

On vertices of exterior powers of the natural simple module for the symmetric group in odd characteristic

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Abstract

We determine the vertices of certain exterior powers of the natural simple $F\mathfrak{S}_n$ -module in odd characteristic p and, in particular, of the irreducible $F\mathfrak{S}_n$ -module $D^{(n-p+1, p-1)}$ for the case $n = pw$ and $w \not\equiv 1 \pmod{p}$.

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1 Introduction

One of the main focuses in modular representation theory lies on the investigation of certain structural invariants of indecomposable and, in particular, simple FG -modules, where F is an algebraically closed field of prime characteristic p and G is a finite group. Among those invariants are the vertices of indecomposable FG -modules which had been introduced by J. A. Green in [4].

The representation theory of the symmetric group \mathfrak{S}_n of degree $n \in \mathbb{N}$ is known to have strong relations to combinatorics, such as the parametrization of simple $F\mathfrak{S}_n$ -modules by the p -regular partitions of n and the characterization of blocks in terms of p -cores and p -weights. However, though closely connected to blocks and their defect groups, vertices of simple $F\mathfrak{S}_n$ -modules could so far not be classified in a similar combinatorial way in general. Nevertheless, for certain families of simple $F\mathfrak{S}_n$ -modules this is indeed possible. One of those consists of the simple $F\mathfrak{S}_n$ -modules parametrized by p -regular *hook partitions*, i.e. p -regular partitions of shape $(n-r, 1^r)$. In [11] J. Müller and R. Zimmermann dealt with these modules and determined their vertices except for the case where p is odd, $p^2 \leq n \equiv 0 \pmod{p}$ and $r = p-1$. They showed, that the vertices of $D^{(n-r, 1^r)}$ are precisely the defect groups of its block except for $n = 4$, $p = 2$ and $r = 1$, where the Sylow 2-subgroup of the alternating group \mathfrak{A}_4 is a vertex of $D^{(3,1)}$.

In odd characteristic one has a generalization of the simple modules corresponding to hook partitions. Therefore, let $p > 2$ and consider the *natural simple $F\mathfrak{S}_n$ -module* $D = D^{(n-1, 1)}$, i.e. the only nontrivial composition factor of the natural permutation module. Then $\dim(D) = n-1$ if $n \not\equiv 0 \pmod{p}$ and $\dim(D) = n-2$ if $n \equiv 0 \pmod{p}$. Moreover, for $r \leq \dim(D)$, the r -fold exterior power $\bigwedge^r D =: D_r$ is again a simple $F\mathfrak{S}_n$ -module. In particular, one has $D_r \cong D^{(n-r, 1^r)}$ for $r < p$.

If $n \not\equiv 0 \pmod{p}$, the module D_r with $r \leq n-1$ is isomorphic to the simple Specht module $S^{(n-r, 1^r)}$, and its vertices are then known. They are exactly the \mathfrak{S}_n -conjugates of the Sylow p -subgroups of the Young subgroup $\mathfrak{S}_{n-r-1} \times \mathfrak{S}_r \leq \mathfrak{S}_n$, as has been shown in [14], Thm. 2.

In this paper we are therefore basically concerned with the case $n = pw$ for some $w \in \mathbb{N}$, so that D_r , for $r \leq n - 2$, is then contained in the principal block of $F\mathfrak{S}_n$. With these assumptions, in Section 5 we will prove the following:

Theorem. *Let p be odd, and let $n = pw$ for some $w \in \mathbb{N}$. Moreover, let $k \in \{1, \dots, w - 1\}$ and $r := kp - 1$. If the dimension of D_r is not divisible by p^2 , then the vertices of D_r are the Sylow p -subgroups of \mathfrak{S}_n .*

From this we then also immediately get:

Corollary. *If p is odd and $n = pw$ for some $w \not\equiv 1 \pmod{p}$ then the vertices of $D_{p-1} \cong D^{(n-p+1, p-1)}$ are the Sylow p -subgroups of \mathfrak{S}_n .*

Thus, in this situation, there is a positive answer to Conjecture (1.7) (a) in [11]. After all, in order to determine the vertices of all simple $F\mathfrak{S}_n$ -modules corresponding to hook partitions $(n - r, 1^r)$, it remains to treat the case where p is odd, $n = wp$ for some $w \equiv 1 \pmod{p}$ and $r = p - 1$.

