

**CENTRAL IDEALS AND CARTAN INVARIANTS
OF SYMMETRIC ALGEBRAS**

László Héthelyi
Department of Algebra
Budapest University of Technology and Economics
H-1521 Budapest
Műegyetem Rkp. 3-9
Hungary
`hethelyi@math.bme.hu`

and

Erzsébet Horváth
Department of Algebra
Budapest University of Technology and Economics
H-1521 Budapest
Műegyetem Rkp. 3-9
Hungary
`he@math.bme.hu`

and

Burkhard Külshammer
Mathematisches Institut
Friedrich-Schiller-Universität
07740 Jena
Germany
`kuelshammer@uni-jena.de`

and

John Murray
Mathematics Department
National University of Ireland
Maynooth, C. Kildare
Ireland
`John.Murray@maths.may.ie`

Abstract. In this paper, we investigate certain ideals in the center of a symmetric algebra A over an algebraically closed field of characteristic $p > 0$. These ideals include the Higman ideal and the Reynolds ideal. They are closely related to the p -power map on A . We generalize some results concerning these ideals from group algebras to symmetric algebras, and we obtain some new results as well. In case $p = 2$, these ideals detect odd diagonal entries in the Cartan matrix of A . In a sequel to this paper, we will apply our results to group algebras.

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1. Introduction

Let A be a symmetric algebra over an algebraically closed field F of characteristic $p > 0$, with symmetrizing bilinear form $(\cdot | \cdot)$. In this paper we investigate the following chain of ideals of the center $\mathbf{Z}A$ of A :

$$\mathbf{Z}A \supseteq \mathbf{T}_1 A^\perp \supseteq \mathbf{T}_2 A^\perp \supseteq \dots \supseteq \mathbf{R}A \supseteq \mathbf{H}A \supseteq \mathbf{Z}_0 A \supseteq 0;$$

here $\mathbf{Z}_0A := \sum_B \mathbf{Z}B$ where B ranges over the set of blocks of A which are simple F -algebras. Thus \mathbf{Z}_0A is a direct product of copies of F , one for each simple block B of A . Furthermore, $\mathbf{H}A$ denotes the *Higman ideal* of A , defined as the image of the *trace map*

$$\tau : A \longrightarrow A, \quad x \longmapsto \sum_{i=1}^n b_i x a_i;$$

here a_1, \dots, a_n and b_1, \dots, b_n are a pair of dual bases of A . Moreover, $\mathbf{R}A$ is the *Reynolds ideal* of A , defined as the intersection of the socle $\mathbf{S}A$ of A and the center $\mathbf{Z}A$ of A . The ideals \mathbf{T}_nA^\perp ($n \in \mathbb{N}$) were introduced in [6 II]; they can be viewed as generalizations of the Reynolds ideal. In fact, $\mathbf{R}A$ is their intersection. These ideals are defined in terms of the p -power map $A \longrightarrow A$, $x \longmapsto x^p$, and the bilinear form $(\cdot | \cdot)$. The precise definition will be given below. Motivated by the special case of group algebras [8,9], we show that

$$\mathbf{Z}_0A \subseteq (\mathbf{T}_1A^\perp)^2 \subseteq \mathbf{H}A,$$

so that $(\mathbf{T}_1A^\perp)^2$ fits nicely into the chain of ideals above. When p is odd then

$$(\mathbf{T}_1A^\perp)^2 = \mathbf{Z}_0A.$$

The case $p = 2$ behaves differently and turns out to have some interesting special features. We show that, in this case,

$$(\mathbf{T}_1A^\perp)^3 = (\mathbf{T}_1A^\perp)(\mathbf{T}_2A^\perp) = \mathbf{Z}_0A,$$

but that $(\mathbf{T}_1A^\perp)^2 \neq \mathbf{Z}_0A$ in general. We prove that, in case $p = 2$, the mysterious ideal $(\mathbf{T}_1A^\perp)^2$ is a principal ideal of $\mathbf{Z}A$. It is generated by the element $\zeta_1(1)^2$ where $\zeta_1 : \mathbf{Z}A \longrightarrow \mathbf{Z}A$ is a certain natural semilinear map related to the p -power map. The map ζ_1 was first defined in [6 IV].

Moreover, in case $p = 2$, the dimension of $(\mathbf{T}_1A^\perp)^2$ is the number of blocks B of A with the property that the Cartan matrix $C_B = (c_{ij})$ of B contains an odd diagonal entry c_{ii} . A primitive idempotent e in A satisfies $e\zeta_1(1)^2 \neq 0$ if and only if the dimension of eAe is odd.

At the end of the paper, we investigate the behaviour of the ideals \mathbf{T}_nA^\perp under Morita and derived equivalences, and we dualize some of the results obtained in the previous sections. In a sequel [2] to this paper, we will apply our results to group algebras of finite groups. We will see that a finite group G contains a real conjugacy class of 2-defect zero if and only if the Cartan matrix of G in characteristic 2 contains an odd diagonal entry. We will also prove a number of related facts.

2. The Reynolds ideal and its generalizations

In the following, let F be an algebraically closed field of characteristic $p > 0$, and let A be a symmetric F -algebra with symmetrizing bilinear form $(\cdot | \cdot)$. Thus A is a finite-dimensional associative unitary F -algebra, and $(\cdot | \cdot)$ is a non-degenerate symmetric bilinear form on A which is associative, in the sense that $(ab|c) = (a|bc)$ for $a, b, c \in A$. We denote the center of A by $\mathbf{Z}A$, the Jacobson radical of A by $\mathbf{J}A$, the socle of A by $\mathbf{S}A$ and the commutator subspace of A by $\mathbf{K}A$. Thus $\mathbf{K}A$ is the F -subspace of A spanned by all commutators $ab - ba$ ($a, b \in A$). For $n \in \mathbb{N}$,

$$\mathbf{T}_nA := \{x \in A : x^{p^n} \in \mathbf{K}A\}$$

is a $\mathbf{Z}A$ -submodule of A , so that

$$\mathbf{K}A = \mathbf{T}_0A \subseteq \mathbf{T}_1A \subseteq \mathbf{T}_2A \subseteq \dots$$

and

$$\sum_{n=0}^{\infty} \mathbf{T}_nA = \mathbf{J}A + \mathbf{K}A$$

(cf. [7]). For any F -subspace X of A , we set

$$X^\perp := \{y \in A : (x|y) = 0 \text{ for } x \in X\}.$$

Then

$$\mathbf{Z}A = \mathbf{K}A^\perp = \mathbf{T}_0A^\perp \supseteq \mathbf{T}_1A^\perp \supseteq \mathbf{T}_2A^\perp \supseteq \dots$$

is a chain of ideals of $\mathbf{Z}A$ such that

$$\bigcap_{n=0}^{\infty} \mathbf{T}_nA^\perp = \mathbf{S}A \cap \mathbf{Z}A.$$

We call $\mathbf{R}A := \mathbf{S}A \cap \mathbf{Z}A$ the *Reynolds ideal* of $\mathbf{Z}A$, in analogy to the terminology used for group algebras. For $n \in \mathbb{N}$ and $z \in \mathbf{Z}A$, there is a unique element $\zeta_n(z) \in \mathbf{Z}A$ such that

$$(\zeta_n(z)|x)^{p^n} = (z|x^{p^n}) \text{ for } x \in A.$$

This defines a map $\zeta_n = \zeta_n^A : \mathbf{Z}A \longrightarrow \mathbf{Z}A$ with the following properties:

Lemma 2.1. *Let $m, n \in \mathbb{N}$, and let $y, z \in \mathbf{Z}A$. Then the following holds:*

- (i) $\zeta_n(y + z) = \zeta_n(y) + \zeta_n(z)$ and $\zeta_n(y)z = \zeta_n(yz^{p^n})$.
- (ii) $\zeta_m \circ \zeta_n = \zeta_{m+n}$.
- (iii) $\text{Im}(\zeta_n) = \mathbf{T}_nA^\perp$.
- (iv) $\zeta_n^A(z)e = \zeta_n^{eAe}(ze)$ for every idempotent e in A .

