CARDINALITY ESTIMATES FOR PIECEWISE CONGRUENCES OF CONVEX POLYGONS (EXTENSIVE VERSION)

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Abstract. Two convex polygons $P, P' \subseteq \mathbb{R}^2$ are congruent by dissection with respect to a given group G of transformations of $\mathbb{R}^2$ if both can be dissected into the same finite number k of polygonal pieces $Q_1, \ldots, Q_k$ and $Q'_1, \ldots, Q'_k$ such that corresponding pieces Q_i, Q'_i are congruent with respect to G, $1 \leq i \leq k$. In that case $\deg_G(P, P')$ denotes the smallest k with the above property.

For the group Isom^+ of proper Euclidean isometries we give two general upper estimates for $\deg_{\mathrm{Isom}^+}(P, P')$, the first one in terms of the numbers of vertices and the diameters of P, P', the second one depending moreover on the radii of inscribed circles. A particular result concerns regular polygons P, P'.

For the groups Sim^+ and Sim of proper and general similarities we establish upper bounds for $\deg_{\mathrm{Sim}^+}(P, P')$ and $\deg_{\mathrm{Sim}}(P, P')$ in terms of the numbers of vertices.

1. Introduction

Given a group G of affine transformations of the Euclidean plane $\mathbb{R}^2$, two polygons $P, P' \subseteq \mathbb{R}^2$ are called *congruent by dissection* (or *equidissectable*) *with respect to* G if there exist a number $k \in \{1, 2, \ldots\}$ and dissections of P into polygons $Q_1, \ldots, Q_k$ and of P' into polygons $Q'_1, \ldots, Q'_k$ such that, for every $i \in \{1, \ldots, k\}$, Q_i and Q'_i are congruent with respect to G. Here a *polygon* is meant to be a union of finitely many triangles. We say that P is *dissected* into $Q_1, \ldots, Q_k$ if $P = Q_1 \cup \ldots \cup Q_k$ and the interiors of distinct pieces Q_i, Q_j, $i \neq j$, are disjoint. If P and P' are equidissectable, the minimal number k admitting dissections with the above property is called the *degree* of the congruence by dissection of P and P'. This optimal number of pieces is denoted by $\deg_G(P, P')$.

Of course, if P and P' are congruent by dissection with respect to a subgroup H of G, then P and P' are equidissectable with respect to G, too, and

$$\deg_G(P, P') \leq \deg_H(P, P').$$

The present paper is devoted to degree estimates for congruences by dissection of convex polygons with respect to the groups Isom of Euclidean isometries and Sim of similarities. In fact, most results concern the subgroups Isom^+ and Sim^+ formed by the proper transformations of Isom and Sim, respectively.

The classical Wallace-Bolyai-Gerwien Theorem says that any two polygons of the same area are equidissectable with respect to Isom (see [4] and [9, Chapter 3] for historical remarks). The group containing all translations and all central reflections is known to be the smallest subgroup of Isom satisfying the above property (see [5, 1]). However, the question for the degree of congruences by dissection is rather open. The following theorem by Hertel seems to give the first upper estimate for $\deg_{\mathrm{Isom}}(P, P')$ concerning general polygons of equal area.

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Theorem 1 ([6, Satz 2]). *Let P_m and P'_n be an m-gon and an n-gon of the same area whose diameters are d and d', respectively. Suppose that there exist dissections of P_m into $m-2$ triangles $T_1, \ldots, T_{m-2}$ and of P'_n into $n-2$ triangles $T'_1, \ldots, T'_{n-2}$ and define*

$$c = \min\{\operatorname{diam}(T_1), \ldots, \operatorname{diam}(T_{m-2}), \operatorname{diam}(T'_1), \ldots, \operatorname{diam}(T'_{n-2})\}.$$

Then

$$\deg_{\mathrm{Isom}}(P_m, P'_n) \le 4(m-2)(n-2)\left(\tfrac{\max\{d,d'\}}{c} + 2\right)^2.$$