At this point we shall emphasize that the assumption of p being odd in the situation above is essential. Namely, in characteristic 2 the exterior powers of the natural simple $F\mathfrak{S}_n$ -module are in general not simple, often not even indecomposable. However, in case $p = 2 < n$ and $2 \nmid n$, we again have $\bigwedge^r D^{(n-1, 1)} \cong S^{(n-r, 1^r)}$ for $r = 0, \dots, n - 1$. Moreover, if $S^{(n-r, 1^r)}$ is then also indecomposable, its vertices are, as in odd characteristic, conjugate to the Sylow p -subgroups of $\mathfrak{S}_{n-r-1} \times \mathfrak{S}_r$, by [14], Thm. 2.

Now, this paper is organized as follows. We begin by summarizing some notation and basic facts about exterior powers of modules over group algebras, about Sylow p -subgroups of the symmetric groups and vertices of simple modules. Then, in Section 5 we deal with the exterior powers of the natural simple $F\mathfrak{S}_n$ -module in odd characteristic and present our results on vertices of such exterior powers.

Throughout this paper, a group G is always supposed to be a finite group, F is an algebraically closed field of prime characteristic p , and any FG -module is a left module of finite F -dimension. As far as the modular representation theory of the symmetric groups is concerned, we take the notation needed from [6] and [7].

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2 Sylow p -subgroups of the symmetric groups

Let p be a prime number, and consider the cyclic group $C_p := \langle (1, \dots, p) \rangle$ of order p . We set $P_1 := 1$, $P_p := C_p$ and $P_{p^i} := P_{p^{i-1}} \wr C_p$ for $i \geq 2$. We then regard P_{p^i} as a subgroup of \mathfrak{S}_{p^i} , by identifying an element $(\sigma_1, \dots, \sigma_p; \pi) \in P_{p^i}$ with $(\overline{\sigma_1, \dots, \sigma_p}; \pi) \in \mathfrak{S}_{p^i}$ which is defined as

follows. Let $j \in \{1, \dots, p^i\}$ with $j = p^{i-1}(a-1) + b$ for $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, p^{i-1}\}$. Then we set:

$$\overline{(\sigma_1, \dots, \sigma_p; \pi)}(j) := p^{i-1}(\pi(a) - 1) + \sigma_{\pi(a)}(b).$$

For $n \in \mathbb{N}$ we consider the p -adic expansion $n = \sum_{i=0}^r \alpha_i p^i$ of n . Then the Sylow p -subgroups of \mathfrak{S}_n are isomorphic to $\prod_{i=0}^r (P_{p^i})^{\alpha_i}$, by [7], 4.1.22 and 4.1.24. In the following, P_n will denote a fixed Sylow p -subgroup of \mathfrak{S}_n which is constructed as follows: Let

$$P_{p^i, j_i} := (1, 1 + k(j_i)) \cdots (p^i, p^i + k(j_i)) \cdot P_{p^i} \cdot (1, 1 + k(j_i)) \cdots (p^i, p^i + k(j_i)),$$

for $i \in \{1, \dots, r\}$, $j_i \in \{1, \dots, \alpha_i\}$ and $k(j_i) := \sum_{a=i+1}^r \alpha_a p^a + (j_i - 1)p^i$, and set

$$P_n = P_{p^r, 1} \times \cdots \times P_{p^r, \alpha_r} \times P_{p^{r-1}, 1} \times \cdots \times P_{p^{r-1}, \alpha_{r-1}} \times \cdots \times P_{p, 1} \times \cdots \times P_{p, \alpha_1}.$$

For the sake of simplicity, with this convention, we will then just write $P_n = \prod_{i=1}^r (P_{p^i})^{\alpha_i}$ instead.

Lemma 2.1. *Let $k, l \in \mathbb{N}$ such that $p^k \leq p^l \leq n$ and $P_{p^k} \leq P_{p^l} \leq \mathfrak{S}_n$. Then P_{p^l} possesses exactly p^{l-k} subgroups which are in \mathfrak{S}_n conjugate to P_{p^k} . Furthermore, if $P := xP_{p^k}x^{-1} \leq P_{p^l}$ for some $x \in \mathfrak{S}_n$, then there also exists an element $y \in P_{p^l}$ with $P = yP_{p^k}y^{-1}$.*

