

# Vertices of low dimensional simple modules for symmetric groups

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## Abstract

We determine the vertices of all simple modules for the symmetric groups over an algebraically closed field of prime characteristic whose dimensions are at most 1000 and give combinatorial descriptions of both the modules and their vertices. Moreover, we formulate some general conjectures on vertices of several series of simple modules for symmetric groups, and provide some evidence for these to hold true.

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## 1 Introduction and result

Consider an algebraically closed field  $F$  of prime characteristic  $p$ , and denote the symmetric group of degree  $n \in \mathbb{N}$  by  $\mathfrak{S}_n$ . The problem of determining vertices of simple  $F\mathfrak{S}_n$ -modules has been open for more than 30 years now. Astonishingly enough, at present one is aware of only very few general results in this direction. A natural approach towards the determination of vertices of simple modules for the symmetric groups in general is to confine one's investigation to certain "small" examples first, for instance by putting restrictions on the characteristic of  $F$  or on the degrees of the symmetric groups.

Our aim in this paper is to limit the dimensions of the simple modules in question, determine their vertices and give a combinatorial description of these. More precisely, denoting the  $F\mathfrak{S}_n$ -Specht module labelled by the partition  $\mu$  of  $n$  by  $S^\mu$ , the simple  $F\mathfrak{S}_n$ -module labelled by the  $p$ -regular partition  $\lambda$  of  $n$  by  $D^\lambda$ , and the Mullineux conjugate partition of  $\lambda$  by  $\mathbf{m}(\lambda)$ , in Section 5 we will prove the following result:

**Theorem 1.1.** *Let  $\lambda$  be a  $p$ -regular partition of  $n$  such that  $\dim(D^\lambda) \leq 1000$ . Suppose further that  $D^\lambda$  is contained in the block  $B$  of weight  $w$ , and let  $P \leq \mathfrak{S}_{pw}$  be a vertex of  $D^\lambda$ . Then  $P$  is a defect group of  $B$ , unless one of the following cases occurs:*

- (a)  $\mu \in \{\lambda, \mathbf{m}(\lambda)\}$  such that  $D^\mu \cong S^\mu$  and  $P \sim_{\mathfrak{S}_n} \prod_{i=1}^{\infty} (P_{\mu_i - \mu_{i+1}})^i$ ;
- (b)  $p = 2$ ,  $\lambda$  is a spin partition, i.e.  $\lambda = (\frac{n}{2} + 1, \frac{n}{2} - 1)$  if  $n$  is even and  $\lambda = (\frac{n+1}{2}, \frac{n-1}{2})$  if  $n$  is odd, and

$$P \sim_{\mathfrak{S}_n} \begin{cases} Q_n, & \text{if } n \equiv 0 \pmod{4} \\ \prod_{i=0}^s (Q_{2^i})^{\alpha_i}, & \text{if } n \equiv 1, 3 \pmod{4} \\ P_n, & \text{if } n \equiv 2 \pmod{4}, \end{cases}$$

where  $n = \sum_{i=0}^s \alpha_i 2^i$  denotes the 2-adic expansion of  $n$ ;

- (c)  $p = 2$ ,  $\lambda$  is an  $S$ -partition but not a spin partition and  $P \sim_{\mathfrak{S}_n} Q_{pw}$ ;
- (d)  $p = 2$ ,  $m \geq 2$ ,  $\lambda \in \{(2m, 3, 2), (2m + i, 3 + i, 2 + i, i - 1, i - 2, \dots, 1) \mid i \geq 1\}$  and  $P \sim_{\mathfrak{S}_n} P_{2m-2} \times Q_4$ ;
- (e)  $p = 3$ ,  $p \nmid n$ ,  $\lambda = (n - r, 1^r)^R$  for some  $r \in \{0, \dots, n - 1\}$  and  $P \sim_{\mathfrak{S}_n} P_{n-r-1} \times P_r$ ;
- (f)  $p = 3$ ,  $n = m + 6 \geq 11$  for some  $m \equiv 2 \pmod{3}$ ,  $(m, 3, 2, 1) \in \{\lambda, \mathbf{m}(\lambda)\}$  and  $P \sim_{\mathfrak{S}_n} P_m \times (P_3)^2$ .

In this context,  $P_n$  and  $Q_n$  are understood to be Sylow  $p$ -subgroups of the symmetric group  $\mathfrak{S}_n$  and the alternating group  $\mathfrak{A}_n$ , respectively. Furthermore, given any partition  $\mu$  of  $n$ , we denote its *regularization* as defined in James, Kerber (1981), 6.3.48 by  $\mu^R$ . Following Benson (1988), an  $S$ -partition is understood to be a 2-regular partition  $\lambda$  of  $n$  such that  $\lambda_{2j-1} - \lambda_{2j} \leq 2$  and  $\lambda_{2j-1} + \lambda_{2j} \not\equiv 2 \pmod{4}$ , for all  $j \in \mathbb{N}$ . The relatively  $\mathfrak{A}_n$ -projective simple  $F\mathfrak{S}_n$ -modules in characteristic 2 are precisely those labelled by  $S$ -partitions. Note that Carter's Criterion for a Specht module being simple enables us to decide whether we are in case (a) of the theorem. Our proof of the above theorem will require two steps. Firstly, we will apply a result by James on minimal dimensions of simple  $F\mathfrak{S}_n$ -modules (cf. James (1983 a)), in order to classify the simple  $F\mathfrak{S}_n$ -modules of dimension at most 1000 combinatorially. To settle the case  $p = 2$  we will, in addition, make use of a slight modification of James' Theorem. Secondly, we will determine the vertices of those modules theoretically and computationally. Our computations have been carried out using the computer algebra system MAGMA (cf. Cannon et al. (1997)) and build on algorithms developed by R. Zimmermann in Zimmermann (2004) and the author in Danz (2007). Further details on the computational methods applied will appear in Danz, Külshammer, Zimmermann (preprint).

Throughout this note, given a finite group  $G$ , an  $FG$ -module will always be a finitely generated left module. The category of finitely generated  $FG$ -modules will be denoted by  $FG\text{-mod}$ . Furthermore, the *socle* of an  $FG$ -module  $M$  will be denoted by  $\text{Soc}(M)$  and the *head* of  $M$  by  $\text{Hd}(M) := M/\text{Rad}(M)$ . We assume familiarity with the representation theory of the symmetric groups and refer to James (1978) and James, Kerber (1981) for a detailed introduction to this subject.

The structure of the present note is as follows: In Section 2 we introduce some further notation and summarize the necessary facts about vertices of indecomposable modules and modular branching rules. Afterwards, in Sections 3 and 4 we give a combinatorial description of the simple  $F\mathfrak{S}_n$ -modules of dimension at most 1000. We finally present our proof of Theorem 1.1 above in Section 5.

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## 2 Preliminaries

### 2.1 Modules over group algebras

We begin by fixing some notation and recalling a series of known facts about vertices of indecomposable modules over group algebras which will be needed in the next sections. Given a finite group  $G$ , an  $FG$ -module  $M$  and a simple  $FG$ -module  $D$ , from now on, we will use the notation  $[M : D]$  to indicate the multiplicity of  $D$  as a composition factor of  $M$ . Moreover, provided that  $D_1, \dots, D_l$  are precisely the composition factors of  $M$ , we write  $M \sim D_1 + \dots + D_l$ . If  $M_1$  is an  $FG$ -module isomorphic to a direct summand of  $M$ , we write  $M_1 \mid M$ . Furthermore, given another finite group  $H$  and an  $FH$ -module  $N$ , we denote the  $F[G \times H]$ -module  $M \otimes_F N$  by  $M \boxtimes N$ .

Provided that the  $FG$ -module  $M$  is indecomposable, a subgroup  $P$  of  $G$  which is minimal with respect to the condition  $M \mid \text{Ind}_P^G(\text{Res}_P^G(M))$  is called a *vertex* of  $M$ . It is well known that the vertices of  $M$  form a  $G$ -conjugacy class of  $p$ -subgroups of  $G$ . Furthermore, given a fixed vertex  $P$  of  $M$ , there exist a defect group  $\Delta$  of the block containing  $M$  and a Sylow  $p$ -subgroup  $R$  of  $G$  such that  $P \leq \Delta \leq R$  and  $|R : P| \mid \dim(M)$ . The following result concerning outer tensor products of indecomposable modules will be applied several times throughout Section 5. For a proof we refer to Külshammer (1993), Prop. 1.2.

**Proposition 2.1.** *Let  $G$  and  $H$  be finite groups. Let further  $M$  be an indecomposable  $FG$ -module with vertex  $P$ , and let  $N$  be an indecomposable  $FH$ -module with vertex  $Q$ . Then  $M \boxtimes N$  is an indecomposable  $F[G \times H]$ -module with vertex  $P \times Q$ .*

As far as the representation theory of the symmetric groups is concerned, given a partition  $\lambda$  of  $n$ , by  $S^\lambda$  and  $Y^\lambda$  we will denote the corresponding  $F\mathfrak{S}_n$ -Specht module and Young module, respectively.

### 2.2 Modular Branching Rules

Throughout the next sections we will make use of Kleshchev's modular branching rules several times. For that reason, we briefly summarize the basic facts in this direction. For further details, the reader is referred to Kleshchev (2005). For convenience, in the following we identify  $\mathbb{F}_p$  with  $I := \{0, \dots, p-1\}$  and denote the residue class of  $z \in \mathbb{Z}$  modulo  $p$  by  $\bar{z}$ . Moreover,  $\mathcal{P}_{n,p}$  will be the set of  $p$ -regular partitions of  $n$ .