Proof. (i), (ii) and (iii) are proved in [7, (44)-(47)].

(iv) Recall that eAe is a symmetric F -algebra; a corresponding symmetric bilinear form is obtained by restricting $(\cdot | \cdot)$ to eAe . Note that $ez = z e \in e\mathbf{Z}Ae \subseteq \mathbf{Z}(eAe)$ and that, similarly, $\zeta_n^A(z)e \in \mathbf{Z}(eAe)$. Moreover, for $x \in eAe$, we have

$$\begin{aligned} (\zeta_n^A(z)e|x)^{p^n} &= (\zeta_n^A(z)|ex)^{p^n} = (\zeta_n^A(z)|x)^{p^n} = (z|x^{p^n}) \\ &= (z|ex^{p^n}) = (ze|x^{p^n}) = (\zeta_n^{eAe}(ze)|x)^{p^n}, \end{aligned}$$

and the result follows.

We apply these properties in order to prove:

Lemma 2.2. *Let $m, n \in \mathbb{N}$. Then*

$$(\mathbf{T}_mA^\perp)(\mathbf{T}_nA^\perp) \subseteq \zeta_{m+n}((\mathbf{T}_nA^\perp)^{p^n(p^m-1)}) \subseteq \mathbf{T}_{m+n}A^\perp.$$

Proof. Let $y, z \in \mathbf{Z}A$. Then Lemma 2.1 implies that

$$\begin{aligned} \zeta_m(y)\zeta_n(z) &= \zeta_m(y\zeta_n(z)^{p^m}) = \zeta_m(\zeta_n(y^{p^n}z)\zeta_n(z)^{p^m-1}) \\ &= \zeta_m(\zeta_n(y^{p^n}z\zeta_n(z)^{p^n(p^m-1)})) \in \zeta_{m+n}((\mathbf{T}_nA^\perp)^{p^n(p^m-1)}). \end{aligned}$$

Thus the result follows from Lemma 2.1 (iii).

Let B_1, \dots, B_r denote the blocks of A , so that $A = B_1 \oplus \dots \oplus B_r$. Each B_i is itself a symmetric F -algebra. If a block B_i is a simple F -algebra then $B_i \cong \text{Mat}(d_i, F)$ for a positive integer d_i , and thus $\mathbf{Z}B_i \cong F$. We set

$$\mathbf{Z}_0A := \sum_i \mathbf{Z}B_i$$

where the sum ranges over all $i \in \{1, \dots, r\}$ such that B_i is a simple F -algebra. Then \mathbf{Z}_0A is an ideal of $\mathbf{Z}A$ and an F -algebra which is isomorphic to a direct sum of copies of F . Its dimension is the number of simple blocks of A . We exploit Lemma 2.2 in order to prove:

Theorem 2.3. (i) $(\mathbf{T}_1A^\perp)^2 \subseteq \mathbf{R}A$.
(ii) $(\mathbf{T}_1A^\perp)(\mathbf{T}_2A^\perp) = (\mathbf{T}_1A^\perp)^3 = \mathbf{Z}_0A$.
(iii) If p is odd then $(\mathbf{T}_1A^\perp)^2 = \mathbf{Z}_0A$.

Proof. (i) Lemma 2.2 implies

$$(\mathbf{T}_1A^\perp)^2 \subseteq \zeta_2((\mathbf{T}_1A^\perp)^{p(p-1)}) \subseteq \zeta_2((\mathbf{T}_1A^\perp)^2).$$

Iteration yields

$$(\mathbf{T}_1A^\perp)^2 \subseteq \zeta_2(\zeta_2((\mathbf{T}_1A^\perp)^2)) = \zeta_4((\mathbf{T}_1A^\perp)^2) \subseteq \zeta_6((\mathbf{T}_1A^\perp)^2) \subseteq \dots$$

Thus

$$(\mathbf{T}_1A^\perp)^2 \subseteq \bigcap_{n=0}^{\infty} \text{Im}(\zeta_{2n}) = \bigcap_{n=0}^{\infty} \mathbf{T}_{2n}A^\perp = \mathbf{S}A \cap \mathbf{Z}A = \mathbf{R}A,$$

by Lemma 2.1 (iii).

(ii) It is easy to see that $\mathbf{T}_nA = \mathbf{T}_nB_1 \oplus \dots \oplus \mathbf{T}_nB_r$ and $\mathbf{T}_nA^\perp = \mathbf{T}_nB_1^\perp \oplus \dots \oplus \mathbf{T}_nB_r^\perp$ for $n \in \mathbb{N}$ where $\mathbf{T}_nB_i^\perp = \{x \in B_i : (x|\mathbf{T}_nB_i) = 0\}$ for $i = 1, \dots, r$. So we may assume that A itself is a block.

If A is simple then $\mathbf{J}A = 0$, so $\mathbf{T}_nA = \mathbf{K}A$ and $\mathbf{T}_nA^\perp = \mathbf{Z}A$ for all $n \in \mathbb{N}$. Hence

$$\mathbf{Z}A = (\mathbf{T}_1A^\perp)(\mathbf{T}_2A^\perp) = (\mathbf{T}_1A^\perp)^3$$

in this case.

Now suppose that A is non-simple. Then $\mathbf{J}A \neq 0$. So $\mathbf{Z}A \neq \mathbf{R}A$. It follows that $\mathbf{J}A + \mathbf{K}A \neq \mathbf{K}A$, whence $\mathbf{J}A$ is not contained in $\mathbf{K}A$. So $\mathbf{T}_1A \neq \mathbf{K}A$. This means that \mathbf{T}_1A^\perp is a proper ideal of $\mathbf{Z}A$. Since $\mathbf{Z}A$ is a local F -algebra this implies that $\mathbf{T}_1A^\perp \subseteq \mathbf{J}A \subseteq \mathbf{Z}A$. Thus we may conclude, using (i), that $(\mathbf{T}_1A^\perp)^3 \subseteq (\mathbf{R}A)(\mathbf{J}A) = 0$. Hence Lemma 2.2 yields

$$(\mathbf{T}_1A^\perp)(\mathbf{T}_2A^\perp) \subseteq \zeta_3((\mathbf{T}_2A^\perp)^{p^2(p-1)}) \subseteq \zeta_3((\mathbf{T}_1A^\perp)^3) = \zeta_3(0) = 0.$$

(iii) Suppose that p is odd. As in the proof of (ii), we may assume that A is a block, and that A is non-simple. Then Lemma 2.2 and (ii) imply that

$$(\mathbf{T}_1A^\perp)^2 \subseteq \zeta_2((\mathbf{T}_1A^\perp)^{p(p-1)}) \subseteq \zeta_2((\mathbf{T}_1A^\perp)^3) = \zeta_2(0) = 0,$$

and the result is proved.