Proof. We prove the assertion by induction on $l - k$. If $l - k = 0$ we have $P_{p^k} = P_{p^l}$ and may set $y := 1$. Let now $l > l - k > 0$. Then, by definition, $P_{p^l} = P_{p^{l-1}} \wr C_p$. Moreover, denoting the base group of the wreath product $P_{p^{l-1}} \wr C_p$ by B , we have $B = \prod_{i=0}^{p-1} g^i P_{p^{l-1}} g^{-i}$ with $g := (1, \dots, 1; (1, \dots, p))$. Let $P := xP_{p^k}x^{-1} \leq P_{p^l}$ for some $x \in \mathfrak{S}_n$. Since $k < l$, every element in P has a fixed point on $\{1, \dots, p^l\}$, and so we get $P \leq B$. Moreover, P has exactly one nontrivial orbit on $\{1, \dots, p^l\}$, and that orbit has length p^k . Consequently, it has to be contained in one of the p orbits of B , and P itself has to be a subgroup of one of the factors $g^i P_{p^{l-1}} g^{-i}$. Hence $g^{-i} x P_{p^k} x^{-1} g^i \leq P_{p^{l-1}}$ for some $i \in \{0, \dots, p-1\}$. By induction, $P_{p^{l-1}}$ possesses precisely p^{l-1-k} subgroups which are conjugate to P_{p^k} , and there is an $h \in P_{p^{l-1}}$ with $g^{-i} x P_{p^k} x^{-1} g^i = h P_{p^k} h^{-1}$. We now set $y := g^i h \in P_{p^l}$ and finally get $P = y P_{p^k} y^{-1}$. Furthermore, P_{p^l} has exactly $p \cdot p^{l-1-k} = p^{l-k}$ subgroups that are conjugate to P_{p^k} . \square

Remark 2.2. Again consider $n \in \mathbb{N}$ and the fixed Sylow p -subgroup P_n of \mathfrak{S}_n as defined above. Moreover, let $P \leq P_n$ be in \mathfrak{S}_n conjugate to P_{p^k} for some $k \in \{1, \dots, r\}$. Then, first of all, we get $P \leq P_{p^l, j_l}$ for some $l \geq k$ and $1 \leq j_l \leq \alpha_l$, since P has exactly one nontrivial orbit on $\{1, \dots, n\}$. Furthermore, the previous lemma shows that P_{p^l, j_l} contains exactly p^{l-k} subgroups which are in \mathfrak{S}_n conjugate to P_{p^k} . Thus, in particular, the p -cycles which are contained in P_n are precisely the elements $(1, \dots, p), (p+1, \dots, 2p), \dots, (n - \alpha_0 - p + 1, \dots, n - \alpha_0)$ and their nontrivial powers.

3 Vertices of simple FG -modules

Let V be an indecomposable FG -module. Then a subgroup P of G which is minimal subject to the condition that V is *relatively P -projective*, i.e. $V | \text{Ind}_P^G(\text{Res}_P^G(V))$, is called a *vertex* of V . The vertices of an indecomposable FG -module are known to form a G -conjugacy class of p -subgroups of G . Moreover, if V is contained in the block B of FG , and if P is a vertex of V , then there are a Sylow p -subgroup R of G and a defect group Δ of B with $P \leq \Delta \leq R$

and $|R : P| \mid \dim(V)$. A proof for this can be found in [4]. Given a vertex P of an indecomposable FG -module V , there exists an indecomposable FP -module L with vertex P such that $V \mid \text{Ind}_P^G(L)$ and $L \mid \text{Res}_P^G(V)$. Then L is called a *source* of V , and L is determined up to isomorphism and $N_G(P)$ -conjugacy.

The following lemma will play an important role in the proof of our main result in Section 5.

Lemma 3.1. *Let $K \leq H \leq G$ such that $p \mid |K|$, and let V be a relatively H -projective FG -module. If $\text{Res}_K^G(V)$ is projective we have $p \cdot |G : H|_p \mid \dim(V)$ where $|G : H|_p$ denotes the highest p -power dividing $|G : H|$.*

Proof. We show that the assertion holds true if V is indecomposable. The general statement then follows by considering each indecomposable direct summand of V . Furthermore, assume G to be a p -group first, let $P \leq H$ be a vertex of V , and let L be a source of V . Thus $V \mid \text{Ind}_P^G(L)$, and Green's Indecomposability Theorem (cf. [4]) yields $V \cong \text{Ind}_P^G(L)$. If $P < H$ we therefore immediately get $p \cdot |G : H| \mid |G : P| \dim(L) = \dim(V)$. If $P = H$ we have $K \leq P$ so that $\text{Res}_K^P(L) \mid \text{Res}_K^G(V)$ is projective which implies $p \mid |K| \mid \dim(L)$. Thus also in this case we obtain $p \cdot |G : H| \mid |G : P| \dim(L) = \dim(V)$.

