$, and $D^{(n-m,m)}$ has thus vertex P_n, by Lemma 2.3.

Finally, let $r > 1$ or $n \not\equiv 0 \pmod{p}$. By Proposition 4.4, there is an appropriate $j \in \{0, 1\}$ such that $D^{(n-m,m)}$ and the simple $F\mathfrak{S}_{n-1-j}$-module $D^{(n-m-j,m-1)}$ have common vertices. Furthermore, $D^{(n-m,m)} \cong S^{(n-m,m)}$ if and only if $D^{(n-m-j,m-1)} \cong S^{(n-m-j,m-1)}$, by Proposition 4.5. Hence, by induction, $D^{(n-m,m)}$ has the defect groups of its block as vertices, unless $D^{(n-m,m)} \cong S^{(n-m,m)}$. In the latter case, $D^{(n-m,m)}$ has clearly vertex $P_{n-2m} \times (P_m)^2$, by Theorem 2.4. This completes the proof of the proposition. □

Remark 5.4. (a) Hence, for $p = 3$, the first module whose vertices cannot be determined via the above methods is the simple $F\mathfrak{S}_{18}$-module $D^{(12,6)}$ of dimension $7752 \equiv 0 \pmod{3}$. We have investigated this module with the computer. By Theorem 2.8, we have $\mathrm{Res}^{\mathfrak{S}_{18}}_{\mathfrak{S}_{17}}(D^{(12,6)}) \cong D^{(12,5)} \oplus D^{(11,6)}$. Furthermore, by Theorem 4.1, the simple $F\mathfrak{S}_{17}$-module $D^{(12,5)}$ has vertex P_{15}. Thus

$$P_9 \times (P_3)^2 = P_{15} <_{\mathfrak{S}_{18}} P \leq P_{18} = P_9 \times P_9,$$

for some vertex P of $D^{(12,6)}$, by Lemma 2.1. From Lemma 2.3 we now deduce that either $P = P_{18}$, or $P = P_9 \times (P_3)^3$. Next, consider the cyclic group $C = \langle (1, \ldots, 9)(10, \ldots, 18) \rangle \leq_{\mathfrak{S}_{18}} P_{18}$. Our computations show that

$$\mathrm{Res}^{\mathfrak{S}_{18}}_{C}(D^{(12,6)}) \cong F \oplus 5M_1 \oplus M_2 \oplus \mathrm{proj},$$

where M_1 and M_2 are indecomposable of dimension 3 and 5, respectively. The indecomposable projective direct summands of $\mathrm{Res}^{\mathfrak{S}_{18}}_{C}(D^{(12,6)})$ are easily detected using the algorithm "ProjSummands" presented in [4]. The trivial module F has clearly vertex C, and hence C has to be contained in some vertex of $D^{(12,6)}$, by Lemma 2.1. Since $C \not\leq_{\mathfrak{S}_{18}} P_9 \times (P_3)^3$, we finally conclude that $D^{(12,6)}$ has vertex P_{18}.

(b) For $p = 3$, the next module whose vertices cannot be derived from Theorem 4.1, or from Proposition 5.2, is the simple $F\mathfrak{S}_{18}$-module $D^{(11,7)}$ of dimension $4845 \equiv 0 \pmod{3}$ which we have also treated with the computer. Here we have $\mathrm{Res}^{\mathfrak{S}_{18}}_{\mathfrak{S}_{16}}(D^{(11,7)}) \cong D^{(10,6)} \oplus D^{(9,7)}$, by Theorem 2.8. Moreover, $D^{(10,6)}$ has vertex P_{15}, by Proposition 5.2. Thus, with the same arguments as above, we obtain that $D^{(11,7)}$ has either vertex P_{18} or vertex $P_9 \times (P_3)^3$. When restricting $D^{(11,7)}$ to the cyclic group C of order 9 mentioned in (a), we also get $F \mid \mathrm{Res}^{\mathfrak{S}_{18}}_{C}(D^{(11,7)})$, so that $D^{(11,7)}$ in fact has vertex P_{18}.

(c) In fact, we can now determine the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions, for $p = 3$ and $n \leq 26$. The alternating $F\mathfrak{S}_n$-module $D^{(n-\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)}$ has clearly vertex P_n, for all n. Furthermore, we observe the following:

$\boxed{n = 19:}$ $D^{(13,6)} \longleftrightarrow_0 D^{(12,6)}$, $D^{(12,7)} \longleftrightarrow_2 D^{(11,6)}$, $D^{(11,8)} \longleftrightarrow_0 D^{(11,7)}$. Hence these modules have the defect groups of their blocks as vertices.