Theorem 2.3 extends [9, Theorem 9] from group algebras to symmetric algebras. We will later improve on part (i). But first we note the following consequence.

Corollary 2.4. Suppose that A is a block, and denote the central character of A by $\omega : \mathbf{Z}A \rightarrow F$. Moreover, let $m, n \in \mathbb{N}$ with $m \neq 0 \neq n$, and let $x, y \in \mathbf{Z}A$. Then

$$\zeta_m(x)\zeta_n(y) = \omega(x)^{p^{-m}}\omega(y)^{p^{-n}}\zeta_m(1)\zeta_n(1).$$

In particular, we have

$$(\mathbf{T}_mA^\perp)(\mathbf{T}_nA^\perp) = F\zeta_m(1)\zeta_n(1),$$

so that $\dim(\mathbf{T}_mA^\perp)(\mathbf{T}_nA^\perp) \leq 1$.

Proof. Theorem 2.3 (i) implies that $\zeta_m(x)^{p^n} \in \mathbf{R}A \subseteq \mathbf{S}A$. Thus

$$\zeta_m(x)^{p^n}y = \omega(y)\zeta_m(x)^{p^n}.$$

Similarly, we have $x\zeta_n(1)^{p^m} = \omega(x)\zeta_n(1)^{p^m}$. So we conclude that

$$\begin{aligned}\zeta_m(x)\zeta_n(y) &= \zeta_n(\zeta_m(x)^{p^n}y) = \zeta_n(\omega(y)\zeta_m(x)^{p^n}) = \omega(y)^{p^{-n}}\zeta_m(x)\zeta_n(1) \\ &= \omega(y)^{p^{-n}}\zeta_m(x\zeta_n(1)^{p^m}) = \omega(y)^{p^{-n}}\zeta_m(\omega(x)\zeta_n(1)^{p^m}) = \omega(y)^{p^{-n}}\omega(x)^{p^{-m}}\zeta_m(1)\zeta_n(1).\end{aligned}$$

The remaining assertions follow from Lemma 2.1 (iii).

We can generalize part of Corollary 2.4 in the following way.

Proposition 2.5. *Let $m, n \in \mathbb{N}$ with $m \neq 0 \neq n$. Then*

$$(\mathbf{T}_m A^\perp)(\mathbf{T}_n A^\perp) = \mathbf{Z}A \cdot \zeta_m(1)\zeta_n(1)$$

is a principal ideal of $\mathbf{Z}A$. If p is odd, or if $m + n > 2$, then the dimension of $(\mathbf{T}_m A^\perp)(\mathbf{T}_n A^\perp)$ equals the number of simple blocks of A and in particular does not depend on $m + n$.

Proof. It is easy to see that we may assume that A is a block. In this case the assertion follows from Corollary 2.4 and Theorem 2.3.

In the next two sections, we will handle the remaining case $p = 2$ and $m = n = 1$. Here we just illustrate this exceptional case by an example.

Let G be a finite group. Then the group algebra FG is a symmetric F -algebra; a symmetrizing bilinear form on FG satisfies

$$(g|h) = \begin{cases} 1, & \text{if } gh = 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $g, h \in G$. An element $g \in G$ is called *real* if g is conjugate to its inverse g^{-1} , and g is said to be of *p -defect zero* if $|\mathbf{C}_G(g)|$ is not divisible by p . We denote the set of all real elements of 2-defect zero in G by R_G . For a subset X of G , we set

$$X^+ := \sum_{x \in X} x \in FG.$$

It was proved in [8, Proposition 4.1] that $R_G^+ = \zeta_1(1)^2 \in (\mathbf{T}_1 FG^\perp)^2$, in case $p = 2$.

Example 2.6. Let $p = 2$, and suppose that G is the symmetric group S_4 of degree 4. Then FG has no simple blocks; in fact, FG has just one block, the principal one. Thus $\mathbf{Z}_0 FG = 0$. On the other hand, R_G is precisely the set of all 3-cycles in S_4 . Thus $0 \neq R_G^+ \in (\mathbf{T}_1 FG^\perp)^2$. (In fact, $(\mathbf{T}_1 FG^\perp)^2$ is one-dimensional, by Corollary 2.4.) This example shows that $(\mathbf{T}_1 A^\perp)^2 \neq \mathbf{Z}_0 A$, in general.

3. Odd Cartan invariants

Let F be an algebraically closed field of characteristic $p = 2$, and let A be a symmetric F -algebra with symmetrizing bilinear form $(\cdot | \cdot)$. In this section, we will prove some remarkable properties of the ideal $(\mathbf{T}_1 A^\perp)^2$ of $\mathbf{Z}A$. We start by recalling some known facts concerning symmetric bilinear forms over F .

Lemma 3.1. *Let V be a finite-dimensional vector space over F , and let $\langle \cdot | \cdot \rangle$ be a non-degenerate symmetric bilinear form on V . Then either $\langle \cdot | \cdot \rangle$ is symplectic (i.e. $\langle v|v \rangle = 0$ for every $v \in V$), or there exists an orthonormal basis v_1, \dots, v_n of V (i.e. $\langle v_i|v_j \rangle = \delta_{ij}$ for $i, j = 1, \dots, n$).*

Proof. This can be found in [4, Hauptsatz V.3.5], for example.

If $\langle \cdot | \cdot \rangle$ is symplectic then there exists a symplectic basis $v_1, \dots, v_m, v_{m+1}, \dots, v_{2m}$ of V , i.e.

$$\begin{aligned}\langle v_i|v_{m+i} \rangle &= \langle v_{m+i}|v_i \rangle = 1 & \text{for } i = 1, \dots, m, \\ \langle v_i|v_j \rangle &= 0 & \text{otherwise,}\end{aligned}$$

(cf. [4, Hauptsatz V.4.10]). Thus there exist only two types of non-degenerate symmetric bilinear forms on a finite-dimensional vector space V over F , a symplectic one and a non-symplectic one. In the symplectic case, the dimension of V has to be even.

We now apply Lemma 3.1 to the symmetrizing bilinear form $(\cdot | \cdot)$ on A .