**Remark 2.2.** Let  $\lambda$  be a partition of  $n$  and  $i \in I$ .

(a) Let  $(x, y)$  be a removable node of the Young diagram  $[\lambda]$  of  $p$ -residue  $i$ , and let  $(u, v)$  be an addable node of  $[\lambda]$  of  $p$ -residue  $i$ , i.e.  $\overline{y-x} = i = \overline{v-u}$ . As usual, we call  $(x, y)$  an  *$i$ -removable* node and  $(u, v)$  an  *$i$ -addable* node of  $[\lambda]$ . Moreover, we label each  $i$ -removable node of  $[\lambda]$  by a “ $-$ ” and each  $i$ -addable node by a “ $+$ ”. The sequence obtained by walking along the rim of  $[\lambda]$  from bottom left to top right reading off all signs is called the  *$i$ -signature* of  $[\lambda]$ . Repeatedly cancelling all terms “ $-+$ ” in the  $i$ -signature, we obtain the *reduced  $i$ -signature* of  $[\lambda]$ . We will also speak of the *reduced  $i$ -signature* of the module  $D^\lambda$ .

A node of  $[\lambda]$  corresponding to a “ $-$ ” in the reduced  $i$ -signature of  $[\lambda]$  is called  *$i$ -normal*, and an addable node of  $[\lambda]$  corresponding to a “ $+$ ” in the reduced  $i$ -signature of  $[\lambda]$  is called

*i-conormal*. Furthermore, the node of  $[\lambda]$  represented by the leftmost “ $-$ ” in the reduced  $i$ -signature is called *i-good*, and the addable node of  $[\lambda]$  represented by the rightmost “ $+$ ” in the reduced  $i$ -signature is called *i-cogood*.

The total numbers of “ $-$ ” and “ $+$ ”, respectively, in the reduced  $i$ -signature of  $[\lambda]$  are denoted by  $\varepsilon_i(\lambda)$  and  $\varphi_i(\lambda)$ , respectively.

(b) Suppose that  $B$  is a block of  $F\mathfrak{S}_n$  with  $p$ -content  $\gamma = (\gamma_0, \dots, \gamma_{p-1})$ . Furthermore, let  $M$  be an  $F\mathfrak{S}_n$ -module belonging to  $B$ . Then the functor  $e_i : F\mathfrak{S}_n\text{-mod} \rightarrow F\mathfrak{S}_{n-1}\text{-mod}$  assigns to  $M$  the component of  $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(M)$  belonging to the block with  $p$ -content  $(\gamma_0, \dots, \gamma_{i-1}, \gamma_i - 1, \gamma_{i+1}, \dots, \gamma_{p-1})$  provided such a block exists. Otherwise, we set  $e_i(M) := 0$ . Similarly, the functor  $f_i : F\mathfrak{S}_n\text{-mod} \rightarrow F\mathfrak{S}_{n+1}\text{-mod}$  assigns to  $M$  the component of  $\text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(M)$  belonging to the block with  $p$ -content  $(\gamma_0, \dots, \gamma_{i-1}, \gamma_i + 1, \gamma_{i+1}, \dots, \gamma_{p-1})$  provided such a block exists. Otherwise, we set  $f_i(M) := 0$ .

Given  $r \in \mathbb{N}$ , the higher divided power functors  $e_i^{(r)} : F\mathfrak{S}_n\text{-mod} \rightarrow F\mathfrak{S}_{n-r}\text{-mod}$  and  $f_i^{(r)} : F\mathfrak{S}_n\text{-mod} \rightarrow F\mathfrak{S}_{n+r}\text{-mod}$  are obtained via the following construction. Regard the  $F$ -vector space  $M^{\mathfrak{S}_r}$  of fixed points of  $\mathfrak{S}(\{n-r+1, \dots, n\})$  as  $F\mathfrak{S}_{n-r}$ -module. Then  $e_i^{(r)}(M)$  denotes the block component of  $M^{\mathfrak{S}_r}$  belonging to the block with  $p$ -content  $(\gamma_0, \dots, \gamma_{i-1}, \gamma_i - r, \gamma_{i+1}, \dots, \gamma_{p-1})$  if such a block exists. Otherwise we again set  $e_i^{(r)}(M) := 0$ . On the other hand, regard  $M$  as  $F[\mathfrak{S}_n \times \mathfrak{S}_r]$ -module with trivial  $\mathfrak{S}(\{n+1, \dots, n+r\})$ -action. Then  $f_i^{(r)}(M)$  is the block component of  $\text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_r}^{\mathfrak{S}_{n+r}}(M)$  belonging to the block with  $p$ -content  $(\gamma_0, \dots, \gamma_{i-1}, \gamma_i + r, \gamma_{i+1}, \dots, \gamma_{p-1})$  if such a block exists, and  $f_i^{(r)}(M) := 0$  otherwise.

Keeping this notation, we will make extensive use of the following theorems:

**Theorem 2.3** (Kleshchev (2005), Thms. 11.2.10, 11.2.11). *Let  $\lambda \in \mathcal{P}_{n,p}$ ,  $i \in I$  and  $r \in \mathbb{N}$ . Then the following hold:*

(i)  $e_i^r(D^\lambda) \cong \bigoplus_{r!} e_i^{(r)}(D^\lambda)$ . Moreover,  $e_i^{(r)}(D^\lambda) \neq 0$  if and only if  $[\lambda]$  possesses at least  $r$  normal nodes of residue  $i$ . In that case,  $e_i^{(r)}(D^\lambda)$  is indecomposable and selfdual with  $\text{Hd}(e_i^{(r)}(D^\lambda)) \cong \text{Soc}(e_i^{(r)}(D^\lambda)) \cong D^\mu$  where  $[\mu]$  is the Young diagram resulting after removing the  $r$  lowest  $i$ -normal nodes from  $[\lambda]$ .

(ii) The module  $e_i^{(r)}(D^\lambda)$  is simple if and only if  $r = \varepsilon_i(\lambda)$ .

(iii)  $f_i^r(D^\lambda) \cong \bigoplus_{r!} f_i^{(r)}(D^\lambda)$ . Moreover,  $f_i^{(r)}(D^\lambda) \neq 0$  if and only if  $[\lambda]$  possesses at least  $r$  conormal nodes of residue  $i$ . In that case,  $f_i^{(r)}(D^\lambda)$  is indecomposable and selfdual with  $\text{Hd}(f_i^{(r)}(D^\lambda)) \cong \text{Soc}(f_i^{(r)}(D^\lambda)) \cong D^\nu$  where  $[\nu]$  is the Young diagram resulting after adding the  $r$  highest  $i$ -conormal nodes to  $[\lambda]$ .

(iv) The module  $f_i^{(r)}(D^\lambda)$  is simple if and only if  $\varphi_i(\lambda) = r$ .

**Theorem 2.4** (Brundan, Kleshchev (2003), Thm. 2.11). *Let  $\lambda \in \mathcal{P}_{n,p}$  and  $i \in I$ .*

(i) Let  $(x, y)$  be an  $i$ -removable node of  $[\lambda]$  such that the partition  $\mu$  of  $n-1$  corresponding to  $[\alpha] := [\lambda] \setminus \{(x, y)\}$  is again  $p$ -regular. If  $(x, y)$  is  $i$ -normal, then  $[e_i(D^\lambda) : D^\alpha]$  equals the number of  $i$ -normal nodes to the right of  $(x, y)$ , counting  $(x, y)$  itself. If  $(x, y)$  is not  $i$ -normal then  $[\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) : D^\alpha] = [e_i(D^\lambda) : D^\alpha] = 0$ .

(ii) Let  $(u, v)$  be an  $i$ -addable node of  $[\lambda]$  such that the partition  $\beta$  of  $n + 1$  corresponding to  $[\beta] := [\lambda] \cup \{(u, v)\}$  is again  $p$ -regular. If  $(u, v)$  is  $i$ -conormal, then  $[f_i(D^\lambda) : D^\beta]$  equals the number of  $i$ -conormal nodes of  $[\lambda]$  to the left of  $(u, v)$ , counting  $(u, v)$  itself. If  $(u, v)$  is not  $i$ -conormal, then  $[\text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(D^\lambda) : D^\beta] = [f_i(D^\lambda) : D^\beta] = 0$ .

### 2.3 Exterior powers of the natural simple module

Let  $p$  be odd and  $n \geq 2$ . Moreover, consider the natural  $F\mathfrak{S}_n$ -Specht module  $S^{(n-1,1)}$  and the natural simple  $F\mathfrak{S}_n$ -module  $D^{(n-1,1)}$ . For  $r \leq n - 1 = \dim(S^{(n-1,1)})$ , we have  $S^{(n-r,1^r)} \cong \bigwedge^r S^{(n-1,1)}$ . A proof for this can for instance be found in Müller, Zimmermann (2007), Prop. 2.3. For each  $r \leq n - 1$ , the natural  $F\mathfrak{S}_n$ -epimorphism  $S^{(n-1,1)} \longrightarrow \text{Hd}(S^{(n-1,1)}) \cong D^{(n-1,1)}$  induces an  $F\mathfrak{S}_n$ -epimorphism

$$\bigwedge^r (S^{(n-1,1)}) \longrightarrow \bigwedge^r (D^{(n-1,1)}) =: D_r.$$

Furthermore, if  $n$  is not divisible by  $p$  then  $D^{(n-1,1)} \cong S^{(n-1,1)}$ , by James (1978), Ex. 5.1 and thus  $D_r \cong S^{(n-r,1^r)}$  which is simple, by Peel (1971) (see also James (1978), Thm. 24.1). Consequently, in that case we obtain  $D_r \cong D^{(n-r,1^r)^R}$ , by James, Kerber (1981), 6.3.51. In particular,  $\dim(D_r) = \binom{n-1}{r}$ .