$\boxed{n = 20:}$ $D^{(14,6)} \cong S^{(14,6)}$ with vertex $(P_3)^6$ which is not a defect group of the block containing $D^{(14,6)}$. Moreover, we have $D^{(13,7)} \longleftrightarrow_2 D^{(13,6)}$, $D^{(12,8)} \longleftrightarrow_2 D^{(11,8)}$, $\dim(D^{(11,9)}) =$

$\binom{20}{9} - \binom{20}{8} - \binom{20}{3} + \binom{20}{2} + 1 = 41041 \equiv 1 \pmod 3$. Thus all these modules have vertex P_{18}.

$\boxed{n = 21:}$ $\mathrm{Res}^{\mathfrak{S}_{21}}_{\mathfrak{S}_{20}}(D^{(15,6)}) \cong D^{(15,5)} \oplus D^{(14,6)}$, $\mathrm{Res}^{\mathfrak{S}_{21}}_{\mathfrak{S}_{20}}(D^{(14,7)}) \cong D^{(13,7)}$ and $\mathrm{Res}^{\mathfrak{S}_{21}}_{\mathfrak{S}_{20}}(D^{(12,9)}) \cong D^{(12,8)} \oplus D^{(11,9)}$. Thus these modules have vertex $P_{21} = P_{18} \times P_3$, by Lemma 2.3. Moreover, $D^{(13,8)} \longleftrightarrow_0 D^{(12,7)}$ with vertex P_{15}.

$\boxed{n = 22:}$ $D^{(16,6)} \longleftrightarrow_0 D^{(15,6)}$ with vertex P_{21}, and $D^{(15,7)} \cong S^{(15,7)}$ with vertex $(P_3)^6$ which is not a defect group of the block containing $D^{(15,7)}$. Moreover, $D^{(14,8)} \longleftrightarrow_0 D^{(14,7)}$ with vertex P_{21}. Finally, $D^{(12,10)} \longleftrightarrow_2 D^{(11,9)}$ and $D^{(13,9)} \longleftrightarrow_0 D^{(12,9)}$. Hence $D^{(12,10)}$ has vertex P_{18}, and $D^{(13,9)}$ has vertex P_{21}.

$\boxed{n = 23:}$ Since $\dim(D^{(17,6)}) = \binom{23}{6} - \binom{23}{5} - 1 = 67297 \equiv 1 \pmod 3$, the module $D^{(17,6)}$ has vertex P_{21}. Furthermore, $D^{(16,7)} \longleftrightarrow_2 D^{(16,6)}$ with vertex P_{21} and $D^{(15,8)} \longleftrightarrow_2 D^{(14,8)}$, $D^{(13,10)} \longleftrightarrow_2 D^{(13,9)}$, and $\dim(D^{(14,9)}) = \binom{23}{9} - \binom{23}{8} - \binom{23}{6} + \binom{25}{5} + 1 = 259579 \equiv 1 \pmod 3$. Also these modules have thus vertex P_{21}.

$\boxed{n = 24:}$ Here we have $\mathrm{Res}^{\mathfrak{S}_{24}}_{\mathfrak{S}_{23}}(D^{(18,6)}) \cong D^{(18,5)} \oplus D^{(17,6)}$ and $\mathrm{Res}^{\mathfrak{S}_{24}}_{\mathfrak{S}_{23}}(D^{(17,7)}) \cong D^{(16,7)}$ so that both modules have vertex P_{24}, by Lemma 2.3. Furthermore, $D^{(16,8)} \cong S^{(16,8)}$ with vertex $(P_3)^6$ which not a defect group of the block containing $D^{(16,8)}$, and $D^{(13,11)} \longleftrightarrow_0 D^{(12,10)}$ has vertex P_{18}. Finally, $\mathrm{Res}^{\mathfrak{S}_{24}}_{\mathfrak{S}_{23}}(D^{(14,10)}) \cong D^{(13,10)}$ and $\mathrm{Res}^{\mathfrak{S}_{24}}_{\mathfrak{S}_{23}}(D^{(15,9)}) \cong D^{(15,8)} \oplus D^{(14,9)}$. Consequently, both modules have vertex $P_{24} = P_{21} \times P_3$, by Lemma 2.3.

$\boxed{n = 25:}$ $D^{(19,6)} \longleftrightarrow_0 D^{(18,6)}$, $D^{(18,7)} \longleftrightarrow_2 D^{(17,6)}$, $D^{(17,8)} \longleftrightarrow_0 D^{(17,7)}$, $D^{(14,11)} \longleftrightarrow_0 D^{(14,10)}$, $D^{(15,10)} \longleftrightarrow_2 D^{(14,9)}$, and $D^{(16,9)} \longleftrightarrow_0 D^{(15,9)}$. Hence all these modules have the defect groups of their blocks as vertices.