Now let G be arbitrary. Furthermore, let R_1, R_2 and R_3 be Sylow p -subgroups of G, H and K , respectively, such that $R_3 \leq R_2 \leq R_1$. Moreover, let $\text{Res}_{R_1}^G(V) = \bigoplus_{i=1}^r W_i$ with indecomposable FR_1 -modules W_1, \dots, W_r . Since V is relatively H -projective, $\text{Res}_{R_1}^G(V)$ and therefore also W_1, \dots, W_r are relatively R_2 -projective. Since $\text{Res}_K^G(V)$ is projective, $\text{Res}_{R_3}^G(V)$ and hence $\text{Res}_{R_3}^{R_1}(W_i)$ is projective for $i = 1, \dots, r$. As we have already proved, the assertion of the lemma holds for p -groups, so that we get $p \cdot |R_1 : R_2| \mid \dim(W_i)$, for $i = 1, \dots, r$. This finally yields $p \cdot |G : H|_p = p \cdot |R_1 : R_2| \mid \dim(V)$. \square

As far as simple FG -modules are concerned, the following theorem is essential:

Theorem 3.2 (Knörr [8]). *Let V be a simple FG -module belonging to a block B of FG , and let P be a vertex of V . Then there exists a block b of $F[PC_G(P)]$ with defect group P such that $b^G = B$. Consequently there is a defect group Δ of B with $C_\Delta(P) \leq P \leq \Delta$.*

Thus Knörr's Theorem implies that the vertices of simple FG -modules in blocks with abelian defect groups are exactly those defect groups. As another direct consequence of the above theorem and Prop. 1.4 in [12] we obtain:

Lemma 3.3. *Let D be a simple $F\mathfrak{S}_n$ -module belonging to a block B of weight w . Moreover, let $\Delta \leq \mathfrak{S}_{pw}$ be a defect group of B , and let $P \leq \Delta$ be a vertex of D . Then $C_{\mathfrak{S}_{pw}}(P) = Z(P)$.*

Corollary 3.4. *Let D be a simple $F\mathfrak{S}_n$ -module belonging to a block B of weight w . Then D is not relatively \mathfrak{S}_{pw-1} -projective.*

Proof. If D were relatively \mathfrak{S}_{pw-1} -projective, then it would also be relatively $\mathfrak{S}_{p(w-1)}$ -projective. Hence there would be a Sylow p -subgroup R of $\mathfrak{S}_{p(w-1)}$ and a vertex P of D with $P \leq R$. But then the p -cycle $(p(w-1) + 1, \dots, pw)$ would have to be contained in $C_{\mathfrak{S}_{pw}}(P) \setminus P$, a contradiction to the previous lemma. \square

4 Exterior powers

In the following we recall some of the basic facts about exterior powers of modules over group algebras.

Remark 4.1. (a) Let V be an FG -module with $\dim(V) = m$ and $s \geq 2$. Then the s -fold exterior power $\bigwedge^s V$ becomes an FG -module of dimension $\binom{m}{s}$ via $g(v_1 \wedge \dots \wedge v_m) := gv_1 \wedge \dots \wedge gv_m$, for $g \in G$ and $v_1, \dots, v_m \in V$, and we set $\bigwedge^0 V := F$ and $\bigwedge^1 V := V$. Moreover, $\dim(\bigwedge^m V) = 1$, and $g \in G$ acts on $\bigwedge^m V$ via multiplication with $\det(\Gamma(g))$ where Γ is the matrix representation of G over F afforded by V .

- For $s \in \{0, \dots, m\}$ one always has an isomorphism of FG -modules

$$\left(\bigwedge^m V\right)^* \otimes \bigwedge^s V \cong \left(\bigwedge^{m-s} V\right)^*,$$

where W^* denotes the dual of an FG -module W . An isomorphism Ψ is obtained by sending an element of the form $\lambda \otimes v$, with $\lambda \in (\bigwedge^m V)^*$ and $v \in \bigwedge^s V$, to the linear map $\gamma_{\lambda, v} \in (\bigwedge^{m-s} V)^*$ defined by $\gamma_{\lambda, v}(w) := \lambda(v \wedge w)$, for $w \in \bigwedge^{m-s} V$. This result is probably well known. However, we could not find an appropriate reference.

In particular, if $\bigwedge^s V$ is indecomposable then so is $\bigwedge^{m-s} V$, and both modules have the same vertices.