Our first goal in Section 2 is an upper estimate for $\deg_{\mathrm{Isom}}(P_m, P'_n)$ for arbitrary convex P_m and P'_n only depending on m, n, d, and d'. We shall see that Theorem 1 gives a bound of that kind depending on m and n like a polynomial of degree 4 and of quadratic behaviour in $\max\left\{\frac{d}{d'}, \frac{d'}{d}\right\}$ (see Corollary 1). Then we establish a stronger estimate for for $\deg_{\mathrm{Isom}^+}(P_m, P'_n)$ quadratic in m, n and linear in $\max\left\{\frac{d}{d'}, \frac{d'}{d}\right\}$ (see Theorem 2). An important technical step to this main result concerns the piecewise congruence of triangles (see Lemma 3).

The estimate of Theorem 2 can be improved if P_m and P'_n are known to contain inscribed circles of sufficiently large radii (see Theorem 3). In particular, $\deg_{\mathrm{Isom}^+}(P^r_m, P^r_n) \le 7(m+n-1)$ for regular polygons of the same area with m and n vertices, respectively (see Theorem 4). This improves the bound

$$\deg_{\mathrm{Isom}}(P^r_m, P^r_n) \le (2m+4)(n+1) \quad \text{for} \quad 3 \le m < n \tag{1}$$

given by Doyen and Landuyt without proof (see [2]).

Nevertheless, our estimates do not use to be sharp in particular situations. For example, isometric congruences by dissection of regular P^r_m and P^r_n with very small m, n, say $m, n \le 12$, are known to require much less than $7(m+n-1)$ pieces. We refer to Theobald's frequently updated web page [8].

The second major part of this paper concerns the groups Sim and Sim$^+$ (Section 3). Any two polygons P, P' are congruent by dissection with respect to Sim, since P and the similar image $\sqrt{\frac{\lambda(P)}{\lambda(P')}}P'$ of P' have the same area and hence are congruent by dissection with respect to Isom by the Wallace-Bolyai-Gerwien Theorem. It is shown in [7] that any convex m-gon P_m and any convex n-gon P'_n satisfy $\deg_{\mathrm{Sim}}(P_m, P'_n) \le 3(\max\{m, n\} - 2)$. This motivates the definition

$$\deg_{\mathrm{Sim}}(m, n) = \max\{\deg_{\mathrm{Sim}}(P_m, P'_n) : P_m \text{ a convex } m\text{-gon}, P'_n \text{ a convex } n\text{-gon}\}$$

for $m, n \ge 3$. We introduce $\deg_{\mathrm{Sim}^+}(m, n)$ analogously. Of course,

$$\deg_{\mathrm{Sim}}(m, n) = \deg_{\mathrm{Sim}}(n, m) \le \deg_{\mathrm{Sim}^+}(m, n) = \deg_{\mathrm{Sim}^+}(n, m).$$

The above mentioned estimate from [7] now reads as

$$\deg_{\mathrm{Sim}}(m, n) \le 3(n-2) \quad \text{for all} \quad 3 \le m \le n. \tag{2}$$

After a characterization of all pairs of triangles T, T' satisfying $\deg_{\mathrm{Sim}}(T, T') \le 2$ (see Corollary 2), we prove several properties concerning congruence by dissection with respect to Sim$^+$, which yield an upper estimate for $\deg_{\mathrm{Sim}^+}(m, n)$ strictly stronger than (2) for all $(m, n) \ne (3, 3)$ (see Theorem 5). A final improvement concerning $\deg_{\mathrm{Sim}}(m, n)$ for larger m, n can be obtained by the use of improper similarities (see Theorem 6).

All our estimates for $\deg_G(P_m, P'_n)$ of convex polygons P_m, P'_n can be realized by dissections into convex pieces if $G \in \{\mathrm{Isom}^+, \mathrm{Isom}, \mathrm{Sim}^+\}$. Only Lemma 17 and Theorem 6 on $G = \mathrm{Sim}$ are based on non-convex pieces of dissection.