Lemma 3.2.

$$(\zeta_1(1)|\zeta_1(1)) = (\dim A) \cdot 1_F.$$

Proof. By Lemma 3.1, there exists an F -basis

$$a_1, \dots, a_m, a_{m+1}, \dots, a_{2m}, a_{2m+1}, \dots, a_n$$

of A such that

$$\begin{aligned} (a_i|a_{m+i}) &= (a_{m+i}|a_i) = 1 && \text{for } i = 1, \dots, m, \\ (a_i|a_i) &= 1 && \text{for } i = 2m+1, \dots, n, \\ (a_i|a_j) &= 0 && \text{otherwise,} \end{aligned}$$

(and either $n = 2m$ or $m = 0$). Then the dual basis b_1, \dots, b_n of a_1, \dots, a_n is given by

$$a_{m+1}, \dots, a_{2m}, a_1, \dots, a_m, a_{2m+1}, \dots, a_n.$$

Thus $(\zeta_1(1)|a_i)^2 = (1|a_i^2) = (a_i|a_i) = (a_i|a_i)^2$ for $i = 1, \dots, n$, so

$$\zeta_1(1) = \sum_{i=1}^n (\zeta_1(1)|a_i) b_i = \sum_{i=1}^n (a_i|a_i) b_i = \sum_{i=2m+1}^n a_i$$

and

$$(\zeta_1(1)|\zeta_1(1)) = \sum_{i,j=2m+1}^n (a_i|a_j) = \sum_{i=2m+1}^n (a_i|a_i) = (n-2m) \cdot 1_F = n \cdot 1_F = (\dim A) \cdot 1_F,$$

and the result is proved.

The next statement holds in arbitrary characteristic. It is essentially taken from [11, Corollary (1.G)].

Lemma 3.3. *Let e be a primitive idempotent in A , and let $r \in \mathbf{R}A$. Then $er = 0$ if and only if $(e|r) = 0$.*

Proof. If $er = 0$ then $0 = (er|1) = (e|r)$. Conversely, if $(e|r) = 0$ then

$$(eAe|ere) = (eAe|r) = (Fe + \mathbf{J}(eAe)|r) \subseteq F(e|r) + (\mathbf{J}A \cdot r|1) = 0.$$

Thus $0 = ere = er$ since the restriction of $(\cdot | \cdot)$ to eAe is non-degenerate.

Now we choose representatives $a_1 = e_1, \dots, a_l = e_l$ for the conjugacy classes of primitive idempotents in A . (This means that Ae_1, \dots, Ae_l are representatives for the isomorphism classes of indecomposable projective left A -modules.) Moreover, we let a_{l+1}, \dots, a_n denote an F -basis of $\mathbf{J}A + \mathbf{K}A$. Then a_1, \dots, a_n form an F -basis of A .

Let b_1, \dots, b_n denote the dual basis of a_1, \dots, a_n . Then $r_1 := b_1, \dots, r_l := b_l$ are contained in $(\mathbf{J}A + \mathbf{K}A)^\perp = \mathbf{S}A \cap \mathbf{Z}A = \mathbf{R}A$, so they form an F -basis of $\mathbf{R}A$. Moreover, Lemma 3.3 implies that $e_i r_j = 0$ for $i \neq j$ and $e_i r_i \neq 0$ for $i = 1, \dots, l$.

Lemma 3.4. *With e_1, \dots, e_l as above, we have $\zeta_1(1)^2 = \sum_{i=1}^l (\dim e_i A e_i) \cdot r_i$ and $e_i \zeta_1(1)^2 = (\dim e_i A e_i) \cdot e_i r_i$ for $i = 1, \dots, l$.*

Proof. Lemma 2.1 (iii) and Theorem 2.3 (i) imply that $\zeta_1(1)^2 \in (\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{R}A$. By making use of Lemma 2.1 (iv) and Lemma 3.2, we obtain

$$\begin{aligned}\zeta_1(1)^2 &= \sum_{i=1}^l (\zeta_1(1)^2 | e_i) r_i = \sum_{i=1}^l (\zeta_1(1) e_i | \zeta_1(1) e_i) r_i \\ &= \sum_{i=1}^l (\zeta_1^{e_i A e_i}(e_i) | \zeta_1^{e_i A e_i}(e_i)) r_i = \sum_{i=1}^l (\dim e_i A e_i) \cdot r_i.\end{aligned}$$

Since $e_i r_j = 0$ for $i \neq j$ the result follows.

The next theorem is the main result of this section.

Theorem 3.5. *For A a symmetric algebra over an algebraically closed field F of characteristic 2 and for e a primitive idempotent in A , the following assertions are equivalent:*

- (1) $\dim e A e$ is even.
- (2) $e \zeta_1(1)^2 = 0$.
- (3) $(e | \zeta_1(1)^2) = 0$.

Proof. We may assume that $e = e_i$ for some $i \in \{1, \dots, l\}$. Then $e_i \zeta_1(1)^2 = (\dim e_i A e_i) \cdot e_i r_i$ with $e_i r_i \neq 0$, by Lemma 3.4. This shows that (1) and (2) are equivalent. Since $\zeta_1(1)^2 \in \mathbf{R}A$, Lemma 3.3 implies that (2) and (3) are equivalent.

The Cartan matrix $C := (c_{ij})_{i,j=1}^l$ of A is defined by

$$c_{ij} := \dim e_i A e_j \quad \text{for } i, j = 1, \dots, l.$$

Thus C is a symmetric matrix with non-negative integer coefficients, the Cartan invariants of A . Hence Theorem 3.5 has the following consequence.

Corollary 3.6. *With the notation for the Cartan matrix of A as above, $\zeta_1(1)^2 \neq 0$ if and only if c_{ii} is odd for some i . More precisely, for a block B of A , we have $\zeta_1(1)^2 1_B \neq 0$ if and only if the Cartan matrix of B contains an odd diagonal entry.*

In order to illustrate Corollary 3.6 recall that, by Example 2.6, the group algebra FG , for $G = S_4$, satisfies $\zeta_1(1)^2 = R_G^+ \neq 0$. Thus the Cartan matrix of FG contains an odd diagonal entry, by Corollary 3.6. Indeed, the Cartan matrix of FG is

$$C := \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix},$$

as is well-known. More substantial examples will be presented in [2].

It may be of interest to note that the existence of odd diagonal Cartan invariants in characteristic 2 is invariant under derived equivalences (cf. [5]).

Proposition 3.7. *Let A' be a symmetric F -algebra which is derived equivalent to A . Then the Cartan matrix of A' contains an odd diagonal entry if and only if the Cartan matrix of A does.*

Proof. It is known that the Cartan matrices $C = (c_{ij})_{i,j=1}^l$ of A and $C' = (c'_{ij})_{i,j=1}^l$ of A' have the same format, and that they are related by an equation

$$C' = Q \cdot C \cdot Q^\top$$

where $Q = (q_{ij})_{i,j=1}^l$ is an integral matrix with determinant ± 1 (cf. [5]). Thus

$$c'_{ii} = \sum_{j,k=1}^l q_{ij} q_{ik} c_{jk} \equiv \sum_{j=1}^l q_{ij}^2 c_{jj} \pmod{2}$$

for $i = 1, \dots, l$. If c'_{ii} is odd then c_{jj} has to be odd for some $j \in \{1, \dots, l\}$ (and conversely).