$\boxed{n = 26:}$ $\dim(D^{(20,6)}) = \binom{26}{6} - \binom{26}{5} - \binom{26}{3} + \binom{26}{2} = 162175 \equiv 1 \pmod 3$ so that $D^{(20,6)}$ has vertex P_{24}. We also have $D^{(19,7)} \longleftrightarrow_2 D^{(19,6)}$ and $D^{(18,8)} \longleftrightarrow_2 D^{(17,8)}$ so that both modules have vertex P_{24}. Furthermore, $\dim(D^{(14,12)}) = \binom{26}{12} - \binom{26}{11} - \binom{26}{6} + \binom{26}{5} + \binom{26}{3} - \binom{26}{2} = 1769365 \equiv 1 \pmod 3$, $D^{(15,11)} \longleftrightarrow_2 D^{(14,11)}$, $D^{(16,10)} \longleftrightarrow_2 D^{(16,9)}$, and $D^{(17,9)} \cong S^{(17,9)}$. Thus all these modules have vertex P_{24}.

The simple $F\mathfrak{S}_{27}$-module $D^{(21,6)}$ of dimension $200655 \equiv 0 \pmod 9$ is the first module whose vertices cannot be determined with the above methods, and which we currently cannot treat with the computer either. We suspect that the vertices of $D^{(21,6)}$ are the Sylow 3-subgroups of $\mathfrak{S}_{27}$.

To summarize, Proposition 5.3 and our additional considerations in characteristic 3 lead to the following:

Corollary 5.5. *Let $p > 2$ and $1 \leq n \leq f(p)$ where $f(3) = 26$, and $f(p) := 2p^2 - 1$ for $p > 3$. Then the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions are known. If $m \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$ then $D^{(n-m,m)}$ has the defect groups of its block as its vertices, unless $S^{(n-m,m)} \cong D^{(n-m,m)}$. In the latter case $D^{(n-m,m)}$ has vertex $P_{n-2m} \times (P_m)^2$.*

6 Closing remarks

We close with some remarks and questions arising from our results in the previous sections. In view of Theorem 4.1 and Corollary 5.5, we are led to the following general question concerning simple $F\mathfrak{S}_n$-modules labelled by two part partitions:

Question 6.1. Let $p > 2$ and $n \in \mathbb{N}$. Consider the simple $F\mathfrak{S}_n$-module $D^{(n-m,m)}$, for $m \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$. Provided $D^{(n-m,m)} \ncong S^{(n-m,m)}$, are the vertices of $D^{(n-m,m)}$ then precisely the defect groups of the block containing $D^{(n-m,m)}$? Of course, by Theorem 2.4, $D^{(n-m,m)}$ has vertex $P_{n-2m} \times (P_m)^2$ if $D^{(n-m,m)} \cong S^{(n-m,m)}$.

Remark 6.2. Suppose that $p > 2$, and let μ be any p-regular partition. Then the simple $F\mathfrak{S}_n$-modules D^μ and $D^\mu \otimes S^{(1^n)} =: D^{\mathbf{m}(\mu)}$ have common vertices. Here, $\mathbf{m}(\mu)$ denotes the *Mullineux conjugate* partition of μ. In this way, once knowing the vertices of simple $F\mathfrak{S}_n$-modules labelled by two part partitions, we also have the vertices of the simple $F\mathfrak{S}_n$-modules labelled by the respective Mullineux conjugate partitions. The rather technical procedure for constructing the Mullineux conjugate of a given p-regular partition of n, is the content of the Mullineux conjecture (cf. [19]), proved by Ford and Kleshchev in [6].

The tables below display the vertices of all simple $F\mathfrak{S}_n$-modules labelled by two part partitions and of those labelled by their Mullineux conjugate partitions, for $p = 3$ and $n \leq 26$. Blocks are parametrized by their 3-cores, and simple modules by their labelling partitions; neighbouring partitions are always Mullineux conjugate to each other. Moreover, column "def." contains the defect groups of the blocks under consideration. Whenever a simple $F\mathfrak{S}_n$-module is isomorphic to some Specht module, the corresponding partition is displayed in column "Specht".