We use the following notations: Given three points $x_1, x_2, x_3 \in \mathbb{R}^2$, the symbols $l(x_1, x_2)$, x_1x_2, $|x_1x_2|\big(= d(x_1, x_2)\big)$, $\angle x_1x_2x_3$, and $|\angle x_1x_2x_3|$ stand for the straight line passing through x_1, x_2, the line segment between x_1, x_2, the length of that segment (which is the Euclidean distance of x_1, x_2), the angle determined by

x_1, x_2, x_3, and the size of that angle, respectively. $\mathrm{int}(A)$, $\mathrm{cl}(A)$, $\mathrm{bd}(A)$, $\mathrm{conv}(A)$, $\mathrm{diam}(A) = \sup\{d(x_1, x_2) : x_1, x_2 \in A\}$, and $\lambda(A)$ denote the interior, the closure, the boundary, the convex hull, the diameter, and the area measure, respectively, of a (measurable) subset $A \subseteq \mathbb{R}^2$. $\lfloor \xi \rfloor$ and $\lceil \xi \rceil$ are the largest lower and the smallest upper integer bound of $\xi \in \mathbb{R}$.

2. Congruence by dissection with respect to isometries

2.1. An estimate in terms of vertex numbers and diameters based on Theorem 1.

Corollary 1. *Let P_m and P'_n be convex polygons of the same area whose numbers of vertices are m and n and whose diameters are d and d', respectively. Then*

$$\deg_{\mathrm{Isom}}(P_m, P'_n) \le 4(m-2)(n-2)\left(\max\{d, d'\}\max\left\{\tfrac{\lfloor \frac{m}{2} \rfloor}{d}, \tfrac{\lfloor \frac{n}{2} \rfloor}{d'}\right\} + 2\right)^2 .$$

In particular

$$\deg_{\mathrm{Isom}}(P_m, P'_n) < mn(m+n)^2\left(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\right)^2 .$$

For proving Corollary 1, we need a lower bound for the value c from Theorem 1.

Lemma 1. *Every convex m-gon P_m admits a dissection into $m-2$ triangles $T_1, \dots, T_{m-2}$ such that*

$$\min\{\mathrm{diam}(T_1), \dots, \mathrm{diam}(T_{m-2})\} \ge \tfrac{1}{\lfloor \frac{m}{2} \rfloor}\,\mathrm{diam}(P_m).$$

One can obtain a strict inequality if $m \ne 3$.

Proof. We proceed by induction. The case $m = 3$ is trivial.

In the case $m = 4$ let P_4 have the vertices x_1, x_2, x_3, x_4. Let $d_1 = d(x_1, x_3)$ and $d_2 = d(x_2, x_4)$ be the lengths of the diagonals, say $d_1 \ge d_2$. The convexity of P_4 and the triangle inequality yield

$$\mathrm{diam}(P_4) = \max\nolimits_{1 \le i < j \le 4} d(x_i, x_j) < d_1 + d_2.$$

Cutting P_4 along x_1x_3 gives a dissection into two triangles each having a diameter of at least

$$d(x_1, x_3) = d_1 \ge \tfrac{d_1 + d_2}{2} > \tfrac{\mathrm{diam}(P_4)}{2} = \tfrac{1}{\lfloor \frac{4}{2} \rfloor}\,\mathrm{diam}(P_4).$$

Now let $m \ge 5$. Let $x_1, \dots, x_m$ be the vertices of P_m in their order along the boundary of P_m.

Case 1. $\mathrm{diam}(P_m)$ *is the length of a diagonal, say of x_1x_k, $k \in \{3, \dots, m-1\}$.* Then x_1x_k dissects P_m into a k-gon $P_{k,1} = \mathrm{conv}\{x_1, \dots, x_k\}$ and an $(m-k+2)$-gon $P_{m-k+2,2} = \mathrm{conv}\{x_k, x_{k+1}, \dots, x_m, x_1\}$, both with diameter $d(x_1, x_k) = \mathrm{diam}(P_m)$. Application of the induction hypothesis to $P_{k,1}$ and $P_{m-k+2,2}$ yields a dissection of P_m with the required properties.