Now suppose that  $p|n$ . Then  $\dim(D^{(n-1,1)}) = n - 2$ , by James (1978), Ex. 5.1, and by Peel (1971) the Specht module  $S^{(n-r,1^r)}$  is uniserial with precisely two composition factors, for  $0 < r \leq n - 2$ . More precisely,  $\text{Hd}(S^{(n-r,1^r)}) \cong D_r \cong \text{Soc}(S^{(n-r-1,1^{r+1})})$ , for  $0 \leq r \leq n - 2$ . In particular, we then have  $D^{(n-r,1^r)} \cong D_r$  in case that  $r < p$ . If  $r \geq p$  the  $p$ -regular partition of  $n$  labelling  $D_r$  is again obtained by the technique of regularizing a partition. In fact, the following holds true:

**Proposition 2.5** (Danz (2007), Kor. 6.3.6). *(i) Let  $p$  be odd, and let  $n = pw$  for some  $w \geq 1$ . Let further  $r \in \{0, \dots, n - 2\}$ . Then*

$$D_r = \bigwedge^r D^{(n-1,1)} \cong \begin{cases} D^{(n-r,1^r)^R}, & \text{if } n - r \geq w + 1 \text{ or } r < p, \\ D^{(n-r-1,1^{r+1})^R}, & \text{if } n - r \leq w \text{ and } r \geq p. \end{cases}$$

*(ii) Let  $p$  be odd and  $n \not\equiv 0 \pmod{p}$ . Then*

$$D_r \cong S^{(n-r,1^r)} \cong D^{(n-r,1^r)^R},$$

for  $r \in \{0, \dots, n - 1\}$ .

### 2.4 James' Theorem

Given  $m \in \{0, \dots, n\}$ , by  $R_n(m)$  we denote the class of simple  $F\mathfrak{S}_n$ -modules  $D$  such that  $D \cong D^\lambda$  or  $D \cong D^{\mathbf{m}(\lambda)}$  for some  $p$ -regular partition  $\lambda$  of  $n$  satisfying  $\lambda_1 \geq n - m$ . Here,  $\mathbf{m}(\lambda)$  shall again be the Mullineux conjugate partition of  $\lambda$ . Note that  $R_n(m_1) \subseteq R_n(m_2)$ , for all  $m_1, m_2 \in \{0, \dots, n\}$  such that  $m_1 \leq m_2$ . With this notation, James has proved the following result on minimal dimensions of simple  $F\mathfrak{S}_n$ -modules:

**Theorem 2.6** (James (1983 a), L. 4). *Let  $m, N \in \mathbb{N}_0$ , and let  $f : \mathbb{Z} \longrightarrow \mathbb{Z}$  be a map satisfying the following conditions:*

(i)  $2f(n-2) > f(n)$  for all  $n \geq N+2$ .

(ii) If  $n \in \{N, N+1\}$ , and if  $D$  is a simple  $F\mathfrak{S}_n$ -module, then either  $D$  belongs to  $R_n(m)$  or  $\dim(D) > f(n)$ .

(iii) For all  $n \geq N$  and every simple  $F\mathfrak{S}_n$ -module  $D \in R_n(m+2) \setminus R_n(m)$  we have  $\dim(D) > f(n)$ .

Then for all  $n \geq N$ , every simple  $F\mathfrak{S}_n$ -module  $D$  either belongs to  $R_n(m)$  or has dimension  $\dim(D) > f(n)$ .

**Remark 2.7.** In order to apply Theorem 2.6 in the next sections, we will need to know the dimensions of the simple  $F\mathfrak{S}_n$ -modules in characteristic 2 and 3, for  $n \leq 18$  and  $n \leq 17$ , respectively. For  $n \leq 17$  the decomposition numbers for  $\mathfrak{S}_n$  are known (cf. <http://www.math.rwth-aachen.de/homes/MOC/decomposition/>) and thus so are the dimensions of the respective simple modules. The decomposition numbers for  $\mathfrak{S}_{18}$  in characteristic 2 are so far still incomplete. The table in the appendix of this paper therefore contains lower bounds on the dimensions of simple  $F\mathfrak{S}_{18}$ -modules in characteristic 2.

In that manner we derive the following: Suppose that  $p = 2$ . Then any simple  $F\mathfrak{S}_{17}$ -module either has dimension at least 1582, or belongs to  $R_{17}(3) \cup \{D^{(9,8)}\}$  and has dimension less than 1155. Moreover, any simple  $F\mathfrak{S}_{18}$ -module either has dimension at least 1582, or belongs to  $R_{18}(3) \cup \{D^{(10,8)}\}$  and has dimension less than 1580. Now let  $p = 3$ . Then any simple  $F\mathfrak{S}_{16}$ -module either has dimension at least 1260, or belongs to  $R_{16}(3)$  and has dimension less than 819. Any simple  $F\mathfrak{S}_{17}$ -module either has dimension at least 1597, or belongs to  $R_{17}(3)$  and has dimension less than 1155.

### 3 Simple $F\mathfrak{S}_n$ -modules in characteristic 3

We begin by investigating the simple  $F\mathfrak{S}_n$ -modules where  $\text{char}(F) = 3$ . Then we have the following:

**Proposition 3.1.** *Suppose that  $p = 3$ ,  $n \geq 16$ , and let  $D^\lambda$  be a simple  $F\mathfrak{S}_n$ -module. Then either  $D^\lambda$  belongs to  $R_n(3)$  or  $\dim(D^\lambda) > f(n)$  where*

$$f(n) := \frac{n^4 - 14n^3 + 47n^2 - 34n - 24}{24},$$

for  $n \in \mathbb{Z}$ .

**Proof.** We will show that the map  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined above satisfies conditions (i)-(iii) of Theorem 2.6, for  $N := 16$  and  $m := 3$ .

An easy calculation shows that (i) holds true. Since  $f(16) = 819$  and  $f(17) = 1155$ , Remark 2.7 shows that condition (ii) is fulfilled. We finally prove that (iii) is satisfied as well. For  $n = 16, 17$ , this holds true, by (ii). Therefore, let  $n \geq 18$ , and let  $D^\lambda$  be a simple  $F\mathfrak{S}_n$ -module in  $R_n(5) \setminus R_n(3)$ .

(a) In the case where  $D^\lambda \in R_n(4) \setminus R_n(3)$  we may assume that  $\lambda \in \{(n-4, 4), (n-4, 3, 1), (n-4, 2^2), (n-4, 2, 1^2)\}$ . The assertion then immediately follows from the dimension formulas in

the appendix of James (1983 a). In fact,  $f(n) = \min\{\dim(D^\lambda) | D^\lambda \in R_n(4) \setminus R_n(3)\} - 1$ .

(b) We now consider the case where  $D^\lambda \in R_n(5) \setminus R_n(4)$  and may assume that  $\lambda \in \{(n-5, 5), (n-5, 4, 1), (n-5, 3, 2), (n-5, 3, 1^2), (n-5, 2^2, 1)\}$ . If  $\lambda = (n-5, 3, 2)$  then  $\lambda = (n-5, 1^5)^R$  and  $D^\lambda \cong \bigwedge^5(D^{(n-1,1)})$ , by Proposition 2.5 and thus

$$\dim(D^\lambda) = \begin{cases} \binom{n-1}{5}, & \text{if } n \not\equiv 0 \pmod{3} \\ \binom{n-2}{5}, & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

This shows  $\dim(D^\lambda) > f(n)$  in either case. For  $\lambda \in \{(n-5, 5), (n-5, 4, 1), (n-5, 3, 1^2), (n-5, 2^2, 1)\}$  we consider the case  $n = 18$  first. By Theorem 2.3 we then obtain:

$$\begin{aligned} 2D^{(12,4)} | \text{Res}_{\mathfrak{S}_{16}}^{\mathfrak{S}_{18}}(D^{(13,5)}), & \quad \text{Res}_{\mathfrak{S}_{17}}^{\mathfrak{S}_{18}}(D^{(13,4,1)}) \cong D^{(12,4,1)} \oplus D^{(13,3,1)}, \\ 2D^{(12,3,1)} | \text{Res}_{\mathfrak{S}_{16}}^{\mathfrak{S}_{18}}(D^{(13,3,1^2)}), & \quad 2D^{(12,2^2)} | \text{Res}_{\mathfrak{S}_{16}}^{\mathfrak{S}_{18}}(D^{(13,2^2,1)}). \end{aligned}$$

This shows that  $\dim(D^\lambda) > 1580 = f(18)$ , for  $\lambda \in \{(13, 5), (13, 4, 1), (13, 3, 1^2), (13, 2^2, 1)\}$ . From now on, we may thus assume that  $n > 18$  and prove the assertion by induction on  $n$ . Application of Theorem 2.3 and Theorem 2.4 yields the following:

$\lambda$	$n \equiv 1 \pmod{3}$
$(n-5, 5)$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-6,5)} \oplus D^{(n-5,4)}$
$(n-5, 4, 1)$	$2D^{(n-6,4)}   \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^\lambda)$
$(n-5, 3, 1^2)$	$2D^{(n-6,2,1^2)}   \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^\lambda)$
$(n-5, 2^2, 1)$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-6,2^2,1)} \oplus D^{(n-5,2^2)}$
$\lambda$	$n \equiv 2 \pmod{3}$
$(n-5, 5)$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-6,5)}$
$(n-5, 4, 1)$	$[\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) : D^{(n-5,4)}] = 1$ $[\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) : D^{(n-5,3,1)}] = 2$
$(n-5, 3, 1^2)$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-6,3,1^2)} \oplus D^{(n-5,2,1^2)}$
$(n-5, 2^2, 1)$	$2D^{(n-6,2,1^2)}   \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^\lambda)$
$\lambda$	$n \equiv 0 \pmod{3}$
$(n-5, 5)$	$2D^{(n-6,4)}   \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^\lambda)$
$(n-5, 4, 1)$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-6,4,1)} \oplus D^{(n-5,3,1)}$
$(n-5, 3, 1^2)$	$2D^{(n-6,3,1)}   \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^\lambda)$
$(n-5, 2^2, 1)$	$2D^{(n-6,2^2)}   \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^\lambda)$

By (i), induction and part (a) above, we deduce that also condition (iii) is fulfilled. The assertion of the proposition now immediately follows from Theorem 2.6.  $\square$

## 4 Simple $F\mathfrak{S}_n$ -modules in characteristic 2

In this section we settle the case where  $p = 2$ . Firstly, we will prove a result which is slightly more general than Theorem 2.6. Therefore, given  $n \geq 3$ , in the following we denote the basic spin module for  $F\mathfrak{S}_n$  by  $D(n)$ . That is, the simple  $F\mathfrak{S}_n$ -module labelled by the partition  $(m+1, m-1)$  in case that  $n = 2m$ , and the one labelled by the partition  $(m+1, m)$  in case that  $n = 2m+1$ , for some  $m \in \mathbb{N}$ . Furthermore,  $D(n)$  has dimension  $2^m$ . This has been shown by Wales in Wales (1979), and a proof can also be found in Benson (1988), Thm. 5.1.

**Lemma 4.1.** *Let  $n \geq 8$ .*

- (i) *If  $n \equiv 0 \pmod{4}$  then  $\text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(D(n)) \cong f_0(D(n)) \sim 2D(n+1) + D^{(\frac{n}{2}+1, \frac{n}{2}-1, 1)}$  and  $\dim(D^{(\frac{n}{2}+1, \frac{n}{2}-1, 1)}) = (n-3)2^{\frac{n-2}{2}}$ .*
- (ii) *If  $n \equiv 1 \pmod{4}$  then  $\text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(D(n)) \cong f_0(D(n)) \oplus f_1(D(n))$ , where  $f_0(D(n)) \cong D^{(\frac{n+1}{2}, \frac{n-1}{2}, 1)}$  and  $f_1(D(n)) \sim 2D(n+1)$ . Moreover,  $\dim(D^{(\frac{n+1}{2}, \frac{n-1}{2}, 1)}) = (n-1)2^{\frac{n-1}{2}}$ .*
- (iii) *If  $n \equiv 2 \pmod{4}$  then  $\text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(D(n)) \cong f_0(D(n)) \oplus f_1(D(n))$ , where  $f_1(D(n)) \cong D(n+1)$  and  $f_0(D(n)) \cong D^{(\frac{n}{2}+1, \frac{n}{2}-1, 1)}$ . Moreover,  $\dim(D^{(\frac{n}{2}+1, \frac{n}{2}-1, 1)}) = (n-1)2^{\frac{n-2}{2}}$ .*
- (iv) *If  $n \equiv 3 \pmod{4}$  then  $\text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(D(n)) \cong f_0(D(n)) \sim 3D(n+1) + D^{(\frac{n+1}{2}, \frac{n-1}{2}, 1)}$  and  $\dim(D^{(\frac{n+1}{2}, \frac{n-1}{2}, 1)}) = (n-2)2^{\frac{n-1}{2}}$ .*

**Proof.** Let  $n \geq 8$  and set  $M_{n+1} := \text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(D(n))$ . Moreover, let  $N_{n+1} := D^{(\frac{n}{2}+1, \frac{n}{2}-1, 1)}$  if  $n$  is even and  $N_{n+1} := D^{(\frac{n+1}{2}, \frac{n-1}{2}, 1)}$  if  $n$  is odd. Then, applying Theorem 2.3 and Theorem 2.4, we observe the following:

$n \equiv 0$ (mod 4)	$0 : + + -$ $1 : \emptyset$	$[M_{n+1} : D(n+1)] = 2$ $[M_{n+1} : N_{n+1}] = 1$	$0 : -$ $1 : + +$	$\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(N_{n+1}) \cong N_n$
$n \equiv 1$ (mod 4)	$0 : + - -$ $1 : + +$	$[M_{n+1} : N_{n+1}] = 1$ $[M_{n+1} : D(n+1)] = 2$	$0 : - - -$ $1 : + + + +$	$[\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(N_{n+1}) : D(n)] = 3$ $[\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(N_{n+1}) : N_n] = 2$
$n \equiv 2$ (mod 4)	$0 : +$ $1 : + -$	$M_{n+1} \cong D(n+1) \oplus N_{n+1}$		
$n \equiv 3$ (mod 4)	$0 : + + +$ $1 : - -$	$[M_{n+1} : D(n+1)] = 3$ $[M_{n+1} : N_{n+1}] = 1$	$0 : +$ $1 : + + - -$	$[\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(N_{n+1}) : N_n] = 2$

The entries of that table above should be read as follows: The second and fourth columns contain the reduced 0-signature and the reduced 1-signature for  $D(n)$  and  $N_{n+1}$ , respectively. The third and fifth columns display those composition factors of  $M_{n+1}$  and  $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(N_{n+1})$ , respectively, we deduce from Theorem 2.4. From the knowledge of  $\dim(D(n))$ , in this way we derive both upper and lower bounds for  $\dim(N_{n+1})$ .

We now argue by induction on  $n$  and consider the cases  $n = 8, 9, 10, 11$  first where the dimensions of all modules under consideration are known. At each step of the induction we simply compare the dimensions of the respective modules against each other and obtain that the composition factors of  $M_{n+1}$  and  $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(N_{n+1})$ , respectively, are precisely the ones displayed in the table above. From this, the assertion of the lemma follows immediately.  $\square$



As a direct consequence, we now obtain:

**Corollary 4.2.** *Let  $n \geq 9$ . Furthermore, let  $D = D^{(\frac{n+1}{2}, \frac{n-3}{2}, 1)}$  if  $n$  is odd, and let  $D = D^{(\frac{n}{2}, \frac{n-2}{2}, 1)}$  if  $n$  is even. Then:*

$$\dim(D) = g(n) := \begin{cases} (n-3)2^{\frac{n-2}{2}}, & \text{if } n \equiv 0 \pmod{4} \\ (n-4)2^{\frac{n-3}{2}}, & \text{if } n \equiv 1 \pmod{4} \\ (n-2)2^{\frac{n-2}{2}}, & \text{if } n \equiv 2 \pmod{4} \\ (n-2)2^{\frac{n-3}{2}}, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

**Theorem 4.3.** *Let  $m, N \in \mathbb{N}$  such that  $N \geq 9$ . Furthermore, let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be a map satisfying the following conditions.*

- (i)  $f(n) > f(n-1)$ , for  $n \geq N+1$ , and  $2f(n-2) > f(n)$ , for  $n \geq N+2$ .
- (ii) If  $n \in \{N, N+1\}$  and  $D$  is a simple  $F\mathfrak{S}_n$ -module, then  $D$  belongs to  $R_n(m) \cup \{D(n)\}$  or  $\dim(D) > f(n)$ .
- (iii) For  $n \geq N+1$ , we have  $g(n) > 2f(n-1)$  where  $g$  denotes the map given in Corollary 4.2.
- (iv) For  $n \geq N$  and every simple  $F\mathfrak{S}_n$ -module  $D$  belonging to  $R_n(m+2) \setminus R_n(m)$ , we have  $\dim(D) > f(n)$ .

Then every simple  $F\mathfrak{S}_n$ -module  $D$  belongs to  $R_n(m) \cup \{D(n)\}$  or has dimension  $\dim(D) > f(n)$ , for all  $n \geq N$ .

**Proof.** We argue by induction on  $n$ . Condition (ii) implies that the assertion holds true for  $n \in \{N, N+1\}$ , and we may thus assume  $n \geq N+2$ . Moreover, let  $D$  be a simple  $F\mathfrak{S}_n$ -module such that  $D \notin R_n(m)$  and  $D \not\cong D(n)$ . In order to show that  $\dim(D) > f(n)$ , we distinguish between two cases.

*Case 1:*  $L := \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D) \cong e_0(D) \oplus e_1(D)$  is reducible.

If  $L$  has at least two composition factors of dimension at least  $f(n-1)$  then  $\dim(D) \cong \dim(L) \geq 2f(n-1) > 2f(n-2) > f(n)$ , by (i). If  $L$  has exactly one composition factor  $\tilde{D}$  of dimension at least  $f(n-1)$ , and if  $\tilde{D}$  is contained in the socle of  $L$  then we may assume that it is contained in the socle of  $e_0(D)$ . Hence  $\tilde{D} = \text{Soc}(e_0(D)) \cong \text{Hd}(e_0(D))$ , by Theorem 2.3. Since  $e_0(D)$  is indecomposable, this implies  $e_0(D) \cong \tilde{D}$ . Since  $L$  is assumed to be reducible,  $e_1(D) \neq 0$  and thus  $\dim(\text{Soc}(e_1(D))) < f(n-1)$ . If all composition factors of  $L$  have dimension less than  $f(n-1)$  then this particularly holds for those in the socle of  $L$ . In any case  $L$  contains a simple submodule  $\hat{D}$  such that  $\dim(\hat{D}) < f(n-1)$ . Hence  $D$  is isomorphic to a composition factor of  $\text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\hat{D})$ , and  $\hat{D} \in R_{n-1}(m) \cup \{D(n-1)\}$ , by the inductive hypothesis.