Remark 6.3. Suppose that $p = 2$, and fix $n \in \mathbb{N}$. As stated earlier, in this case the vertices of simple $F\mathfrak{S}_n$-modules parametrized by two part partitions are rather poorly understood. However, some partial results on vertices of the simple $F\mathfrak{S}_n$-modules $D^{(n-m,m)}$, for $m \in \{2, 3\}$ and $m \in \{4, 5\}$, respectively, can be found in [23] and [7], respectively. Some further computational results concerning vertices of simple $F\mathfrak{S}_n$-modules $D^{(n-m,m)}$ are given in [2], [3] and [23]. Building on our computational data already mentioned in the introduction, we suspect that the simple $F\mathfrak{S}_n$-module $D^{(n-2,2)}$ has either vertex P_n, or $n = 5$, or $n \equiv 3 \pmod 4$. As explained earlier, the Sylow 2-subgroup of $\mathfrak{A}_4$ is a vertex of $D^{(3,2)}$. Moreover, for $n \equiv 3 \pmod 4$, we obtain $S^{(n-2,2)} \cong D^{(n-2)}$, by Theorem 2.6, and $D^{(n-2,2)}$ has then vertex $P_{n-4} \times (P_2)^2 \notin \mathrm{Syl}_2(\mathfrak{S}_n)$, by Theorem 2.4.

References

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[3] S. Danz, *Vertices of low dimensional simple modules for symmetric groups*, to appear in Comm. Algebra

Simple $F\mathfrak{S}_n$-modules in characteristic 3

n	blocks		def.	parts.		vtx.	Specht	
1	(1)		1	$(\mathbf{1})$		1	(1)	
2	(2)	(1^2)	1	$(\mathbf{2})$	$(\mathbf{1^2})$	1	(2)	(1^2)
3	$\emptyset$		P_3	$(\mathbf{3})$	$(\mathbf{2,1})$	P_3	(3)	(1^3)
4	(1)		P_3	$(\mathbf{4})$	$(\mathbf{2^2})$	P_3	(4)	(1^4)
	$(3,1)$	$(2,1^2)$	1	$(\mathbf{3,1})$	$(2,1^2)$	1	$(3,1)$	$(2,1^2)$
5	(2)	(1^2)	P_3	$(\mathbf{5})$	$(\mathbf{3,2})$	P_3	(5)	(1^5)
				$(2^2,1)$	$(\mathbf{4,1})$	P_3	$(2,1^3)$	$(4,1)$
6	$\emptyset$		P_3^2	$(\mathbf{6})$	$(\mathbf{3^2})$	P_3^2	(6)	(1^6)
				$(\mathbf{5,1})$	$(3,2,1)$	P_3^2	-	
	$(4,2)$	$(2^2,1^2)$	1	$(\mathbf{4,2})$	$(2^2,1^2)$	1	$(4,2)$	$(2^2,1^2)$
7	(1)		P_3^2	$(\mathbf{7})$	$(\mathbf{4,3})$	P_3^2	(7)	(1^7)
				$(\mathbf{5,2})$	$(3,2,1^2)$	P_3^2	-	
	$(3,1)$	$(2,1^2)$	P_3	$(\mathbf{6,1})$	$(3^2,1)$	P_3	$(6,1)$	$(2,1^5)$
8	(2)	(1^2)	P_3^2	$(\mathbf{8})$	$(\mathbf{4^2})$	P_3^2	(8)	(1^8)
				$(\mathbf{5,3})$	$(3,2^2,1)$	P_3^2	$(5,3)$	$(2^3,1^2)$
				$(4,3,1)$	$(\mathbf{7,1})$	P_3^2	$(2,1^6)$	$(7,1)$
				$(3^2,1^2)$	$(\mathbf{6,2})$	P_3^2	-	-
9	$\emptyset$		P_9	$(\mathbf{9})$	$(\mathbf{5,4})$	P_9	(9)	(1^9)
				$(\mathbf{8,1})$	$(4^2,1)$	P_9	-	
				$(\mathbf{6,3})$	$(3^2,2,1)$	P_9	-	
	$(4,2)$	$(2^2,1^2)$	P_3	$(\mathbf{7,2})$	$(4,3,1^2)$	P_3	$(7,2)$	$(2^2,1^5)$
10	(1)		P_9	$(\mathbf{10})$	$(\mathbf{5^2})$	P_9	(10)	(1^{10})
				$(\mathbf{8,2})$	$(4^2,1^2)$	P_9	-	
				$(\mathbf{7,3})$	$(4,3,2,1)$	P_9	-	
	$(3,1)$	$(2,1^2)$	P_3^2	$(\mathbf{9,1})$	$(5,4,1)$	P_3^2	$(9,1)$	$(2,1^8)$
				$(\mathbf{6,4})$	$(3^2,2^2)$	P_3^2	$(6,4)$	$(2^4,1^2)$
11	(2)	(1^2)	P_9	$(\mathbf{11})$	$(\mathbf{6,5})$	P_9	(11)	(1^{11})
				$(\mathbf{8,3})$	$(4^2,2,1)$	P_9	-	-
				$(5^2,1)$	$(\mathbf{10,1})$	P_9	$(2,1^9)$	$(10,1)$
				$(5,4,1^2)$	$(\mathbf{9,2})$	P_9	-	-
				$(4,3,2^2)$	$(\mathbf{7,4})$	P_9	-	-