Case 2. $\mathrm{diam}(P_m)$ *is the length of an edge of P_m, say of x_1x_m.*

Case 2.1. $d(x_1, x_3) > \frac{1}{\lfloor \frac{m}{2} \rfloor}\,\mathrm{diam}(P_m)$. We cut P_m along x_1x_3 into the triangle $T = \triangle x_1x_2x_3$ and the $(m-1)$-gon $P_{m-1} = \mathrm{conv}\{x_3, x_4, \dots, x_m, x_1\}$ of diameter $d(x_1, x_m) = \mathrm{diam}(P_m)$. Then $\mathrm{diam}(T) \ge d(x_1, x_3) > \frac{1}{\lfloor \frac{m}{2} \rfloor}\,\mathrm{diam}(P_m)$. Dissection of P_{m-1} according to the induction hypothesis gives $m-3$ additional triangles of sufficiently large diameters.

Case 2.2. $d(x_1, x_3) \le \frac{1}{\lfloor \frac{m}{2} \rfloor}\,\mathrm{diam}(P_m)$. Then

$$d(x_3, x_m) > d(x_1, x_m) - d(x_1, x_3) \ge \left(1 - \tfrac{1}{\lfloor \frac{m}{2} \rfloor}\right)\mathrm{diam}(P_m) = \tfrac{\lfloor \frac{m-2}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}\,\mathrm{diam}(P_m).$$

We split P_m along x_3x_m into the quadrilateral $P_{4,1} = \mathrm{conv}\{x_1, x_2, x_3, x_m\}$ with $\mathrm{diam}(P_{4,1}) = d(x_1, x_m) = \mathrm{diam}(P_m)$ and into $P_{m-2,2} = \mathrm{conv}\{x_3, \dots, x_m\}$ with $\mathrm{diam}(P_{m-2,2}) \ge d(x_3, x_m) > \frac{\lfloor \frac{m-2}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor}\,\mathrm{diam}(P_m)$. The induction hypothesis gives dissections of $P_{4,1}$ and of $P_{m-2,2}$ into two and into $m-4$ triangles, respectively. The two pieces T of $P_{4,1}$ satisfy

$$\operatorname{diam}(T) > \tfrac{1}{2}\operatorname{diam}(P_{4,1}) = \tfrac{1}{2}\operatorname{diam}(P_m) \geq \frac{1}{\lfloor\frac{m}{2}\rfloor}\operatorname{diam}(P_m).$$

The $m-2$ pieces U of $P_{m-2,2}$ have diameters

$$\operatorname{diam}(U) \geq \frac{1}{\lfloor\frac{m-2}{2}\rfloor}\operatorname{diam}(P_{m-2,2}) > \frac{1}{\lfloor\frac{m-2}{2}\rfloor}\,\frac{\lfloor\frac{m-2}{2}\rfloor}{\lfloor\frac{m}{2}\rfloor}\operatorname{diam}(P_m) = \frac{1}{\lfloor\frac{m}{2}\rfloor}\operatorname{diam}(P_m).$$

This completes the proof. □

Proof of Corollary 1. We apply Theorem 1 to P_m and P'_n. According to Lemma 1 there exist dissections of P_m and P'_n such that the value c from the theorem satisfies

$$c \geq \min\left\{\frac{d}{\lfloor\frac{m}{2}\rfloor}, \frac{d'}{\lfloor\frac{n}{2}\rfloor}\right\}, \qquad \text{that is} \qquad \frac{1}{c} \leq \max\left\{\frac{\lfloor\frac{m}{2}\rfloor}{d}, \frac{\lfloor\frac{n}{2}\rfloor}{d'}\right\}. \tag{3}$$