- If $s < p$, then

$$\bigwedge^s V \mid \bigotimes_{i=1}^s V,$$

by [7], 5.2. For any FG -modules U and V and any $s \in \mathbb{N}$ one also has

$$\bigwedge^s (V \oplus U) \cong \bigoplus_{i=0}^s \left[\bigwedge^i V \otimes \bigwedge^{s-i} U \right],$$

by [2], (12.2).

(b) Consider a cyclic group C of order p^l for some $l \in \mathbb{N}_0$. Then FC possesses exactly p^l isomorphism classes of indecomposable modules. Namely, up to isomorphism, for any $i = 1, \dots, p^l$ there is exactly one indecomposable FC -module T_i of dimension i . A proof for this is given in [5], Thm. VII.5.3. With this notation we now get:

Lemma 4.2. *For a cyclic group C of order p , let V be a projective FC -module, and let $s \in \mathbb{N}$ with $p \nmid s$. Then $\bigwedge^s V$ is a projective FC -module as well.*

Proof. We prove the assertion by induction on $\dim(V)$. Suppose first that $\dim(V) = p$, i.e. $V \cong FC$. Then $\bigwedge^s V = 0$ for $s > p$, and $\bigwedge^s V \mid \bigotimes_{i=1}^s V$, for $s < p$, by Remark 4.1. Thus $\bigwedge^s V$ is a projective FC -module. We may now assume that $\dim(V) > p$ and $V = V_1 \oplus V_2$ with submodules $V_1 \neq 0 \neq V_2$. Then Remark 4.1 yields

$$\bigwedge^s V = \bigwedge^s (V_1 \oplus V_2) \cong \bigoplus_{i=0}^s \left[\bigwedge^i V_1 \otimes \bigwedge^{s-i} V_2 \right].$$

The hypothesis $p \nmid s$ implies $p \nmid i$ or $p \nmid s - i$, for all $i = 0, \dots, s$. Hence, by induction, $\bigwedge^i V$ or $\bigwedge^{s-i} V$ is projective, and consequently also $\bigwedge^i V \otimes \bigwedge^{s-i} V$ is projective, for $i = 0, \dots, s$. This finally proves the lemma. \square

5 Results

Remark 5.1. In this section we assume $\text{char}(F) = p > 2$, we fix an integer $n \geq 2$ and a non-negative integer $r \leq n - 1$. Furthermore, let $M := M^{(n-1,1)}$ be the natural $F\mathfrak{S}_n$ -permutation module, $S := S^{(n-1,1)}$ the natural $F\mathfrak{S}_n$ -Specht module and $D := D^{(n-1,1)}$ the natural simple $F\mathfrak{S}_n$ -module. Moreover, we denote the head $V/\text{Rad}(V)$ of an $F\mathfrak{S}_n$ -module V by $\text{Hd}(V)$.

(a) The $F\mathfrak{S}_n$ -epimorphism $\varphi : S \rightarrow D \cong \text{Hd}(S)$ induces an $F\mathfrak{S}_n$ -epimorphism

$$\bigwedge^r \varphi : \bigwedge^r S \rightarrow \bigwedge^r D =: D_r.$$

By [11], Prop. 2.2, we have $\bigwedge^r S \cong S^{(n-r, 1^r)}$.

(b) If $p \nmid n$ we have $M \cong D^{(n)} \oplus D$ and $S \cong D$, by [6], Ex. 5.1. Hence, in this case, we get $\dim(D) = n - 1$ and also $D_r \cong S^{(n-r, 1^r)}$, for $r \leq n - 1$, which is simple by [13] (see also [6], Thm. 24.1). Moreover, the vertices of D_r are known then. By [14], Thm. 2, they are the \mathfrak{S}_n -conjugates of the Sylow p -subgroups of $\mathfrak{S}_{n-r-1} \times \mathfrak{S}_r$.

(c) For this reason, we now suppose that $n = wp$ with $w \geq 1$ and $r \leq n - 2$. In this case, by [6], Ex. 5.1, M is uniserial with composition series $M \supset M_1 \supset M_2 \supset 0$, where $M/M_1 \cong D^{(n)} \cong M_2$, $M_1 = S$ and $M_1/M_2 \cong D$.

Furthermore, if $\{b_1, \dots, b_n\}$ is a permutation basis for M , then $S = \{\sum_{i=1}^n \alpha_i b_i \mid \alpha_1, \dots, \alpha_n \in F; \sum_{i=1}^n \alpha_i = 0\}$ and $M_2 = \{\alpha \sum_{i=1}^n b_i \mid \alpha \in F\}$. Thus, in particular, $\dim(D) = n - 2$.