4. The Higman ideal

Let F be an algebraically closed field, and let A be a symmetric F -algebra with symmetrizing bilinear form $(\cdot | \cdot)$. Moreover, let a_1, \dots, a_n and b_1, \dots, b_n denote a pair of dual bases of A . In the following, the F -linear map

$$\tau : A \longrightarrow A, \quad x \longmapsto \sum_{i=1}^n b_i x a_i,$$

will be of interest (cf. [3, §66]). We record the following properties of this *trace map* τ :

Lemma 4.1. (i) τ is independent of the choice of dual bases.

(ii) τ is self-adjoint with respect to $(\cdot | \cdot)$.

(iii) $\text{Im}(\tau) \subseteq \mathbf{SA} \cap \mathbf{ZA} = \mathbf{RA}$ and $\mathbf{JA} + \mathbf{KA} \subseteq \text{Ker}(\tau)$.

Proof. (i) Let a'_1, \dots, a'_n and b'_1, \dots, b'_n be another pair of dual bases of A . Then $b'_i = \sum_{j=1}^n (a_j | b'_i) b_j$ and $a_i = \sum_{j=1}^n (a_i | b'_j) a'_j$ for $i = 1, \dots, n$. Thus

$$\sum_{i=1}^n b'_i x a'_i = \sum_{i,j=1}^n (a_j | b'_i) b_j x a'_i = \sum_{j=1}^n b_j x \sum_{i=1}^n (a_j | b'_i) a'_i = \sum_{j=1}^n b_j x a_j$$

for $x \in A$.

(ii) Let $x, y \in A$. Then, by (i), we get

$$(\tau(x) | y) = \sum_{i=1}^n (b_i x a_i | y) = \sum_{i=1}^n (x | a_i y b_i) = (x | \tau(y)).$$

(iii) Let $x, y \in A$. Then

$$\begin{aligned} \tau(x)y &= \sum_{i=1}^n b_i x a_i y = \sum_{i,j=1}^n b_i x (a_i y | b_j) a_j = \sum_{i,j=1}^n (a_i | y b_j) b_i x a_j \\ &= \sum_{j=1}^n y b_j x a_j = y \tau(x). \end{aligned}$$

Hence $\text{Im}(\tau) \subseteq \mathbf{ZA}$. In order to prove $\text{Im}(\tau) \subseteq \mathbf{SA}$, we choose a_1, \dots, a_n appropriately. Indeed, we may assume that $a_1 + \mathbf{JA}, \dots, a_r + \mathbf{JA}$ form an F -basis of A/\mathbf{JA} , that $a_{r+1} + (\mathbf{JA})^2, \dots, a_s + (\mathbf{JA})^2$ form an F -basis of $(\mathbf{JA})/(\mathbf{JA})^2$, that $a_{s+1} + (\mathbf{JA})^3, \dots, a_t + (\mathbf{JA})^3$ form an F -basis of $(\mathbf{JA})^2/(\mathbf{JA})^3$, etc. Then b_1, \dots, b_r are contained in $(\mathbf{JA})^\perp$, b_{r+1}, \dots, b_s are contained in $((\mathbf{JA})^2)^\perp$, b_{s+1}, \dots, b_t are contained in $((\mathbf{JA})^3)^\perp$, etc.

Now let $x \in A$ and $y \in \mathbf{JA}$. Then $b_i x a_i y \in (\mathbf{JA})^\perp \cdot A \cdot A \cdot (\mathbf{JA}) = 0$ for $i = 1, \dots, r$, $b_i x a_i y \in ((\mathbf{JA})^2)^\perp \cdot A \cdot (\mathbf{JA}) \cdot (\mathbf{JA}) = 0$ for $i = r+1, \dots, s$, $b_i x a_i y \in ((\mathbf{JA})^3)^\perp \cdot A \cdot (\mathbf{JA})^2 \cdot (\mathbf{JA}) = 0$ for $i = s+1, \dots, t$, etc. We see that $\tau(x)y = 0$, so $\text{Im}(\tau) \subseteq \mathbf{SA}$.

Since τ is self-adjoint (i.e. $\tau^* = \tau$) we conclude that

$$\text{Ker}(\tau) = \text{Ker}(\tau^*) = \text{Im}(\tau)^\perp \supseteq (\mathbf{SA} \cap \mathbf{ZA})^\perp = \mathbf{JA} + \mathbf{KA}.$$

Thus $\mathbf{HA} := \text{Im}(\tau)$ is an ideal of \mathbf{ZA} contained in \mathbf{RA} , called the *Higman ideal* of \mathbf{ZA} . By Lemma 4.1, it is independent of the choice of dual bases. In the following, we write

$$1_A = e_1 + \dots + e_m$$

with pairwise orthogonal primitive idempotents e_1, \dots, e_m of A .

Lemma 4.2. *We have $(\tau(e_i)|e_j) = (\dim e_i A e_j) \cdot 1_F$ for $i, j = 1, \dots, m$.*

Proof. We consider the decomposition $A = \bigoplus_{i,j=1}^m e_i A e_j$. For $i, j = 1, \dots, m$, let X_{ij} be an F -basis of $e_i A e_j$. Then $X := \bigcup_{i,j=1}^m X_{ij}$ is an F -basis of A . We denote the dual basis of X by X^* . For $x \in X$, there is a unique $x^* \in X^*$ such that $(x|x^*) = 1$. Then the map $X \rightarrow X^*$, $x \mapsto x^*$, is a bijection. Moreover, for $i, j = 1, \dots, m$, $X_{ij}^* := \{x^* : x \in X_{ij}\}$ is an F -basis of $e_j A e_i$. Thus

$$\tau(e_i)e_j = e_j \tau(e_i)e_j = \sum_{x \in X} e_j x^* e_i x e_j = \sum_{x \in X_{ij}} e_j x^* e_i x e_j = \sum_{x \in X_{ij}} x^* x$$

and

$$(\tau(e_i)|e_j) = (\tau(e_i)e_j|1) = \sum_{x \in X_{ij}} (x^* x|1) = \sum_{x \in X_{ij}} (x^*|x) = |X_{ij}| \cdot 1_F = (\dim e_i A e_j) \cdot 1_F,$$

so the result is proved.

We may assume that e_1, \dots, e_m are numbered in such a way that $a_1 := e_1, \dots, a_l := e_l$ represent the conjugacy classes of primitive idempotents in A . We choose an F -basis a_{l+1}, \dots, a_n of $\mathbf{J}A + \mathbf{K}A$, so that a_1, \dots, a_n form an F -basis of A . We denote the dual basis of a_1, \dots, a_n by b_1, \dots, b_n . As above, $r_1 := b_1, \dots, r_l := b_l$ form an F -basis of $\mathbf{R}A = \mathbf{S}A \cap \mathbf{Z}A$.

Lemma 4.3. *We have $\tau(e_i) = \sum_{j=1}^l (\dim e_i A e_j) \cdot r_j$ for $i = 1, \dots, l$.*

Proof. Let $i \in \{1, \dots, l\}$. Then $\tau(e_i) \in \mathbf{H}A \subseteq \mathbf{R}A$, so

$$\tau(e_i) = \sum_{j=1}^l (\tau(e_i)|e_j) r_j = \sum_{j=1}^l (\dim e_i A e_j) \cdot r_j$$

by Lemma 4.2.