Combining this with the statement of the theorem we obtain the first estimate of the corollary. The second one is proved by

$$\begin{aligned}
\deg_{\mathrm{Isom}}(P_m, P'_n) &\leq 4(m-2)(n-2)\Big(\max\{d,d'\}\max\left\{\tfrac{\frac{m}{2}}{d}, \tfrac{\frac{n}{2}}{d'}\right\} + 2\Big)^2\\
&\leq 4(m-2)(n-2)\Big(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\max\left\{\tfrac{m}{2}, \tfrac{n}{2}\right\} + 2\Big)^2\\
&= (m-2)(n-2)\Big(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\max\{m,n\} + 4\Big)^2\\
&\leq (m-2)(n-2)\Big(\max\{m,n\} + 4\Big)^2\Big(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\Big)^2\\
&\leq (m-2)(n-2)\Big((m+n-3) + 4\Big)^2\Big(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\Big)^2\\
&= (m-2)(m+n+1)(n-2)(m+n+1)\Big(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\Big)^2\\
&< m(m+n)n(m+n)\Big(\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\Big)^2.
\end{aligned}$$

□

We close this subsection by a claim showing that Lemma 1 is sharp. In this sense the estimate (3) is best possible. This justifies the formulation of Corollary 1.

Lemma 2. *For every $m \in \{3,4,5,\ldots\}$ and every $\varepsilon > 0$, there exists a convex m-gon P_m such that every dissection of P_m into $m-2$ triangles $T_1,\ldots,T_{m-2}$ satisfies*

$$\min\{\operatorname{diam}(T_1),\ldots,\operatorname{diam}(T_{m-2})\} < \Big(\frac{1}{\lfloor\frac{m}{2}\rfloor} + \varepsilon\Big)\operatorname{diam}(P_m).$$

Sketch of the proof. We assume $m \geq 4$, because the case $m = 3$ is trivial. Let $\delta > 0$ be small. We define $P_m = \operatorname{conv}\{x_1,\ldots,x_m\}$ where

$$x_i = \begin{cases} \big(\cos\big((i-1)\delta+\delta^2\big), \sin\big((i-1)\delta+\delta^2\big)\big) & \text{if } i \text{ is odd},\\ \big(\cos(i\delta), \sin(i\delta)\big) & \text{if } i \text{ is even}.\end{cases}$$

Then $\operatorname{diam}(P_m) = d(x_1, x_m)$ (if δ is sufficiently small).

Suppose that P_m is dissected into $m-2$ triangles $T_1,\ldots,T_{m-2}$. It can be shown by Euler's formula that all vertices of the triangles are vertices of P_m, too.[1] Any

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[1] This is based on the following general observation.

Lemma. *Let a convex m-gon P_m be dissected into k triangles $T_1,\ldots,T_k$. Then $k \geq m-2$ with equality if and only if the vertices of all triangles T_i, $1 \leq i \leq k$, are vertices of P_m.*

Proof. Let $V = \{x_1,\ldots,x_m,y_1,\ldots,y_p,z_1,\ldots,z_q,w_1,\ldots,w_r\}$ be the set of vertices of all triangles T_i, $1 \leq i \leq k$. Here $x_1,\ldots,x_m$ are the vertices of P_m, $y_1,\ldots,y_p$ are the vertices in the interior of P_m each of which being contained in the relative interior of an edge of some T_i, $1 \leq i \leq k$, $z_1,\ldots,z_q$ are the remaining vertices in the interior of P_m, and $w_1,\ldots,w_r$ are the vertices in $\operatorname{bd}(P_m)\setminus\{x_1,\ldots,x_m\}$.

The point y_1 belongs to the relative interior of an edge of a unique T_i. We connect y_1 with the opposite vertex of T_i by a line segment. This segment subdivides the original partition of P_m and generates a decomposition into $k+1$ triangles. Next we refine the new decomposition by a segment connecting y_2 with the opposite vertex of the triangle of the new decomposition that contains y_2 in the relative interior of one of its edges. We continue the procedure up to y_p. This