In case that  $\hat{D} \in R_{n-1}(m)$  we have  $\hat{D} \cong D^\mu$  such that  $\mu_1 \geq n-1-m$ . By the classical branching rules,  $\text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(S^\mu)$  admits a filtration

$$0 =: S_l \subset \dots \subset S_1 \subset S_0 := \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(S^\mu)$$

such that  $S_j/S_{j+1} \cong S^{\nu(j)}$  where the Young diagram  $[\nu(j)]$  is obtained from  $[\mu]$  by adding a node, for  $j = 0, \dots, l-1$ . In particular,  $\nu(j)_1 \geq n-m-1$  for  $j = 0, \dots, l-1$ . Since, by James (1978), Cor. 12.2, each composition factor of  $S^{\nu(j)}$  is labelled by some partition which in the dominance order is greater than or equal to  $\nu_j$ , each composition factor of  $\text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(S^\mu)$  is thus contained in  $R_n(m+1)$ . Since  $\widehat{D}$  is one of these, we obtain  $D \in R_n(m+1)$  and hence  $\dim(D) > f(n)$ , by (iv). In case that  $\widehat{D} = D(n-1)$  Lemma 4.1 implies that

$$D \cong \begin{cases} D^{(\frac{n+1}{2}, \frac{n-3}{2}, 1)}, & \text{if } n \text{ is odd} \\ D^{(\frac{n}{2}, \frac{n-2}{2}, 1)}, & \text{if } n \text{ is even.} \end{cases}$$

Consequently,  $\dim(D) = g(n) > 2f(n-1) > 2f(n-2) > f(n)$  by (i), (iii) and Corollary 4.2.

*Case 2:*  $L := \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D)$  is simple, i.e.  $L \cong D^\mu$  for some  $\mu \in \mathcal{P}_{n-1,2}$ . Then  $\widetilde{L} := \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D) \cong e_0(D^\mu) \oplus e_1(D^\mu)$  is reducible, by Lemma 3 in James (1983).

If  $\widetilde{L}$  has at least two composition factors of dimension at least  $f(n-2)$  then  $\dim(D) = \dim(\widetilde{L}) > f(n)$  by (i). Otherwise, similarly as above, we deduce that  $\widetilde{L}$  contains a simple submodule  $\widehat{D} \in R_{n-2}(m) \cup \{D(n-2)\}$ , so that  $D$  is isomorphic to some composition factor of  $\text{Ind}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(\widehat{D})$ . In case that  $\widehat{D} \in R_{n-2}(m)$  we obtain  $D \in R_n(m+2)$  and hence  $\dim(D) > f(n)$ , by (iv). Thus we may now assume  $\widehat{D} = D(n-2)$ . Since  $L$  is isomorphic to a composition factor of  $\text{Ind}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_{n-1}}(\widehat{D})$ , Lemma 4.1 implies that either  $L \cong D(n-1)$  or

$$L \cong \begin{cases} D^{(\frac{n-1}{2}, \frac{n-3}{2}, 1)}, & \text{if } n \text{ is odd} \\ D^{(\frac{n}{2}, \frac{n-4}{2}, 1)}, & \text{if } n \text{ is even.} \end{cases}$$

In the latter case we have  $\dim(D) = \dim(L) = g(n-1) > 2f(n-2) > f(n)$ , by Corollary 4.2 and (i). If  $L \cong D(n-1)$  then Lemma 4.1 in turn yields

$$D \cong \begin{cases} D^{(\frac{n+1}{2}, \frac{n-3}{2}, 1)}, & \text{if } n \text{ is odd} \\ D^{(\frac{n}{2}, \frac{n-2}{2}, 1)}, & \text{if } n \text{ is even,} \end{cases}$$

i.e.  $\dim(D) = g(n) > 2f(n-1) > 2f(n-2) > f(n)$ , by Corollary 4.2 and (i).

Consequently  $\dim(D) > f(n)$  in either case, and the theorem is proved.  $\square$

**Proposition 4.4.** *Suppose that  $p = 2$ ,  $n \geq 17$ , and let  $D$  be a simple  $F\mathfrak{S}_n$ -module. Then  $D$  belongs to  $R_n(3) \cup \{D(n)\}$  or  $\dim(D^\lambda) > f(n)$  where*

$$f(n) := \frac{n^4 - 14n^3 + 47n^2 - 34n - 24}{24},$$

for  $n \in \mathbb{Z}$ .

**Proof.** Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be the map defined above. We show that the conditions of Theorem 4.3 are satisfied, for  $f$  and  $N := 17$ . In the proof of Proposition 3.1 we have already shown that (i) holds true. Furthermore,  $f(17) = 1155$  and  $f(18) = 1580$ . Hence (ii) follows from Remark 2.7, and the validity of (iii) is verified by an easy computation. We finally prove that (iv) is satisfied and proceed by induction on  $n$ . Furthermore, we may assume that  $n \geq 19$ , for

(iv) is known to hold true for  $n = 17, 18$ .

(a) If  $D^\lambda \in R_n(4) \setminus R_n(3)$  then  $\lambda \in \{(n-4, 4), (n-4, 3, 1)\}$ , and (iv) again follows from the dimension formulas in the appendix of James (1983 a). Note that also in this case we have  $f(n) = \min\{\dim(D^\lambda) | D^\lambda \in R_n(4) \setminus R_n(3)\} - 1$ .

(b) We now consider  $D^\lambda \in R_n(5) \setminus R_n(4)$ , i.e.  $\lambda \in \{(n-5, 5), (n-5, 4, 1), (n-5, 3, 2)\}$ . Then Theorem 2.3 and Theorem 2.4 lead to the following:

$\lambda$	$n$ odd	$n$ even
$(n-5, 5)$	$[\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) : D^{(n-5,4)}] = 2$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-6,5)}$
$(n-5, 4, 1)$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-5,4)} \oplus D^{(n-6,4,1)}$	$[\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) : D^{(n-5,4)}] = 3$
$(n-5, 3, 2)$	$[\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) : D^{(n-5,3,1)}] = 3$	$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-6,3,2)} \oplus D^{(n-5,3,1)}$

The validity of (iii) now follows from (i), (a) and the inductive hypothesis. Hence the assertion of the proposition follows from Theorem 4.3.  $\square$

## 5 Proof of the theorem

In the preceding sections we have classified the simple  $F\mathfrak{S}_n$ -modules of “small” dimensions for  $p = 2$  and  $p = 3$ , respectively. In the following we mention some further results which we will need throughout the proof of Theorem 1.1. First of all, we recall Carter’s criterion for a Specht module being simple.

**Theorem 5.1** (James, Kerber (1981), Thm. 7.3.23). *Let  $\lambda$  be a  $p$ -regular partition of  $n$ . Then the corresponding  $F\mathfrak{S}_n$ -Specht module  $S^\lambda$  is simple if and only if the highest  $p$ -power dividing the hook length  $h_{rs}$  equals the highest  $p$ -power dividing the hook length  $h_{ts}$ , for all nodes  $(r, s), (t, s) \in [\lambda]$ .*

As an immediate consequence one obtains:

**Corollary 5.2.** *Let  $\lambda = (n-p-r, p+r)$  be a  $p$ -regular two row partition of  $n$  such that  $r \in \{0, \dots, p-1\}$ . Then the Specht module  $S^\lambda$  is simple if and only if  $n \equiv ip + 2r - 1 \pmod{p^2}$ , for some  $i \in \{0, \dots, p-1\} \setminus \{1\}$ .*

**Lemma 5.3.** *Let  $p = 2$ . Then we have the following:*

$$(i) \dim(D^{(n-2,2)}) = \begin{cases} \frac{1}{2}(n^2 - 5n + 4), & \text{if } n \equiv 0 \pmod{4} \\ \frac{1}{2}(n^2 - 3n - 2), & \text{if } n \equiv 1 \pmod{4} \\ \frac{1}{2}(n^2 - 5n + 2), & \text{if } n \equiv 2 \pmod{4} \\ \frac{1}{2}(n^2 - 3n), & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

$$(ii) \dim(D^{(n-3,3)}) = \begin{cases} \frac{1}{6}(n^3 - 9n^2 + 14n), & \text{if } n \equiv 0 \pmod{4} \\ \frac{1}{6}(n^3 - 6n^2 + 5n), & \text{if } n \equiv 1 \pmod{4} \\ \frac{1}{6}(n^3 - 9n^2 + 20n - 12), & \text{if } n \equiv 2 \pmod{4} \\ \frac{1}{6}(n^3 - 6n^2 - n + 6), & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

$$(iii) \dim(D^{(n-3,2,1)}) = \begin{cases} \frac{1}{3}(n^3 - 6n^2 + 8n), & \text{if } n \equiv 0 \pmod{2} \\ \frac{1}{6}(2n^3 - 15n^2 + 25n), & \text{if } n \equiv 1 \pmod{4} \\ \frac{1}{6}(2n^3 - 15n^2 + 25n - 6), & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** Parts (i) and (ii) have already been proved in Zimmermann (2004). We show that (iii) holds true and therefore set  $\lambda := (n-3, 2, 1)$ . If  $n$  is even we have  $D^\lambda \cong S^\lambda$  by Theorem 5.1, and the Hook Formula thus yields

$$\dim(D^\lambda) = \frac{n!}{(n-1)(n-3)(n-5)! \cdot 3} = \frac{n(n-2)(n-4)}{3} = \frac{n^3 - 6n^2 + 8n}{3}.$$

If  $n$  is odd we obtain  $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^\lambda) \cong D^{(n-3,2)} \oplus D^{(n-4,2,1)}$ , by Theorem 2.3. In combination with (i) we thus have

$$\begin{aligned} \dim(D^\lambda) &= \dim(D^{(n-1-2,2)}) + \dim(D^{(n-1-3,2,1)}) \\ &= \begin{cases} \frac{1}{6}(2n^3 - 15n^2 + 25n), & \text{if } n \equiv 1 \pmod{4} \\ \frac{1}{6}(2n^3 - 15n^2 + 25n - 6), & \text{if } n \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

□

The following result is due to Brundan and Kleshchev and yields an assertion similar to Proposition 3.1 and Proposition 4.4.