In the following, suppose that $\text{char } F = p > 0$. We know from Theorem 2.3 that $(\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{R}A$. We are going to show that, more precisely, $(\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{H}A$. In the proof, we will make use of the following fact.

Lemma 4.4. *Let $C = (c_{ij})$ be a symmetric $n \times n$ -matrix with coefficients in the field \mathbb{F}_2 with two elements. Then its main diagonal $c := (c_{11}, c_{22}, \dots, c_{nn})$, considered as a vector in \mathbb{F}_2^n , is a linear combination of the rows of C .*

Proof. Arguing by induction on n , we may assume that $n > 1$. If $c = 0$ then there is nothing to prove. So we may assume that $c_{ii} = 1$ for some $i \in \{1, \dots, l\}$. Permuting the rows and columns of C , if necessary, we may assume that $c_{11} = 1$. We now perform elementary row operations on C . For $k = 2, \dots, n$, we subtract the first row, multiplied by c_{k1} , from the k -th row. The resulting matrix C' has the entries

$$0, c_{k2} - c_{k1}c_{12}, \dots, c_{kn} - c_{k1}c_{1n}$$

in its k -th row and the entries

$$c_{1k}, c_{2k} - c_{21}c_{1k}, \dots, c_{nk} - c_{n1}c_{1k}$$

in its k -th column. We now remove the first row and the first column from C' and end up with a symmetric $(n-1) \times (n-1)$ -matrix D with diagonal entries

$$c_{kk} - c_{k1}c_{1k} = c_{kk} - c_{1k}^2 = c_{kk} - c_{1k} \quad (k = 2, \dots, n).$$

On the other hand, if we subtract the first row of C from c then we obtain the vector

$$c' := (0, c_{22} - c_{12}, \dots, c_{nn} - c_{1n}).$$

Thus the vector $d := (c_{22} - c_{12}, \dots, c_{nn} - c_{1n})$ coincides with the main diagonal of D . By induction, d is a linear combination of the rows of D , so c is a linear combination of the rows of C .

As Gary McGuire kindly pointed out to us, a different proof of Lemma 4.4 can be found in [1, Proposition 4.6.2]. We apply Lemma 4.4 in the proof of the following result which is a refinement of Theorem 2.3 (i). The special case of group algebras was first proved in [8, Lemma 5.1].

Theorem 4.5. *We always have $(\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{H}A$.*

Proof. If p is odd then, by Theorem 2.3 (iii), we have

$$(\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{Z}_0 A = \sum_B \mathbf{Z}B = \sum_B \mathbf{H}B \subseteq \mathbf{H}A$$

where B ranges over the simple blocks of A ; in fact, if $B = \text{Mat}(d, F)$ for a positive integer d then $\mathbf{H}B = \mathbf{Z}B$.

Thus we may assume that $p = 2$. Then Lemma 2.2 gives us elements $\alpha_1, \dots, \alpha_l$ in the prime field of F such that

$$\sum_{j=1}^l (\dim e_i A e_j) \cdot \alpha_j = (\dim e_i A e_i) \cdot 1_F \quad \text{for } i = 1, \dots, l.$$

Thus Lemma 3.4 and Lemma 4.3 imply that

$$\zeta_1(1)^2 = \sum_{i=1}^l (\dim e_i A e_i) \cdot r_i = \sum_{i,j=1}^l (\dim e_i A e_j) \cdot \alpha_j r_i = \sum_{j=1}^l \alpha_j \tau(e_j) \in \mathbf{H}A.$$

Hence Proposition 2.5 implies that $(\mathbf{T}_1 A^\perp)^2 = \mathbf{Z}A \cdot \zeta_1(1)^2 \subseteq \mathbf{H}A$.

5. Morita invariance

Let F be an algebraically closed field of characteristic $p > 0$, and let A be a symmetric F -algebra. In this section we investigate the behaviour of the ideals $\mathbf{T}_n A^\perp$ of $\mathbf{Z}A$ under Morita equivalences. These results will be used in [2].

Proposition 5.1. *Let e be an idempotent in A such that $AeA = A$. Then the map*

$$f : \mathbf{Z}A \longrightarrow \mathbf{Z}(eAe), \quad z \longmapsto ez = ze,$$

is an isomorphism of F -algebras mapping $\mathbf{T}_n A^\perp$ onto $\mathbf{T}_n(eAe)^\perp$, for $n \in \mathbb{N}$.

Proof. Certainly f is a homomorphism of F -algebras. Let $z \in \mathbf{Z}A$ such that $0 = f(z) = ez$. Then $0 = AezA = AeAz = Az$, so that $z = 0$. Thus f is injective. Since $AeA = A$ the F -algebras A and eAe are Morita equivalent; in particular, their centers are isomorphic. Hence f is an isomorphism of F -algebras. Lemma 2.1 (iv) implies that $f \circ \zeta_n^A = \zeta_n^{eAe} \circ f$, so

$$f(\mathbf{T}_n A^\perp) = f(\zeta_n^A(\mathbf{Z}A)) = \zeta_n^{eAe}(f(\mathbf{Z}A)) = \zeta_n^{eAe}(\mathbf{Z}(eAe)) = \mathbf{T}_n(eAe)^\perp$$

by Lemma 2.1 (iii).

We mention two consequences of Proposition 5.1.

Corollary 5.2. *Let d be a positive integer, and let A_d denote the symmetric F -algebra $\text{Mat}(d, A)$. Then the map*

$$h : \mathbf{Z}A \longrightarrow \mathbf{Z}A_d, \quad z \longmapsto z1_d,$$

is an isomorphism of F -algebras mapping $\mathbf{T}_n A^\perp$ onto $(\mathbf{T}_n A_d)^\perp$, for $n \in \mathbb{N}$.

Proof. We denote the matrix units of A_d by e_{ij} ($i, j = 1, \dots, d$). Then the map

$$f : A \longrightarrow e_{11}A_d e_{11}, \quad a \longmapsto a e_{11},$$

is an isomorphism of F -algebras. This implies that $f(\mathbf{Z}A) = \mathbf{Z}(e_{11}A_d e_{11})$ and $f(\mathbf{T}_n A^\perp) = \mathbf{T}_n(e_{11}A_d e_{11})^\perp$ for $n \in \mathbb{N}$. On the other hand, Proposition 5.1 implies that the map

$$g : \mathbf{Z}A_d \longrightarrow \mathbf{Z}(e_{11}A_d e_{11}), \quad z \longmapsto z e_{11} = e_{11}z,$$

is an isomorphism of F -algebras such that $g((\mathbf{T}_n A_d)^\perp) = \mathbf{T}_n(e_{11}A_d e_{11})^\perp$ for $n \in \mathbb{N}$. Now observe that h is an isomorphism of F -algebras such that $g \circ h$ is the restriction of f to $\mathbf{Z}A$. Thus $h(\mathbf{T}_n A^\perp) = (\mathbf{T}_n A_d)^\perp$ for $n \in \mathbb{N}$.