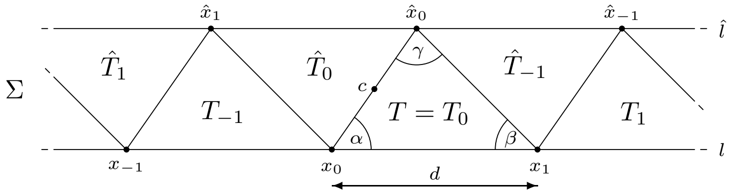

FIGURE 1. The triangle strip Σ

such triangulation contains two different triangles each of them sharing two edges with the boundary of P_m. Only one of them can contain the long edge x_1x_m. Hence at least one of them is of the form $T = \triangle x_ix_{i+1}x_{i+2}$ with $1 \le i \le m-2$. Consequently, $\operatorname{diam}(T) = d(x_i, x_{i+2}) = d(x_1, x_3) = d(x_2, x_4)$. We obtain

$$\tfrac{\operatorname{diam}(T)}{\operatorname{diam}(P_m)} = \tfrac{d(x_1,x_3)}{d(x_1,x_m)} \quad \text{and} \quad \lim_{\delta\downarrow 0} \tfrac{d(x_1,x_3)}{d(x_1,x_m)} = \lim_{\delta\downarrow 0} \tfrac{2\delta}{\lfloor\frac{m}{2}\rfloor 2\delta} = \tfrac{1}{\lfloor\frac{m}{2}\rfloor}.$$

Thus P_m satisfies the claim of Lemma 2 if δ is sufficiently small. □

2.2. Congruence by dissection of triangles.

Lemma 3. *Let T and T' be triangles of the same area having the diameters d and d', respectively. Then*

$$\deg_{\mathrm{Isom}^+}(T, T') \le 4\left\lceil \tfrac{1}{2}\max\left\{\tfrac{d}{d'}, \tfrac{d'}{d}\right\}\right\rceil + 3.$$

Proof. We use the method of crossposing triangle strips (see [3, Chapter 12]).

Let $T = \triangle x_0x_1\hat{x}_0$ with $d = d(x_0, x_1)$. c denotes the centre of $x_0\hat{x}_0$. Let τ be the translation mapping x_0 onto x_1 and let σ be the central reflection with respect to c. We define $x_i = \tau^i(x_0)$, $T_i = \tau^i(T)$, $\hat{x}_i = \sigma(x_i)$, and $\hat{T}_i = \sigma(T_i)$ for $i \in \mathbb{Z}$. Then the triangles $T_i, \hat{T}_i$, $i \in \mathbb{Z}$, form an infinite dissection of a strip Σ bounded by $l = l(x_0, x_1)$ and $\hat{l} = l(\hat{x}_0, \hat{x}_1)$ (see Figure 1). The sizes of the inner angles of T at $x_0, x_1, \hat{x}_0$ are denoted by α, β, γ, respectively. Since $d(x_0, x_1) = d$ is the diameter of T, the width of Σ is at most $\frac{\sqrt{3}}{2}d$. Based on T', we introduce a strip Σ' and respective terms c', τ', σ', x'_i, T'_i, $\hat{x}'_i$, $\hat{T}'_i$, l', $\hat{l}'$, α', β', γ' analogously. We assume $T = \triangle x_0x_1\hat{x}_0$ and $T' = \triangle x'_0x'_1\hat{x}'_0$ to be oriented in the same way.

Without loss of generality, $d \le d'$. Then $\alpha' \le \alpha$ or $\beta' \le \beta$, because T and T' have the same area. Again without loss of generality, $\alpha' \le \alpha$.

We suppose that $c' = c$ (which can be obtained by translating T'). Finally, we assume that the intersection $\Sigma \cap \Sigma'$ is a parallelogram P, whose vertices p_1, p_2, p_3, p_4 represent the intersections $l \cap l'$, $l \cap \hat{l}'$, $\hat{l} \cap \hat{l}'$, $\hat{l} \cap l'$, respectively, such that $d(p_2, p_3) = d'$ and $\delta = |\angle p_1p_2p_3| \le \frac{\pi}{3}$ (see Figure 2). In fact, this situation can be obtained by suitably rotating Σ' around c, because the width of Σ does not exceed $\frac{\sqrt{3}}{2}d \le \frac{\sqrt{3}}{2}d'$.