**Lemma 5.4** (Brundan, Kleshchev (2001), L. 1.20). *Let  $p \geq 5$  and  $n \geq 13$ . Furthermore, let  $D$  be a simple  $F\mathfrak{S}_n$ -module. Then either  $D$  belongs to  $R_n(2)$  or*

$$\dim(D) \geq \frac{n^3 - 9n^2 + 14n}{6}.$$

We finally give our proof of Theorem 1.1.

**Proof. (Theorem 1.1)** We proceed by distinguishing between three cases.

*Case 1:* Suppose that  $p = 2$ . In Danz (2007) and Zimmermann (2004) the vertices of all simple  $F\mathfrak{S}_n$ -modules for  $n \leq 14$ , and for  $15 \leq n \leq 20$  the vertices of simple  $F\mathfrak{S}_n$ -modules having dimension at most 1000 have been determined. Parts of the results will also appear in Danz, Külshammer, Zimmermann (preprint). Given a simple  $F\mathfrak{S}_n$ -module  $D^\lambda$  of dimension at most 1000, for  $n \leq 16$  the computational data yield the following:

If  $\lambda$  is a spin partition then the vertices of  $D^\lambda$  are as in (b). If  $\lambda$  is an  $S$ -partition, but not a spin partition, then the vertices of  $D^\lambda$  are precisely the Sylow 2-subgroups of  $\mathfrak{A}_{pw}$ . Here  $w$  denotes the weight of  $\lambda$ . If  $D^\lambda \cong S^\lambda$  then we also have  $[Y^\lambda : D^\lambda] = 1$  since there is a Specht filtration

$$0 \subset S^\lambda = Y_k \subset Y_{k-1} \subset \dots \subset Y_0 = Y^\lambda,$$

where  $Y_j/Y_{j+1} \cong S^{\nu_j}$ , for some  $\nu_j \triangleright \lambda$  and  $j = 0, \dots, k-1$ . For a proof see Donkin (1987), (2.6) and James (1983 b), Thm. 3.1. Since  $Y^\lambda$  is indecomposable and selfdual we thus have  $D^\lambda \cong S^\lambda \cong Y^\lambda$ . In this case, Grabmeier's Theorem (cf. Grabmeier (1985), Satz 7.8) implies that  $\prod_{i=1}^{\infty} (P_{\lambda_i - \lambda_{i+1}})^i$  is a vertex of  $D^\lambda$ .

Now suppose that  $\lambda$  is neither an  $S$ -partition nor a spin partition nor a partition of the previous form. Then the vertices of  $D^\lambda$  are precisely the defect groups of its block, except in the cases where  $\lambda \in \{(4, 3, 2), (6, 3, 2)\}$ . Namely, we have explicitly computed that

$$D^{(2)} \boxtimes D^{(3,2)} | \text{Res}_{\mathfrak{S}_2 \times \mathfrak{S}_5}^{\mathfrak{S}_9}(D^{(4,3,2)}) \quad \text{and} \quad D^{(4,3,2)} | \text{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_5}^{\mathfrak{S}_9}(D^{(2)} \boxtimes D^{(3,2)}).$$

Since  $(3, 2)$  is a spin partition,  $D^{(3,2)}$  has vertex  $Q_4 \in \text{Syl}_2(\mathfrak{A}_4)$ , as just mentioned above. Consequently, by Proposition 2.1,  $D^{(4,3,2)}$  and  $D^{(2)} \boxtimes D^{(3,2)}$  have  $P_2 \times Q_4$  as a common vertex. Furthermore, we have computed that also

$$D^{(4)} \boxtimes D^{(3,2)} | \text{Res}_{\mathfrak{S}_4 \times \mathfrak{S}_5}^{\mathfrak{S}_{11}}(D^{(6,3,2)}) \quad \text{and} \quad D^{(6,3,2)} | \text{Ind}_{\mathfrak{S}_4 \times \mathfrak{S}_5}^{\mathfrak{S}_{11}}(D^{(4)} \boxtimes D^{(3,2)})$$

holds so that  $P_4 \times Q_4$  is a vertex of  $D^{(6,3,2)}$ . The modules  $D^{(4,3,2)}$  and  $D^{(6,3,2)}$  are both covered by part (d) of the theorem.

Consequently, we may now assume that  $n \geq 17$ , and consider a simple  $F\mathfrak{S}_n$ -module  $D^\lambda$  such that  $\dim(D^\lambda) \leq 1000$ . By Proposition 4.4 we know that then either  $D^\lambda = D(n)$  or  $D \in R_n(3)$ . Suppose that  $D^\lambda = D(n)$ . Then  $n \leq 20$ , since we are assuming  $\dim(D^\lambda) \leq 1000$ . Again, the vertices of  $D(n)$  for  $n \leq 27$  have been computed in Danz (2007) and Zimmermann (2004), giving assertion (b).

Now, let  $D^\lambda \in R_n(3)$ , i.e.  $\lambda \in \{(n), (n-1, 1), (n-2, 2), (n-3, 3), (n-3, 2, 1)\}$ . In case that  $\lambda \in \{(n), (n-1, 1)\}$  the vertices of  $D^\lambda$  are the defect groups of the block containing  $D^\lambda$ , by Müller, Zimmermann (2007). Next, suppose that  $\lambda = (n-2, 2)$ . Then  $n \leq 47$  by the dimension formulas in Lemma 5.3. Moreover,  $D^{(n-2,2)} \cong S^{(n-2,2)} \cong Y^{(n-2,2)}$  for  $n \equiv 3 \pmod{4}$ , by Corollary 5.2. Hence, in that case  $D^{(n-2,2)}$  has vertex  $P_{n-4} \times (P_2)^2$ , again by Grabmeier's Theorem. It remains to settle the cases where  $n < 46$  and  $n \not\equiv 3 \pmod{4}$ . The computations in Danz (2007) and Zimmermann (2004) show that in all of these cases  $D^{(n-2,2)}$  has exactly the Sylow 2-subgroups of  $\mathfrak{S}_n$  as its vertices.

Now we consider the case where  $\lambda = (n-3, 3)$ , so that  $n \leq 20$ , by the dimension formulas in Lemma 5.3. If  $n$  is odd then  $D^{(n-3,3)} | \text{Ind}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^{(n-4,2)})$  as well as  $D^{(n-4,2)} | \text{Res}_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n}(D^{(n-3,3)})$ , by Theorem 2.3. Hence  $D^{(n-3,3)}$  and  $D^{(n-4,2)}$  have common vertices if  $n$  is odd. By what we have just proved above, these are the Sylow 2-subgroups of  $\mathfrak{S}_{n-2}$  if  $n \equiv 3 \pmod{4}$ , i.e. the defect groups of the block containing  $D^{(n-3,3)}$ . If  $n \equiv 1 \pmod{4}$  then  $P_{n-6} \times (P_2)^2$  is a vertex of both  $D^{(n-4,2)}$  and  $D^{(n-3,3)}$ . Note that in the latter case we also have  $D^{(n-3,3)} \cong S^{(n-3,3)} \cong Y^{(n-3,3)}$ , by Corollary 5.2. If  $n \leq 20$  is even then the vertices of  $D^{(n-3,3)}$  have been computed to be the defect groups of the block of  $F\mathfrak{S}_n$  containing  $D^{(n-3,3)}$ .

Finally, consider  $\lambda = (n-3, 2, 1)$  so that  $n \leq 16$ , by Lemma 5.3, and we thus already know that the vertices of  $D^{(n-3,2,1)}$  are then the defect groups of its block. However, we mention the following: If  $n$  is even, then  $D^{(n-3,2,1)} \cong S^{(n-3,2,1)} \cong Y^{(n-3,2,1)}$ , by Theorem 5.1. Thus Grabmeier's Theorem this time implies that  $D^{(n-3,2,1)}$  has vertex  $P_{n-5}$  which is actually a defect group of the block containing  $D^{(n-3,2,1)}$  in this case. If  $n$  is odd we obtain  $D^{(n-3,2)} | \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^{(n-3,2,1)})$  and  $D^{(n-3,2,1)} | \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^{(n-3,2)})$ , by Theorem 2.3. Consequently, the vertices of  $D^{(n-3,2)}$  are also vertices of  $D^{(n-3,2,1)}$ . Since  $D^{(n-3,2)}$  and  $D^{(n-3,2,1)}$  belong to the principal blocks of  $F\mathfrak{S}_{n-1}$  and  $F\mathfrak{S}_n$ , respectively, the vertices of  $D^{(n-3,2,1)}$  are exactly the defect groups of its block by what we have proved above.