For $0 < r \leq n - 2$, the Specht module $S^{(n-r, 1^r)}$ is uniserial with two composition factors, by [13] (see also [6], Thm. 24.1). More precisely, we get $\text{Hd}(S^{(n-r, 1^r)}) \cong D_r \cong \text{Soc}(S^{(n-r-1, 1^{r+1})})$ for $0 \leq r \leq n - 2$ and, in particular, $D_0 \cong D^{(n)} \cong S^{(n)}$ and $D_{n-2} \cong S^{(1^n)}$. Thus, as a composition factor of $S^{(n-r, 1^r)}$, the module D_r is contained in the principal block of $F\mathfrak{S}_n$ and therefore not relatively \mathfrak{S}_{n-1} -projective, by Corollary 3.4.

Finally, for $r < p$ we obtain the simple modules corresponding to hook partitions. Namely, then $D_r \cong D^{(n-r, 1^r)}$. For $r \geq p$, the p -regular partition of n corresponding to D_r can be constructed by using the method of “regularizing” a partition as described in [7], 6.3.48.

Proposition 5.2. *Let $n \equiv 0 \pmod{p}$, let $r \in \{0, \dots, n - 2\}$, and let $P \leq P_n$ be a vertex of D_r . Then*

$$P_{n-r-2} \times P_r <_{\mathfrak{S}_n} P \leq P_n.$$

Proof. First of all, we have $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D) \cong D^{(n-2, 1)}$. Together with the previous remark this yields

$$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D_r) = \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \left(\bigwedge^r D \right) \cong \bigwedge^r (\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D)) \cong \bigwedge^r D^{(n-2, 1)} \cong S^{(n-1-r, 1^r)},$$

and $P_{n-r-2} \times P_r$ is a vertex of $S^{(n-1-r, 1^r)}$, by [14], Thm. 2. Since, by Corollary 3.4, D_r is not relatively \mathfrak{S}_{n-1} -projective we thus obtain $P_{n-r-2} \times P_r <_{\mathfrak{S}_n} P \leq P_n$ for a vertex $P \leq P_n$ of D_r . \square

As already mentioned, the vertices of D_r with $r < p - 1$ have been determined in [11]. For this reason, we are now mainly concerned with the case $r \geq p - 1$.

Theorem 5.3. *Let p be odd, let $n = pw$, for some $w \in \mathbb{N}$, and let $r = kp - 1$, for some $k \in \{1, \dots, w - 1\}$. If the dimension of D_r is not divisible by p^2 , then the vertices of D_r are the Sylow p -subgroups of \mathfrak{S}_n .*

Proof. In case $w < p$ the assertion clearly holds, by Theorem 3.2. Hence we may now suppose that $w \geq p$. Let again $P \leq P_n$ be a vertex of D_r . Since $\dim(D_r)$ is supposed to be not divisible by p^2 , we get $|P_n : P| \in \{1, p\}$. We assume that $P \neq P_n$, i.e. $|P_n : P| = p$. Proposition 5.2 implies that

$$P_{(w-k-1)p} \times P_{(k-1)p} \sim_{\mathfrak{S}_n} Q < P < P_n.$$

On the one hand, $\langle z \rangle \leq Z(P_n) \leq P$ with $z := (1, \dots, p)(p+1, \dots, 2p) \cdots (n-p+1, \dots, n)$, by Lemma 3.3. On the other hand, we have $(w-k-1)p \geq p$ or $(k-1)p \geq p$, so that P , and therefore P_n as well, contains a subgroup which is conjugate to P_p . Thus, from Remark 2.2, we deduce that at least one of the p -cycles $(1, \dots, p), (p+1, \dots, 2p), \dots, (n-p+1, \dots, n)$ has to be contained in P . So, if $((j-1)p+1, \dots, jp) \in P$ for some $j \in \{1, \dots, w\}$, we set $c := z \cdot ((j-1)p+1, \dots, jp)^{-1}$ and have $\langle c \rangle =: C \leq P$. As in Remark 4.1 (b) we again fix a transversal $\{T_1, \dots, T_p\}$ for the isomorphism classes of indecomposable FC -modules and obtain $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M) \cong pT_1 \oplus mT_p$ where $m = w - 1$. In the notation of Remark 5.1 (c), $\text{Soc}(\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M)) \not\subseteq \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_1)$. Thus there exists a submodule U of $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M)$ with $U \cong F$ and $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M) = \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_1) \oplus U$. Moreover, $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_2) \not\subseteq \text{Rad}(\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M)) = \text{Rad}(\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_1))$. Consequently, there is a submodule V of $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_1)$ with $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_1) = \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_2) \oplus V$ and $V \cong \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(D)$. Hence $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M) = \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(M_2) \oplus U \oplus V \cong 2T_1 \oplus \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(D)$ and therefore

$$\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(D) \cong (p-2)T_1 \oplus mT_p.$$