Corollary 5.3. *Let B be a symmetric F -algebra which is Morita equivalent to A . Then there is an isomorphism of F -algebras $\mathbf{Z}A \longrightarrow \mathbf{Z}B$ mapping $\mathbf{T}_n A^\perp$ onto $\mathbf{T}_n B^\perp$, for $n \in \mathbb{N}$.*

Proof. Let e be an idempotent in A such that eAe is a basic algebra of A , and let f be an idempotent in B such that fBf is a basic algebra of B . Then $AeA = A$ and $BfB = B$. Moreover, eAe and fBf are isomorphic since A and B are Morita equivalent. Thus Proposition 5.1 yields a chain of isomorphisms

$$\mathbf{Z}A \longrightarrow \mathbf{Z}(eAe) \longrightarrow \mathbf{Z}(fBf) \longrightarrow \mathbf{Z}B$$

mapping $\mathbf{T}_n A^\perp$ onto $\mathbf{T}_n B^\perp$, for $n \in \mathbb{N}$.

It would be interesting to know whether Corollary 5.3 extends to symmetric F -algebras which are derived equivalent (cf. [5]).

Question 5.4. Suppose that A and B are derived equivalent symmetric F -algebras. Is there an isomorphism of F -algebras $\mathbf{Z}A \longrightarrow \mathbf{Z}B$ mapping $\mathbf{T}_n A^\perp$ onto $\mathbf{T}_n B^\perp$, for $n \in \mathbb{N}$?

6. Some dual results

Let F be an algebraically closed field of characteristic $p > 0$, and let A be a symmetric F -algebra. For $n \in \mathbb{N}$,

$$\mathbf{T}_n \mathbf{Z}A := \{z \in \mathbf{Z}A : z^{p^n} = 0\}$$

is an ideal of $\mathbf{Z}A$. In this way we obtain an ascending chain of ideals

$$0 = \mathbf{T}_0 \mathbf{Z}A \subseteq \mathbf{T}_1 \mathbf{Z}A \subseteq \mathbf{T}_2 \mathbf{Z}A \subseteq \dots \subseteq \mathbf{J} \mathbf{Z}A \subseteq \mathbf{Z}A$$

of $\mathbf{Z}A$ such that

$$\sum_{n=0}^{\infty} \mathbf{T}_n \mathbf{Z}A = \mathbf{J} \mathbf{Z}A.$$

This ascending chain of ideals turns out to be related to the descending chain of ideals

$$\mathbf{Z}A = \mathbf{T}_0 A^\perp \supseteq \mathbf{T}_1 A^\perp \supseteq \mathbf{T}_2 A^\perp \dots \supseteq \mathbf{R}A \supseteq 0$$

of $\mathbf{Z}A$ considered before.

Proposition 6.1. *Let $n \in \mathbb{N}$. Then $(\mathbf{T}_n A^\perp)(\mathbf{T}_n \mathbf{Z}A) = 0$.*

Proof. Let $y \in \mathbf{Z}A$ and $z \in \mathbf{T}_n \mathbf{Z}A$, so that $z^{p^n} = 0$. Then Lemma 2.1 (i) implies that

$$\zeta_n(y)z = \zeta_n(yz^{p^n}) = \zeta_n(y0) = \zeta_n(0) = 0.$$

Hence $(\mathbf{T}_n A^\perp)(\mathbf{T}_n \mathbf{Z}A) = (\text{Im } \zeta_n)(\mathbf{T}_n \mathbf{Z}A) = 0$, by Lemma 2.1 (iii).

The result above is essentially [9, Proposition 4]. We conclude that

$$\mathbf{T}_n \mathbf{Z}A \subseteq \{z \in \mathbf{Z}A : z(\mathbf{T}_n A^\perp) = 0\} \subseteq \{z \in \mathbf{Z}A : z\zeta_n(1) = 0\}.$$

In [2], we will see that these inclusions are proper in general, even for group algebras of finite groups. If n is sufficiently large then $\mathbf{T}_n \mathbf{Z}A = \mathbf{J}\mathbf{Z}A$ and $\mathbf{T}_n A^\perp = \mathbf{R}A$, and certainly

$$\mathbf{J}\mathbf{Z}A = \{z \in \mathbf{Z}A : z \cdot \mathbf{R}A = 0\}.$$

Also, if n is large and $A = FG$ for a finite group G then $\zeta_n(1) = G_p^+$ where G_p denotes the set of p -elements in G (cf. [7, (48)]), and it is known that

$$\mathbf{J}\mathbf{Z}FG = \{z \in \mathbf{Z}FG : zG_p^+ = 0\}$$

(cf. [7, (59)]). However, it is easy to construct an example of a symmetric F -algebra A such that

$$\mathbf{J}\mathbf{Z}A \neq \{z \in \mathbf{Z}A : z\zeta_n(1) = 0\}$$

for all sufficiently large n .

For $n \in \mathbb{N}$, the ideal $\mathbf{T}_n \mathbf{Z}A$ of $\mathbf{Z}A$ is related to a semilinear map $\kappa_n : A/\mathbf{K}A \rightarrow A/\mathbf{K}A$ first constructed in [6 IV]; κ_n is defined in such a way that

$$(z^{p^n} | x) = (z | \kappa_n(x))^{p^n} \quad \text{for } z \in \mathbf{Z}A \quad \text{and} \quad x \in A/\mathbf{K}A;$$

here we set $(z | a + \mathbf{K}A) := (z | a)$ for $z \in \mathbf{Z}A$ and $a \in A$. Also, we set $(a + \mathbf{K}A)^{p^n} := a^{p^n} + \mathbf{K}A$ for $a \in A$. We recall the following properties of κ_n (cf. [7, (50) - (53)]).

Lemma 6.2. *Let $m, n \in \mathbb{N}$, let $x, y \in A/\mathbf{K}A$, and let $z \in \mathbf{Z}A$. Then the following holds:*

- (i) $\kappa_n(x + y) = \kappa_n(x) + \kappa_n(y)$, $z\kappa_n(x) = \kappa_n(z^{p^n}x)$ and $\kappa_n(zx^{p^n}) = \zeta_n(z)x$.
- (ii) $\kappa_m \circ \kappa_n = \kappa_{m+n}$.
- (iii) $\text{Im}(\kappa_n) = \mathbf{T}_n \mathbf{Z}A^\perp / \mathbf{K}A$.

Our next result is a dual version of Theorem 2.3. For simplicity, we concentrate on the case where A is a non-simple block. (If A is a simple block then $\mathbf{T}_1 \mathbf{Z}A = 0$, so $\mathbf{T}_1 \mathbf{Z}A^\perp = A$. Moreover, we have $\mathbf{T}_2 A^\perp = \mathbf{T}_1 A^\perp = \mathbf{Z}A$ in this case.)