*Case 2:* Let now  $p = 3$ , and consider a simple  $F\mathfrak{S}_n$ -module  $D^\lambda$  such that  $\dim(D^\lambda) \leq 1000$ . By Proposition 3.1, either  $n \leq 15$  or  $D^\lambda \in R_n(3)$ . Consider the case  $n \leq 15$  first. The

computational results in Danz, Külshammer, Zimmermann (preprint) show that then there are four cases which actually occur. Firstly, if  $D^\lambda \cong S^\lambda$  then we again deduce that  $D^\lambda \cong S^\lambda \cong Y^\lambda$ , and  $\prod_{i=1}^{\infty} (P_{\lambda_i - \lambda_{i+1}})^i$  is thus a vertex of  $D^\lambda$ , by Grabmeier's Theorem. The same holds true when replacing  $\lambda$  by the Mullineux conjugate partition  $\mathbf{m}(\lambda)$ . Secondly, if  $n \not\equiv 0 \pmod{3}$  and  $\lambda = (n-r, 1^r)^R$  for some  $r \in \{0, \dots, n-1\}$  then  $D^\lambda \cong S^{(n-r, 1^r)}$ , as mentioned in Subsection 2.3. Moreover, in this case  $P_{n-r-1} \times P_r$  is known to be a vertex of  $D^\lambda$ , by Wildon (2003), Thm. 2, and we are in case (e) of the theorem. Thirdly,  $n = 11$  and  $\lambda = (5, 3, 2, 1)$ . Then  $D^\lambda$  has dimension 714 and belongs to the principal block of  $F\mathfrak{S}_{11}$ . Moreover, our computations show that

$$D^{(5)} \boxtimes D^{(3,2,1)} \mid \text{Res}_{\mathfrak{S}_5 \times \mathfrak{S}_6}^{\mathfrak{S}_{11}}(D^{(5,3,2,1)}) \quad \text{and} \quad D^{(5,3,2,1)} \mid \text{Ind}_{\mathfrak{S}_5 \times \mathfrak{S}_6}^{\mathfrak{S}_{11}}(D^{(5)} \boxtimes D^{(3,2,1)}).$$

Since  $D^{(3,2,1)}$  is contained in a block of  $F\mathfrak{S}_6$  with abelian defect group  $(P_3)^2$ , it has vertex  $(P_3)^2$ , by Knörr's Theorem (cf. Knörr (1979)). So  $D^{(5,3,2,1)}$  and  $D^{(5)} \boxtimes D^{(3,2,1)}$  have common vertex  $(P_3)^3$ , by Proposition 2.1, and we are in case (f). In any other case, our computations show that  $D^\lambda$ , for  $n \leq 15$ , has the defect groups of its block as vertices.

Now we consider the case where  $n \geq 16$ , and by Proposition 3.1 we may assume that  $\lambda \in \{(n), (n-1, 1), (n-2, 2), (n-2, 1^2), (n-3, 3), (n-3, 2, 1)\}$ . If  $\lambda \in \{(n), (n-1, 1), (n-2, 2)\}$  then the vertices of  $D^\lambda$  are exactly the defect groups of its block by Zimmermann (2004) (see also Müller, Zimmermann (2007)). The same holds true for  $\lambda = (n-3, 3)$ , by Zimmermann (2004), except in the case where  $n \equiv 5 \pmod{9}$ . Namely then  $D^\lambda \cong S^\lambda \cong Y^\lambda$ , by Corollary 5.2, and has vertex  $P_{n-6} \times (P_3)^2$  by Grabmeier's Theorem. Thus it remains to settle the cases where  $\lambda \in \{(n-2, 1^2), (n-3, 2, 1)\}$ . Since  $(n-3, 2, 1) = (n-3, 1^3)^R$  we obtain  $\bigwedge^3 D^{(n-1, 1)} \cong D^{(n-3, 2, 1)}$  and  $\bigwedge^2 D^{(n-1, 1)} \cong D^{(n-2, 1^2)}$  by Proposition 2.5. Hence, in particular,  $D^{(n-3, 2, 1)} \cong S^{(n-3, 1^3)}$  and  $D^{(n-2, 1^2)} \cong S^{(n-2, 1^2)}$  provided  $n \not\equiv 0 \pmod{3}$ . By Wildon (2003), the vertices of  $D^{(n-3, 2, 1)}$  and  $D^{(n-2, 1^2)}$  are in those cases precisely the  $\mathfrak{S}_n$ -conjugates of  $P_{n-4} \times P_3$  and  $P_{n-3}$ , respectively. Consequently, we may now assume that  $n \equiv 0 \pmod{3}$ . Then

$$\dim(D^{(n-2, 1^2)}) = \binom{n-2}{2} \quad \text{and} \quad \dim(D^{(n-3, 2, 1)}) = \binom{n-2}{3}.$$

This implies  $n \leq 45$  for  $\lambda = (n-2, 1^2)$ , and  $n \leq 21$  for  $\lambda = (n-3, 2, 1)$ . If  $n \not\equiv 3 \pmod{9}$  then, by Danz (to appear), the vertices of  $D^{(n-2, 1^2)}$  are precisely the Sylow 3-subgroups of  $\mathfrak{S}_n$ , i.e. the defect groups of the principal block of  $F\mathfrak{S}_n$ . Furthermore,  $\dim(D^{(15, 2, 1)}) = 560 \not\equiv 0 \pmod{3}$  so that the Sylow 3-subgroups of  $\mathfrak{S}_{18}$  are the vertices of  $D^{(15, 2, 1)}$ . After all, it remains to consider the following cases:

$n$	$\lambda$	$\dim(D^\lambda)$
21	$(19, 1^2)$	$171 \equiv 0 \pmod{9}$
	$(18, 2, 1)$	$969 \equiv 0 \pmod{3}$
30	$(28, 1^2)$	$378 \equiv 0 \pmod{27}$
39	$(37, 1^2)$	$666 \equiv 0 \pmod{9}$

Our computations show that also in all these cases the Sylow 3-subgroups of  $\mathfrak{S}_n$ , i.e. the defect groups of the principal block of  $F\mathfrak{S}_n$ , are the vertices of  $D^\lambda$ . This completes the case  $p = 3$ .

*Case 3:* Finally, let  $p \geq 5$ , and let  $D^\lambda$  be a simple  $F\mathfrak{S}_n$ -module of dimension at most 1000. Lemma 5.4 shows that either  $n \leq 12$  or  $D^\lambda \in R_n(2)$ . By Zimmermann (2004), the vertices of the modules belonging to  $R_n(2)$  are the defect groups of the respective blocks. For  $n \leq 12 < p^2$ , all blocks of  $F\mathfrak{S}_n$  have abelian defect groups. Consequently, in those cases the vertices of the simple  $F\mathfrak{S}_n$ -modules are the defect groups of the respective blocks as well, by Knörr (1979).

This completes the proof of the theorem.  $\square$

We close with some final remarks on Theorem 1.1.

**Remark 5.5.** Let  $p = 2$ .

(a) As already mentioned in the proof above, we have actually computed the vertices of the  $F\mathfrak{S}_n$ -spin module up to  $n = 27$ . Building on our computational results, we conjecture that the vertices of the spin module  $D(n)$  are as in part (b) of Theorem 1.1, for all  $n \geq 3$ .

(b) Let  $m \geq 2$ , and let  $\lambda_{m,0} := (2m, 3, 2)$ . Moreover, consider the simple  $F\mathfrak{S}_{2m+5}$ -module  $D^{\lambda_{m,0}}$ . We now set  $\lambda_{m,1} := (2m+1, 4, 3)$ ,  $\lambda_{m,2} := (2m+2, 5, 4, 1)$  and  $\lambda_{m,i} := (2m+i, 3+i, 2+i, i-1, i-2, \dots, 2, 1)$ , for  $i \geq 3$ . This leads to the modules appearing in part (d) of Theorem 1.1. As an immediate consequence of Theorem 2.3 we then obtain that

$$D^{\lambda_{m,0}} | \operatorname{Res}_{\mathfrak{S}_{2m+5}}^{\mathfrak{S}|\lambda_{m,i}|} (D^{\lambda_{m,i}}) \quad \text{and} \quad D^{\lambda_{m,i}} | \operatorname{Ind}_{\mathfrak{S}_{2m+5}}^{\mathfrak{S}|\lambda_{m,i}|} (D^{\lambda_{m,0}}),$$

for all  $i \geq 1$ . Hence  $D^{\lambda_{m,0}}$  and  $D^{\lambda_{m,i}}$  have common vertices, for all  $i \geq 1$ .

As mentioned in the proof above, our computations show that in fact the following holds true:

- $D^{(2)} \boxtimes D^{(3,2)} | \operatorname{Res}_{\mathfrak{S}_2 \times \mathfrak{S}_5}^{\mathfrak{S}_9} (D^{(4,3,2)})$  and  $D^{(4,3,2)} | \operatorname{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_5}^{\mathfrak{S}_9} (D^{(2)} \boxtimes D^{(3,2)})$ ,
- $D^{(4)} \boxtimes D^{(3,2)} | \operatorname{Res}_{\mathfrak{S}_4 \times \mathfrak{S}_5}^{\mathfrak{S}_{11}} (D^{(6,3,2)})$  and  $D^{(6,3,2)} | \operatorname{Ind}_{\mathfrak{S}_4 \times \mathfrak{S}_5}^{\mathfrak{S}_{11}} (D^{(4)} \boxtimes D^{(3,2)})$ .