Table 1: simple $F\mathfrak{S}_n$-modules in characteristic 3; $n = 1, \dots, 11$

n	blocks		def.	parts.		vtx.	Specht	
12	$\emptyset$		P_{12}	$(\mathbf{12})$	$(\mathbf{6^2})$	P_{12}	(12)	(1^{12})
				$(\mathbf{11},\mathbf{1})$	$(6,5,1)$	P_{12}	-	
				$(\mathbf{9},\mathbf{3})$	$(5,4,2,1)$	P_{12}	-	
				$(\mathbf{8},\mathbf{4})$	$(4^2,2^2)$	P_{12}	-	
	$(4,2)$	$(2^2,1^2)$	P_3^2	$(\mathbf{10},\mathbf{2})$	$(5^2,1^2)$	P_3^2	$(10,2)$	$(2^2,1^8)$
				$(\mathbf{7},\mathbf{5})$	$(4,3^2,2)$	P_3^2	$(7,5)$	$(2^5,1^2)$
13	(1)		P_{12}	$(\mathbf{13})$	$(\mathbf{7},\mathbf{6})$	P_{12}	(13)	(1^{13})
				$(\mathbf{11},\mathbf{2})$	$(6,5,1^2)$	P_{12}	-	
				$(\mathbf{10},\mathbf{3})$	$(5^2,2,1)$	P_{12}	-	
				$(\mathbf{8},\mathbf{5})$	$(4^2,3,2)$	P_{12}	-	
	$(3,1)$	$(2,1^2)$	P_9	$(\mathbf{12},\mathbf{1})$	$(6^2,1)$	P_9	$(12,1)$	$(2,1^{11})$
				$(\mathbf{9},\mathbf{4})$	$(5,4,2^2)$	P_9	-	-
14	(2)	(1^2)	P_{12}	$(\mathbf{14})$	$(\mathbf{7^2})$	P_{12}	(14)	(1^{14})
				$(\mathbf{11},\mathbf{3})$	$(6,5,2,1)$	P_3^4	$(11,3)$	$(2^3,1^8)$
				$(\mathbf{8},\mathbf{6})$	$(4^2,3^2)$	P_{12}	-	-
				$(7,6,1)$	$(\mathbf{13},\mathbf{1})$	P_{12}	$(2,1^{12})$	$(13,1)$
				$(6^2,1^2)$	$(\mathbf{12},\mathbf{2})$	P_{12}	-	-
				$(5^2,2^2)$	$(\mathbf{10},\mathbf{4})$	P_{12}	-	-
				$(5,4,3,2)$	$(\mathbf{9},\mathbf{5})$	P_{12}	-	-
15	$\emptyset$		P_{15}	$(\mathbf{15})$	$(\mathbf{8},\mathbf{7})$	P_{15}	(15)	(1^{15})
				$(\mathbf{14},\mathbf{1})$	$(7^2,1)$	P_{15}	-	
				$(\mathbf{12},\mathbf{3})$	$(6^2,2,1)$	P_{15}	-	
				$(\mathbf{11},\mathbf{4})$	$(6,5,2^2)$	P_{15}	-	
				$(\mathbf{9},\mathbf{6})$	$(5,4,3^2)$	P_{15}	-	
	$(4,2)$	$(2^2,1^2)$	P_9	$(\mathbf{13},\mathbf{2})$	$(7,6,1^2)$	P_9	$(13,2)$	$(2^2,1^{11})$
				$(\mathbf{10},\mathbf{5})$	$(5^2,3,2)$	P_9	-	-
16	(1)		P_{15}	$(\mathbf{16})$	$(\mathbf{8^2})$	P_{15}	(16)	(1^{16})
				$(\mathbf{14},\mathbf{2})$	$(7^2,1^2)$	P_{15}	-	
				$(\mathbf{13},\mathbf{3})$	$(7,6,2,1)$	P_{15}	-	
				$(\mathbf{11},\mathbf{5})$	$(6,5,3,2)$	P_{15}	-	
				$(\mathbf{10},\mathbf{6})$	$(5^2,3^2)$	P_{15}	-	
	$(3,1)$	$(2,1^2)$	P_{12}	$(\mathbf{15},\mathbf{1})$	$(8,7,1)$	P_{12}	$(15,1)$	$(2,1^{14})$
				$(\mathbf{12},\mathbf{4})$	$(6^2,2^2)$	P_3^4	$(12,4)$	$(2^4,1^8)$
				$(\mathbf{9},\mathbf{7})$	$(5,4^2,3)$	P_{12}	-	-