Proposition 6.3. *Suppose that A is a non-simple block. Then the following holds:*

- (i) $(\mathbf{T}_1 A^\perp)(\mathbf{T}_1 \mathbf{Z}A^\perp) \subseteq \mathbf{K}A$ for $p \neq 2$.
- (ii) $(\mathbf{T}_2 A^\perp)(\mathbf{T}_1 \mathbf{Z}A^\perp) \subseteq \mathbf{K}A$ and $(\mathbf{T}_1 A^\perp)(\mathbf{T}_2 \mathbf{Z}A^\perp) \subseteq \mathbf{K}A$ for $p = 2$.
- (iii) $(\mathbf{T}_1 A^\perp)(\mathbf{T}_1 \mathbf{Z}A^\perp) \subseteq \mathbf{J}\mathbf{Z}A^\perp$ for $p = 2$. Moreover, in this case we have $(\mathbf{T}_1 A^\perp)(\mathbf{T}_1 \mathbf{Z}A^\perp) \subseteq \mathbf{K}A$ if and only if $\zeta_1(1)^2 = 0$.

Proof. (i) Let $y \in \mathbf{Z}A$ and $x \in A/\mathbf{K}A$. Then $\zeta_1(y)\kappa_1(x) = \kappa_1(\zeta_1(y)^p x) = 0$ since $\zeta_1(y)^p \in (\mathbf{T}_1 A^\perp)^p = 0$ by Theorem 2.3 (iii). Thus

$$(\mathbf{T}_1 A^\perp)(\mathbf{T}_1 \mathbf{Z}A^\perp / \mathbf{K}A) = (\text{Im } \zeta_1)(\text{Im } \kappa_1) = 0,$$

and (i) is proved.

(ii) Let x, y be as in (i). Then $\zeta_2(y)\kappa_1(x) = \kappa_1(\zeta_2(y)^2x) = 0$ since $\zeta_2(y)^2 \in (\mathbf{T}_2A^\perp)^2 = 0$, by Theorem 2.3 (ii). Thus

$$(\mathbf{T}_2A^\perp)(\mathbf{T}_1\mathbf{Z}A^\perp/\mathbf{K}A) = (\text{Im } \zeta_2)(\text{Im } \kappa_1) = 0.$$

Similarly, we have $\zeta_1(y)\kappa_2(x) = \kappa_2(\zeta_1(y)^4x) = 0$ since $\zeta_1(y)^3 \in (\mathbf{T}_1A^\perp)^3 = 0$ by Theorem 2.3 (ii). Thus

$$(\mathbf{T}_1A^\perp)(\mathbf{T}_2\mathbf{Z}A^\perp/\mathbf{K}A) = (\text{Im } \zeta_1)(\text{Im } \kappa_2) = 0,$$

and (ii) follows.

(iii) Again, let x, y be as in (i). Then

$$\zeta_1(y)\kappa_1(x) = \kappa_1(\zeta_1(y)^2x) = \kappa_1(\zeta_1(y)\kappa_1(yx^2)) \in \kappa_1((\text{Im } \zeta_1)(\text{Im } \kappa_1)).$$

Iteration yields

$$(\text{Im } \zeta_1)(\text{Im } \kappa_1) \subseteq \kappa_1((\text{Im } \zeta_1)(\text{Im } \kappa_1)) \subseteq \kappa_1(\kappa_1((\text{Im } \zeta_1)(\text{Im } \kappa_1))) = \kappa_2((\text{Im } \zeta_1)(\text{Im } \kappa_1)) \subseteq \dots$$

Thus

$$(\mathbf{T}_1A^\perp)(\mathbf{T}_1\mathbf{Z}A^\perp/\mathbf{K}A) = (\text{Im } \zeta_1)(\text{Im } \kappa_1) \subseteq \bigcap_{n=0}^{\infty} \text{Im}(\kappa_n) = \bigcap_{n=0}^{\infty} \mathbf{T}_n\mathbf{Z}A^\perp/\mathbf{K}A = \mathbf{J}\mathbf{Z}A^\perp/\mathbf{K}A,$$

and the first assertion of (iii) is proved. Now note that $(\mathbf{T}_1A^\perp)(\mathbf{T}_1\mathbf{Z}A^\perp) \subseteq \mathbf{K}A$ if and only if

$$0 = ((\mathbf{T}_1A^\perp)(\mathbf{T}_1\mathbf{Z}A^\perp)|\mathbf{Z}A) = (\mathbf{T}_1A^\perp|\mathbf{T}_1\mathbf{Z}A^\perp)$$

if and only if $\mathbf{T}_1A^\perp \subseteq \mathbf{T}_1\mathbf{Z}A$ if and only if $z^2 = 0$ for all $z \in \mathbf{T}_1A^\perp$. But $(\mathbf{T}_1A^\perp)^2 = F\zeta_1(1)^2$ by Corollary 2.4, so $z^2 = 0$ for all $z \in \mathbf{T}_1A^\perp$ if and only if $\zeta_1(1)^2 = 0$.

Note that, in the situation of Proposition 6.3 (iii), we have $\zeta_1(1)^2 = 0$ if and only if all diagonal Cartan invariants of A are even, by Lemma 3.4. Also, we have

$$\dim(\mathbf{T}_1A^\perp)(\mathbf{T}_1\mathbf{Z}A^\perp) + \mathbf{K}A/\mathbf{K}A \leq 1.$$

There is the following dual of Proposition 6.1.

Proposition 6.4. *Let $n \in \mathbb{N}$. Then $(\mathbf{T}_n\mathbf{Z}A)(\mathbf{T}_n\mathbf{Z}A^\perp) \subseteq \mathbf{K}A$.*

Proof. Let $z \in \mathbf{T}_n\mathbf{Z}A$ and $x \in A/\mathbf{K}A$. Then

$$z\kappa_n(x) = \kappa_n(z^{p^n}x) = \kappa_n(0x) = 0.$$

Thus $(\mathbf{T}_n\mathbf{Z}A)(\mathbf{T}_n\mathbf{Z}A^\perp/\mathbf{K}A) = (\mathbf{T}_n\mathbf{Z}A)(\text{Im } \kappa_n) = 0$, and the result follows.

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References

1. E.F. Assmus and J.D. Key, *Designs and their codes*, Cambridge University Press, Cambridge 1992
2. T. Breuer, L. Héthelyi, E. Horváth, B. Külshammer and J. Murray, Cartan invariants and central ideals of group algebras, submitted for publication
3. C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience Publishers, New York - London - Sydney 1962
4. B. Huppert, *Angewandte Lineare Algebra*, Walter de Gruyter, Berlin 1990
5. S. König and A. Zimmermann, *Derived equivalences for group rings*, Springer-Verlag, Berlin 1998
6. B. Külshammer, Bemerkungen über die Gruppenalgebra als symmetrische Algebra I-IV, *J. Algebra* **72** (1981), 1-7; **75** (1982), 59-69; **88** (1984), 273-291, **93** (1985), 310-323
7. B. Külshammer, Group-theoretical descriptions of ring-theoretical invariants of group algebras, *Progress in Math.* **95** (1991), 425-442
8. J.C. Murray, Blocks of defect zero and products of elements of order p , *J. Algebra* **214** (1999), 385-399
9. J. Murray, On a certain ideal of Külshammer in the centre of a group algebra, *Arch. Math.* **77** (2001), 373-377
10. J. Murray, Squares in the centre of the group algebra of a symmetric group, *Bull. London Math. Soc.* **34** (2002), 155-164
11. T. Okuyama, Some studies on group algebras, *Hokkaido Math. J.* **9** (1980), 217-221