Moreover, we have also checked that

- $D^{(6)} \boxtimes D^{(3,2)} | \operatorname{Res}_{\mathfrak{S}_6 \times \mathfrak{S}_5}^{\mathfrak{S}_{13}} (D^{(8,3,2)})$  and  $D^{(8,3,2)} | \operatorname{Ind}_{\mathfrak{S}_6 \times \mathfrak{S}_5}^{\mathfrak{S}_{13}} (D^{(6)} \boxtimes D^{(3,2)})$ ;
- $D^{(8)} \boxtimes D^{(3,2)} | \operatorname{Res}_{\mathfrak{S}_8 \times \mathfrak{S}_5}^{\mathfrak{S}_{15}} (D^{(10,3,2)})$  and  $D^{(10,3,2)} | \operatorname{Ind}_{\mathfrak{S}_8 \times \mathfrak{S}_5}^{\mathfrak{S}_{15}} (D^{(8)} \boxtimes D^{(3,2)})$ .

The question arising from this observation is the following:

**Question 5.6.** Keeping the notation of Remark 5.5 (b), does then the following hold in general?

$$D^{(2m-2)} \boxtimes D^{(3,2)} | \operatorname{Res}_{\mathfrak{S}_{2m-2} \times \mathfrak{S}_5}^{\mathfrak{S}_{2m+5}} (D^{(2m,3,2)}) \quad \text{and} \quad D^{(2m,3,2)} | \operatorname{Ind}_{\mathfrak{S}_{2m-2} \times \mathfrak{S}_5}^{\mathfrak{S}_{2m+5}} (D^{(2m-2)} \boxtimes D^{(3,2)})$$

The spin module  $D^{(3,2)}$  has vertex  $Q_4$ , by Theorem 1.1, and  $D^{(2m-2)} = F$  has clearly vertex  $P_{2m-2}$ . Hence, provided that there is a positive answer to this question,  $P_{2m-2} \times Q_4$  is then a vertex of both  $D^{(2m-2)} \boxtimes D^{(3,2)}$  and  $D^{\lambda_{m,i}}$ , for  $i \geq 0$ .

**Remark 5.7.** Let  $p = 3$ , and let  $n = m + 6$ , with  $m \geq 5$  and  $m \equiv 2 \pmod{3}$ . Consider  $\lambda = (m, 3, 2, 1)$ . We have already mentioned that

$$D^{(5)} \boxtimes D^{(3,2,1)} | \text{Res}_{\mathfrak{S}_5 \times \mathfrak{S}_6}^{\mathfrak{S}_{11}}(D^{(5,3,2,1)}) \quad \text{and} \quad D^{(5,3,2,1)} | \text{Ind}_{\mathfrak{S}_5 \times \mathfrak{S}_6}^{\mathfrak{S}_{11}}(D^{(5)} \boxtimes D^{(3,2,1)}).$$

Furthermore, we have computed that also

$$D^{(8)} \boxtimes D^{(3,2,1)} | \text{Res}_{\mathfrak{S}_8 \times \mathfrak{S}_6}^{\mathfrak{S}_{14}}(D^{(8,3,2,1)}) \quad \text{and} \quad D^{(8,3,2,1)} | \text{Ind}_{\mathfrak{S}_8 \times \mathfrak{S}_6}^{\mathfrak{S}_{14}}(D^{(8)} \boxtimes D^{(3,2,1)})$$

holds true. The simple  $F\mathfrak{S}_{14}$ -module  $D^{(8,3,2,1)}$  of dimension 6369 belonging to the principal block of  $F\mathfrak{S}_{14}$  has thus vertex  $(P_3)^4$ , since both  $D^{(8)}$  and  $D^{(3,2,1)}$  have vertex  $(P_3)^2$ . The question arising from these facts is

**Question 5.8.** Keeping the notation of the previous remark, does then the following hold in general?

$$D^{(m)} \boxtimes D^{(3,2,1)} | \text{Res}_{\mathfrak{S}_m \times \mathfrak{S}_6}^{\mathfrak{S}_{m+6}}(D^{(m,3,2,1)}) \quad \text{and} \quad D^{(m,3,2,1)} | \text{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_6}^{\mathfrak{S}_{m+6}}(D^{(m)} \boxtimes D^{(3,2,1)})$$

Provided that this question has a positive answer,  $P_m \times (P_3)^2$  is then a vertex of  $D^{(m,3,2,1)}$ , for  $m \equiv 2 \pmod{3}$ . One thing to mention here is that we cannot drop the condition  $m \equiv 2 \pmod{3}$ . Namely, if  $m \equiv 0 \pmod{3}$  then  $D^{(m,3,2,1)}$  belongs to the principal block of  $F\mathfrak{S}_n$  and  $n \equiv 0 \pmod{3}$ . Furthermore, in this case  $P_n$  is not conjugate to  $P_m \times (P_3)^2$  if  $m \not\equiv 0 \pmod{9}$ . Since we expect a positive answer to Question 4.2 in Danz, Külshammer, Zimmermann (preprint),  $D^{(m,3,2,1)}$  should then have vertex  $P_n$ . In all our explicit examples this has indeed been the case. Moreover, if  $m \equiv 1 \pmod{3}$  we obtain  $D^{(m-1,3,2,1)} | \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^{(m,3,2,1)})$  and  $D^{(m,3,2,1)} | \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D^{(m-1,3,2,1)})$ , by Theorem 2.3. Hence, in this case, the vertices of  $D^{(m-1,3,2,1)}$  are also vertices of  $D^{(m,3,2,1)}$ .

## A Dimensions of simple $F\mathfrak{S}_{18}$ -modules in characteristic 2

The table below displays some lower bounds for the dimensions of the simple  $F\mathfrak{S}_{18}$ -modules in characteristic 2. The entries of that table should be read as follows: Modules are represented by the corresponding 2-regular partitions. If, for some 2-regular partition  $\lambda$  of 18, there is an entry  $\mu$  in column “ $m$ ” where  $m \in \{14, 15, 16\}$ , then  $(n - m)!D^\mu | \text{Res}_{\mathfrak{S}_m}^{\mathfrak{S}_{18}}(D^\lambda)$ , by Theorem 2.3. If there is an entry  $\mu$  (respectively two entries  $\mu$  and  $\nu$ ) in column “17” opposite  $\lambda$  then  $\text{Res}_{\mathfrak{S}_{17}}^{\mathfrak{S}_{18}}(D^\lambda) \cong D^\mu$  (respectively  $\text{Res}_{\mathfrak{S}_{17}}^{\mathfrak{S}_{18}}(D^\lambda) \cong D^\mu \oplus D^\nu$ ), by Theorem 2.3. All the other dimensions have been computed using known dimension formulas such as in Lemma 5.3, or by constructing the modules in question with the computer.

partition	17	16	15	14	dimension
(18)					1
(17,1)					16
(16,2)					118
(15,3)					544
(15,2,1)					1344
(14,4)	(13,4)				1582



partition	17	16	15	14	dimension
(13,5)	(12,5)				3808
(14,3,1)		(13,2,1)			$\geq 1792$
(12,6)	(11,6)				4488
(10,8)					256
(11,7)	(10,7)				6528
(13,4,1)			(12,3)		$\geq 2016$
(13,3,2)	(12,3,2);(13,3,1)				19124
(12,5,1)		(11,4,1)			$\geq 21504$
(9,8,1)					4096
(11,6,1)			(10,5)		$\geq 7728$
(9,5,4)	(8,5,4);(9,5,3)				170494
(9,6,3)			(8,5,2)		$\geq 192192$
(8,6,4)	(7,6,4)				55808
(8,7,2,1)				(7,6,1)	$\geq 18432$
(10,5,2,1)				(9,4,1)	$\geq 91392$
(10,4,3,1)	(10,4,2,1);(9,4,3,1)				492386
(9,6,2,1)		(8,5,2,1)			$\geq 532480$
(9,5,3,1)	(8,5,3,1)				225760
(7,6,4,1)		(6,5,4,1)			$\geq 442368$
(8,6,3,1)	(8,6,2,1);(7,6,3,1)				202368
(9,4,3,2)				(8,3,2,1)	$\geq 559104$
(8,5,4,1)				(7,4,3)	$\geq 261120$
(6,5,4,3)				(5,4,3,2)	$\geq 860160$
(7,6,3,2)				(6,5,2,1)	$\geq 319488$
(8,5,3,2)		(7,4,3,2)			$\geq 1003520$
(7,5,4,2)	(6,5,4,2);(7,5,3,2)				574464
(8,4,3,2,1)			(8,4,2,1)		$\geq 145560$
(6,5,4,2,1)			(5,4,3,2,1)		$\geq 1757184$
(7,5,3,2,1)			(6,5,3,1)		$\geq 139776$
(12,4,2)	(11,4,2)				26844
(12,3,2,1)				(11,2,1)	$\geq 13440$
(11,5,2)	(10,5,2);(11,5,1)				116280
(11,4,3)			(10,3,2)		$\geq 44688$
(11,4,2,1)		(10,3,2,1)			$\geq 143366$
(10,7,1)		(9,6,1)			$\geq 48128$
(10,6,2)	(9,6,2)				102102
(10,5,3)		(9,4,3)			$\geq 88064$
(9,7,2)	(9,7,1);(8,7,2)				14976
(8,7,3)		(7,6,3)			$\geq 45056$
(7,6,5)			(6,5,4)		$\geq 126912$

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