Table 2: simple $F\mathfrak{S}_n$-modules in characteristic 3; $n=12,\dots,16$

n	blocks		def.	parts.		vtx.	Specht	
17	(2)	(1^2)	P_{15}	$(\mathbf{17})$	$(\mathbf{9},\mathbf{8})$	P_{15}	(17)	(1^{17})
				$(\mathbf{14},\mathbf{3})$	$(7^2,2,1)$	P_{15}	$(14,3)$	$(2^3,1^{11})$
				$(\mathbf{11},\mathbf{6})$	$(6,5,3^2)$	P_{15}	-	-
				$(8^2,1)$	$(\mathbf{16},\mathbf{1})$	P_{15}	$(2,1^{15})$	$(16,1)$
				$(8,7,1^2)$	$(\mathbf{15},\mathbf{2})$	P_{15}	-	-
				$(7,6,2^2)$	$(\mathbf{13},\mathbf{4})$	P_{15}	-	-
				$(6^2,3,2)$	$(\mathbf{12},\mathbf{5})$	P_{15}	-	-
				$(5^2,4,3)$	$(\mathbf{10},\mathbf{7})$	P_{15}	-	-
18	$\emptyset$		P_{18}	$(\mathbf{18})$	$(\mathbf{9^2})$	P_{18}	(18)	(1^{18})
				$(\mathbf{17},\mathbf{1})$	$(9,8,1)$	P_{18}	-	
				$(\mathbf{15},\mathbf{3})$	$(8,7,2,1)$	P_{18}	-	
				$(\mathbf{14},\mathbf{4})$	$(7^2,2^2)$	P_{18}	-	
				$(\mathbf{12},\mathbf{6})$	$(6^2,3^2)$	P_{18}	-	
				$(\mathbf{11},\mathbf{7})$	$(6,5,4,3)$	P_{18}	-	
	$(4,2)$	$(2^2,1^2)$	P_{12}	$(\mathbf{16},\mathbf{2})$	$(8^2,1^2)$	P_{12}	$(16,2)$	$(2^2,1^{14})$
				$(\mathbf{13},\mathbf{5})$	$(7,6,3,2)$	P_3^4	$(13,5)$	$(2^5,1^8)$
				$(\mathbf{10},\mathbf{8})$	$(5^2,4^2)$	P_{12}	-	-
19	(1)		P_{18}	$(\mathbf{19})$	$(\mathbf{10},\mathbf{9})$	P_{18}	(19)	(1^{19})
				$(\mathbf{17},\mathbf{2})$	$(9,8,1^2)$	P_{18}	-	
				$(\mathbf{16},\mathbf{3})$	$(8^2,2,1)$	P_{18}	-	
				$(\mathbf{14},\mathbf{5})$	$(7^2,3,2)$	P_{18}	-	
				$(\mathbf{13},\mathbf{6})$	$(7,6,3^2)$	P_{18}	-	
				$(\mathbf{11},\mathbf{8})$	$(6,5,4^2)$	P_{18}	-	
	$(3,1)$	$(2,1^2)$	P_{15}	$(\mathbf{18},\mathbf{1})$	$(9^2,1)$	P_{15}	$(18,1)$	$(2,1^{17})$
				$(\mathbf{15},\mathbf{4})$	$(8,7,2^2)$	P_{15}	$(15,4)$	$(2^4,1^{11})$
				$(\mathbf{12},\mathbf{7})$	$(6^2,4,3)$	P_{15}	-	-
20	(2)	(1^2)	P_{18}	$(\mathbf{20})$	$(\mathbf{10^2})$	P_{18}	(20)	(1^{20})
				$(\mathbf{17},\mathbf{3})$	$(9,8,2,1)$	P_{18}	-	-
				$(\mathbf{14},\mathbf{6})$	$(7^2,3^2)$	P_3^6	$(14,6)$	$(2^6,1^8)$
				$(\mathbf{11},\mathbf{9})$	$(6,5^2,4)$	P_{18}	-	-
				$(10,9,1)$	$(\mathbf{19},\mathbf{1})$	P_{18}	$(2,1^{18})$	$(19,1)$
				$(9^2,1^2)$	$(\mathbf{18},\mathbf{2})$	P_{18}	-	-
				$(8^2,2^2)$	$(\mathbf{16},\mathbf{4})$	P_{18}	-	-
				$(8,7,3,2)$	$(\mathbf{15},\mathbf{5})$	P_{18}	-	-
				$(7,6,4,3)$	$(\mathbf{13},\mathbf{7})$	P_{18}	-	-
				$(6^2,4^2)$	$(\mathbf{12},\mathbf{8})$	P_{18}	-	-