With Remark 4.1, this shows

$$\begin{aligned} \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(D_r) &= \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n} \left(\bigwedge^r D \right) \cong \bigwedge^r \text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(D) \cong \bigwedge^r ((p-2)T_1 \oplus mT_p) \\ &\cong \bigoplus_{i=0}^r \left[\bigwedge^i ((p-2)T_1) \otimes \bigwedge^{r-i} (mT_p) \right] \\ &= \bigoplus_{i=0}^{p-2} \left[\bigwedge^i ((p-2)T_1) \otimes \bigwedge^{r-i} (mT_p) \right], \end{aligned}$$

since $\bigwedge^i((p-2)T_1) = 0$ for $i > p-2$. Furthermore $r-i \in \{(k-1)p+1, \dots, (k-1)p+(p-1)\}$ for $i \in \{0, \dots, p-2\}$. Thus Lemma 4.2 applies to those cases, so that $\bigwedge^{r-i}(mT_p)$ and therefore also $\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_n}(D_r)$ is projective, for $i = 0, \dots, p-2$. Hence we can now apply Lemma 3.1 with $K = C$, $H = P$ and $G = \mathfrak{S}_n$ and finally get

$$p^2 = |\mathfrak{S}_n : P|_p |C| \mid \dim(D_r),$$

a contradiction to the hypothesis $p^2 \nmid \dim(D_r)$. Consequently $P = P_n$, and the theorem is proved. \square

Remark 5.4. Our original proof of the previous theorem made use of a result by C. Bessenrodt (cf. [1], Thm. 1.1) which involves the complexity of a module. The present, more elementary argument which uses Lemma 3.1 instead had been suggested by the referee.

With Theorem 5.3, we now obtain the following corollary which is a generalization of Thm. (1.2) in [11].

Corollary 5.5. *Let p be odd, let $n = pw$ for some $w \in \mathbb{N}$, and let $(n - r, 1^r)$ be a $(p$ -regular) hook partition of n . Suppose that $r < p - 1$ or that $r = p - 1$ and $w \not\equiv 1 \pmod{p}$. Then the vertices of the simple $F\mathfrak{S}_n$ -module $D^{(n-r, 1^r)}$ are precisely the Sylow p -subgroups of \mathfrak{S}_n .*

Proof. The assertion for $r < p - 1$ had been shown in [11], Thm. (1.2). In case $r = p - 1$ we have $D^{(n-r, 1^r)} = D^{(n-p+1, 1^{p-1})} \cong D_{p-1}$ and

$$\dim(D_{p-1}) = \binom{n-2}{p-1} = \frac{(wp-2) \cdots (w-1)p}{(p-1)!}.$$

Thus, if $w \not\equiv 1 \pmod{p}$, then $\dim(D_{p-1})$ is divisible by p but not by p^2 , and the assertion follows, by Theorem 5.3. \square

Proposition 5.6. *Let p be odd, let $w = ap + l$, $n = pw$ and $r = kp - m$ for some $a, k \in \mathbb{N}$ and $1 \leq m < p$. Moreover, let $1 \leq k \leq l < p$, and suppose further that $k \neq l$ in case $m = 1$. Then the vertices of D_r are the Sylow p -subgroups of \mathfrak{S}_n .*

Proof. With the above assumptions we have $n - r - 2 = ap^2 + (l - k)p + m - 2$. Let further $P \leq P_n$ be a vertex of D_r . Then

$$P_{ap^2+(l-k)p+m-2} \times P_{kp-m} \sim_{\mathfrak{S}_n} Q < P \leq P_n = P_{ap^2+lp} = P_{ap^2} \times (P_p)^l,$$

by Proposition 5.2. For $m \geq 2$, this implies

$$Q \sim_{\mathfrak{S}_n} P_{ap^2} \times (P_p)^{l-k} \times (P_p)^{k-1} \sim_{\mathfrak{S}_n} P_{ap^2} \times (P_p)^{l-1},$$

so that we then get $P = P_n$. If $m = 1$ we have

$$Q \sim_{\mathfrak{S}_n} P_{ap^2} \times (P_p)^{l-k-1} \times (P_p)^{k-1} \sim_{\mathfrak{S}_n} P_{ap^2} \times (P_p)^{l-2}.$$