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way we obtain a dissection of P_m into $k+p$ triangles $T'_1, \dots, T'_{k+p}$ whose set of vertices coincides with V. No element of V belongs to the relative interior of an edge of some T'_i, $1 \le i \le k+p$. Hence the set $E = \{e_1, \dots, e_l\}$ of all edges of the triangles $T'_1, \dots, T'_{k+p}$ defines a planar graph with vertex set V and Euler's formula yields

$$1 = \operatorname{card}(V) - \operatorname{card}(E) + \operatorname{card}(\{T'_1, \dots, T'_{k+p}\}) = (m + p + q + r) - l + (k + p).$$

We write $l = l_1 + l_2$, l_1 being the number of segments from E situated on $\operatorname{bd}(P_m)$ and l_2 being the number of segments meeting $\operatorname{int}(P_m)$. So l_1 edges from E belong to exactly one of the triangles $T'_1, \dots, T'_{k+p}$ and the l_2 other edges belong to two triangles. Hence $3(k+p) = l_1 + 2l_2 = 2l - l_1$. We have $l_1 = m + r$, because $\operatorname{bd}(P_m)$ is subdivided by $x_1, \dots, x_m, w_1, \dots, w_r$. Therefore

$$2l = 3(k+p) + l_1 = 3(k+p) + m + r.$$

Combination of the two formulas gives $k = (m-2) + (p + 2q + r)$. So $k \ge m-2$, because $p, q, r \ge 0$, and $k = m-2$ if and only if $p = q = r = 0$; that is, if and only if $V = \{x_1, \dots, x_m\}$. This yields the claim. □

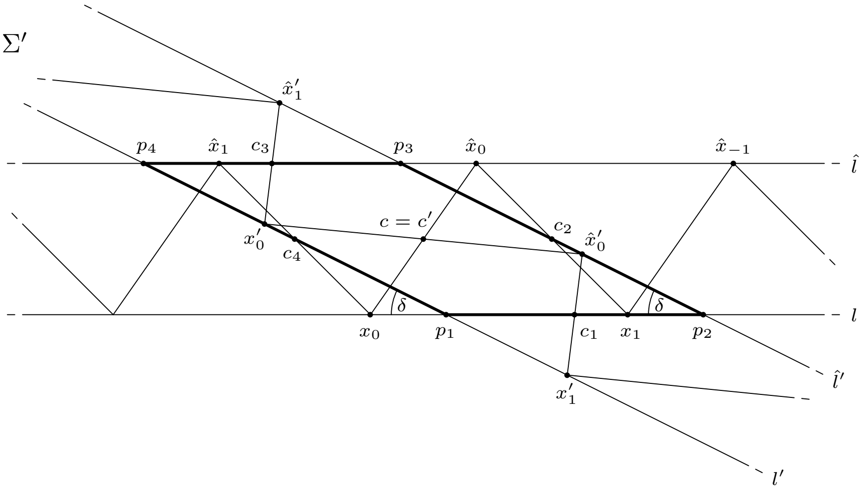

Figure 2. Crossposing the strips Σ and Σ'

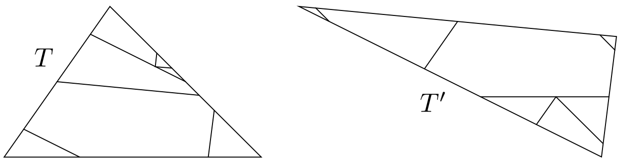

Figure 3. Dissections of T and T'

The midpoints c_1 and c_3 of p_1p_2 and p_3p_4 satisfy $d(c,c_1) = d(c,c_3) = \frac{d'}{2}$ and hence agree with the centres of $x'_1\hat{x}'_0$ and $x'_0\hat{x}'_1$, respectively. The area of P is twice that of T' and so twice that of T, too. This shows that $d(p_1,p_2) = d$ and the midpoints c_2 and c_4 of p_2p_3 and p_1p_4 coincide with those of $x_1\hat{x}_0$ and $x_0\hat{x}_1$, respectively.