Table 3: simple $F\mathfrak{S}_n$-modules in characteristic 3; $n = 17, \ldots, 20$

n	blocks		def.	parts.		vtx.	Specht	
21	$\emptyset$		P_{21}	$(\mathbf{21})$	$(\mathbf{10^2})$	P_{21}	(21)	(1^{21})
				$(\mathbf{20,1})$	$(10^2,1)$	P_{21}	-	
				$(\mathbf{18,3})$	$(9^2,2,1)$	P_{21}	-	
				$(\mathbf{17,4})$	$(9,8,2^2)$	P_{21}	-	
				$(\mathbf{15,6})$	$(8,7,3^2)$	P_{21}	-	
				$(\mathbf{14,7})$	$(7^2,4,3)$	P_{21}	-	
				$(\mathbf{12,9})$	$(6^2,5,4)$	P_{21}	-	
	$(4,2)$	$(2^2,1^2)$	P_{15}	$(\mathbf{19,2})$	$(10,9,1^2)$	P_{15}	$(19,2)$	$(2^2,1^{17})$
				$(\mathbf{16,5})$	$(8^2,3,2)$	P_{15}	$(16,5)$	$(2^5,1^{11})$
				$(\mathbf{13,8})$	$(7,6,4^2)$	P_{15}	-	-
22	(1)		P_{21}	$(\mathbf{22})$	$(\mathbf{11^2})$	P_{21}	(22)	(1^{22})
				$(\mathbf{20,2})$	$(10^2,1^2)$	P_{21}	-	
				$(\mathbf{19,3})$	$(10,9,2,1)$	P_{21}	-	
				$(\mathbf{17,5})$	$(9,8,3,2)$	P_{21}	-	
				$(\mathbf{16,6})$	$(8^2,3^2)$	P_{21}	-	
				$(\mathbf{14,8})$	$(7^2,4^2)$	P_{21}	-	
				$(\mathbf{13,9})$	$(7,6,5,4)$	P_{21}	-	
	$(3,1)$	$(2,1^2)$	P_{18}	$(\mathbf{21,1})$	$(11,10,1)$	P_{18}	$(21,1)$	$(2,1^{20})$
				$(\mathbf{18,4})$	$(9^2,2^2)$	P_{18}	-	-
				$(\mathbf{15,7})$	$(8,7,4,3)$	P_3^6	$(15,7)$	$(2^7,1^8)$
				$(\mathbf{12,10})$	$(6^2,5^2)$	P_{18}	-	-
23	(2)	(1^2)	P_{21}	$(\mathbf{23})$	$(\mathbf{12,11})$	P_{21}	(23)	(1^{23})
				$(\mathbf{20,3})$	$(10^2,2,1)$	$P_9 \times P_3^4$	$(20,3)$	$(2^3,1^{17})$
				$(\mathbf{17,6})$	$(9,8,3^2)$	P_{21}	-	-
				$(\mathbf{14,9})$	$(7^2,5,4)$	P_{21}	-	-
				$(11^2,1)$	$(\mathbf{22,1})$	P_{21}	$(2,1^{21})$	$(22,1)$
				$(11,10,1^2)$	$(\mathbf{21,2})$	P_{21}	-	-
				$(10,9,2^2)$	$(\mathbf{19,4})$	P_{21}	-	-
				$(9^2,3,2)$	$(\mathbf{18,5})$	P_{21}	-	-
				$(8^2,4,3)$	$(\mathbf{16,7})$	P_{21}	-	-
				$(8,7,4^2)$	$(\mathbf{15,8})$	P_{21}	-	-
				$(7,6,5^2)$	$(\mathbf{13,10})$	P_{21}	-	-