By Remark 2.2, any subgroup of P_n which is in \mathfrak{S}_n conjugate to P_{p^k} for some $k > 1$ has to be a subgroup of $P_{ap^2} \leq P_n$. Hence $P_{ap^2} \leq Q < P$ and consequently $P_n = P_{ap^2} \times (P_p)^l \subseteq PC_{\mathfrak{S}_n}(P) \subseteq P$, by Lemma 3.3. Thus again we obtain $P = P_n$. \square

Remark 5.7. In the situation of Corollary 5.5, there is a positive answer to Conjecture (1.7) (a) in [11]. So far, the case $n = pw$ with $w \equiv 1 \pmod{p}$ and $r = p - 1$ still remains open. Using the computer algebra system MAGMA [9], [10] and algorithms which had been developed in [15], we partially computed the vertices of the simple $F\mathfrak{S}_n$ -modules D_r in characteristic 3 for “small” n . The table below contains the results for $n \leq 27$, where the entries denote the vertices of the $F\mathfrak{S}_n$ -module D_r for appropriate n and r . The gray entries correspond to modules whose dimensions are not divisible by 3, and the framed ones have been deduced by computer calculations. The remaining entries could be obtained by applying Theorem 5.3 and Proposition 5.6, respectively.

3	6	9	12	15	18	21	24	27	n/r
P_3	P_6	P_9	P_{12}	P_{15}	P_{18}	P_{21}	P_{24}	P_{27}	0
P_3	P_6	P_9	P_{12}	P_{15}	P_{18}	P_{21}	P_{24}	P_{27}	1
	P_6	P_9	P_{12}	P_{15}	P_{18}	P_{21}	P_{24}	P_{27}	2
	P_6	P_9	P_{12}	P_{15}	P_{18}	P_{21}	P_{24}	P_{27}	3
	P_6	P_9	P_{12}	P_{15}	P_{18}	P_{21}	P_{24}	P_{27}	4
		P_9	P_{12}	P_{15}	P_{18}			P_{27}	5
		P_9	P_{12}	P_{15}	P_{18}			P_{27}	6
		P_9	P_{12}	P_{15}	P_{18}			P_{27}	7
			P_{12}	P_{15}					8
			P_{12}	P_{15}	P_{18}	P_{21}	P_{24}	P_{27}	9
			P_{12}	P_{15}	P_{18}	P_{21}	P_{24}	P_{27}	10
				P_{15}	P_{18}		P_{24}	P_{27}	11
				P_{15}	P_{18}		P_{24}	P_{27}	12
				P_{15}	P_{18}		P_{24}	P_{27}	13
					P_{18}			P_{27}	14
					P_{18}	P_{21}		P_{27}	15
					P_{18}	P_{21}		P_{27}	16
						P_{21}			17
						P_{21}	P_{24}	P_{27}	18
						P_{21}	P_{24}	P_{27}	19
							P_{24}	P_{27}	20
							P_{24}	P_{27}	21
							P_{24}	P_{27}	22
								P_{27}	23
								P_{27}	24
								P_{27}	25

As a generalization of Conjecture (1.7) (a) in [11], we state the following conjecture which is motivated by our results above as well as by the computational data.

Conjecture 5.8. Let p be odd, let $n \equiv 0 \pmod{p}$ and $r \leq n - 2$. Then the vertices of D_r are precisely the Sylow p -subgroups of \mathfrak{S}_n .

For $p = 3$ and $n \leq 15$ with $n \equiv 0 \pmod{3}$, we also computed the sources of the $F\mathfrak{S}_n$ -modules D_r . The results suggest the following question, generalizing Conjecture (1.7) (b) in [11].

Question 5.9. Let p be odd, let $n \equiv 0 \pmod{p}$ and $r \leq n - 2$. Does then the following hold in general?

- (i) If n is even and $r = (n - 2)/2$ then $\text{Res}_{P_n}^{\mathfrak{S}_n}(D_r)$ is a direct sum of two indecomposable modules of dimension $\dim(D_r)/2$ both of which are sources of D_r .
- (ii) Otherwise $\text{Res}_{P_n}^{\mathfrak{S}_n}(D_r)$ is indecomposable and a source of D_r .

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