Table 4: simple $F\mathfrak{S}_n$-modules in characteristic 3; $n = 21, \ldots, 23$

n	blocks		def.	parts.		vtx.	Specht	
24	$\emptyset$		P_{24}	$(\mathbf{24})$	$(\mathbf{12^2})$	P_{24}	(24)	(1^{24})
				$(\mathbf{23},\mathbf{1})$	$(12,11,1)$	P_{24}	-	
				$(\mathbf{21},\mathbf{3})$	$(11,10,2,1)$	P_{24}	-	
				$(\mathbf{20},\mathbf{4})$	$(10^2,2^2)$	P_{24}	-	
				$(\mathbf{18},\mathbf{6})$	$(9^2,3^2)$	P_{24}	-	
				$(\mathbf{17},\mathbf{7})$	$(9,8,4,3)$	P_{24}	-	
				$(\mathbf{15},\mathbf{9})$	$(8,7,5,4)$	P_{24}	-	
				$(\mathbf{14},\mathbf{10})$	$(7^2,5^2)$	P_{24}	-	
	$(4,2)$	$(2^2,1^2)$	P_{18}	$(\mathbf{22},\mathbf{2})$	$(11^2,1^2)$	P_{18}	$(22,2)$	$(2^2,1^{20})$
				$(\mathbf{19},\mathbf{5})$	$(10,9,3,2)$	P_{18}	-	-
				$(\mathbf{16},\mathbf{8})$	$(8^2,4^2)$	P_3^6	$(16,8)$	$(2^8,1^8)$
				$(\mathbf{13},\mathbf{11})$	$(7,6^2,5)$	P_{18}	-	-
25	(1)		P_{24}	$(\mathbf{25})$	$(\mathbf{13},\mathbf{12})$	P_{24}	(25)	(1^{25})
				$(\mathbf{23},\mathbf{2})$	$(12,11,1^2)$	P_{24}	-	
				$(\mathbf{22},\mathbf{3})$	$(11^2,2,1)$	P_{24}	-	
				$(\mathbf{20},\mathbf{5})$	$(10^2,3,2)$	P_{24}	-	
				$(\mathbf{19},\mathbf{6})$	$(10,9,3^2)$	P_{24}	-	
				$(\mathbf{17},\mathbf{8})$	$(9,8,4^2)$	P_{24}	-	
				$(\mathbf{16},\mathbf{9})$	$(8^2,5,4)$	P_{24}	-	
				$(\mathbf{14},\mathbf{11})$	$(7^2,6,5)$	P_{24}	-	
	$(3,1)$	$(2,1^2)$	P_{21}	$(\mathbf{24},\mathbf{1})$	$(12^2,1)$	P_{21}	$(24,1)$	$(2,1^{23})$
				$(\mathbf{21},\mathbf{4})$	$(11,10,2^2)$	$P_9\times P_3^4$	$(21,4)$	$(2^4,1^{17})$
				$(\mathbf{18},\mathbf{7})$	$(9^2,4,3)$	P_{21}	-	-
				$(\mathbf{15},\mathbf{10})$	$(8,7,5^2)$	P_{21}	-	-
26	(2)	(1^2)	P_{24}	$(\mathbf{26})$	$(\mathbf{13^2})$	P_{24}	(26)	(1^{26})
				$(\mathbf{23},\mathbf{3})$	$(12,11,2,1)$	P_{24}	$(23,3)$	$(2^3,1^{20})$
				$(\mathbf{20},\mathbf{6})$	$(10^2,3^2)$	P_{24}	-	-
				$(\mathbf{17},\mathbf{9})$	$(9,8,5,4)$	P_{24}	$(17,9)$	$(2^9,1^8)$
				$(\mathbf{14},\mathbf{12})$	$(7^2,6^2)$	P_{24}	-	-
				$(13,12,1)$	$(\mathbf{25},\mathbf{1})$	P_{24}	$(2,1^{24})$	$(25,1)$
				$(12^2,1^2)$	$(\mathbf{24},\mathbf{2})$	P_{24}	-	-
				$(11^2,2^2)$	$(\mathbf{22},\mathbf{4})$	P_{24}	-	-
				$(11,10,3,2)$	$(\mathbf{21},\mathbf{5})$	P_{24}	-	-
				$(10,9,4,3)$	$(\mathbf{19},\mathbf{7})$	P_{24}	-	-
				$(9^2,4^2)$	$(\mathbf{18},\mathbf{8})$	P_{24}	-	-
				$(8^2,5^2)$	$(\mathbf{16},\mathbf{10})$	P_{24}	-	-
				$(8,7,6,5)$	$(\mathbf{15},\mathbf{11})$	P_{24}	-	-

Table 5: simple $F\mathfrak{S}_n$-modules in characteristic 3; $n = 24, 25, 26$

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