The edges of the triangles $T_i, \hat{T}_i, T'_i, \hat{T}'_i$ dissect P into finitely many polygons that appear in k pairs symmetric with respect to c. Each pair consists of one member contained in $\bigcup_{i\in\mathbb{Z}} T_i$ and one member covered by $\bigcup_{i\in\mathbb{Z}} \hat{T}_i$. Images of the k first members under suitable integer powers of τ form a dissection of T (see Figure 3). Similarly, we find one element in every pair of symmetric pieces of P such that images of these k elements under suitable integer powers of τ' constitute a dissection of T'. This shows that $\deg_{\mathrm{Isom}^+}(T,T') \le k$.

It remains to establish an upper bound for k. We prepare this by proving

$$\alpha + \beta' + \delta \le \pi. \tag{4}$$

Among all triangles $\triangle x_0x_1y$ with $y \in \hat{l}$ and of diameter $d(x_0,x_1) = d$ the isosceles one with $d(x_1,y) = d(x_0,x_1) = d$ and $|\angle x_1x_0y| = |\angle x_0yx_1| = \alpha_0 \ge \frac{\pi}{3}$ maximizes the size of the inner angle at x_0, in particular $\alpha \le \alpha_0$. Its area coincides with that of T and can be computed by $\frac{1}{2}d^2\sin(|\angle x_0x_1y|) = \frac{1}{2}d^2\sin(\pi - 2\alpha_0) = \frac{1}{2}d^2\sin 2\alpha_0$. So

$$\alpha \le \alpha_0, \quad \text{where} \quad \tfrac{\pi}{3} \le \alpha_0 < \tfrac{\pi}{2} \quad \text{and} \quad d^2\sin 2\alpha_0 = 2\lambda(T) = \lambda(P) = dd'\sin\delta.$$

Similarly,

$$\beta' \le \beta'_0, \quad \text{where} \quad \tfrac{\pi}{3} \le \beta'_0 < \tfrac{\pi}{2} \quad \text{and} \quad {d'}^2\sin 2\beta'_0 = dd'\sin\delta.$$

These admit the estimate

$$\begin{aligned} dd' \sin(\pi - \delta) &= \tfrac{1}{2} 2dd' \sin\delta \\ &\leq \tfrac{1}{2}\big(\tfrac{d\cos\alpha_0}{d'\cos\beta'_0} + \tfrac{d'\cos\beta'_0}{d\cos\alpha_0}\big) dd' \sin\delta \\ &= \tfrac{1}{2}\big(\tfrac{d\cos\alpha_0}{d'\cos\beta'_0}\, d'^2 \sin 2\beta'_0 + \tfrac{d'\cos\beta'_0}{d\cos\alpha_0}\, d^2 \sin 2\alpha_0\big) \\ &= dd' \left(\cos\alpha_0 \sin\beta'_0 + \cos\beta'_0 \sin\alpha_0\right) \\ &= dd' \sin(\alpha_0 + \beta'_0). \end{aligned}$$

Hence $\sin(\alpha_0 + \beta'_0) \geq \sin(\pi - \delta)$ and therefore $\alpha_0 + \beta'_0 \leq \pi - \delta$, because $\alpha_0 + \beta'_0, \pi - \delta \in \left[\frac{\pi}{2}, \pi\right]$. This implies (4), namely $\alpha + \beta' + \delta \leq \alpha_0 + \beta'_0 + \delta \leq \pi$.

Since $\delta < \frac{\pi}{2}$ and $d(x_0, c_4) = \frac{d(x_0, \hat{x}_1)}{2} \leq \frac{d}{2} \leq \frac{d'}{2} = d(p_1, c_4)$ in the triangle $\triangle x_0 p_1 c_4$, we have $p_1 \in \bigcup_{i=0}^{\infty} x_i x_{i+1}$. Thus $x_0\hat{x}_0$ meets p_1p_4 as well as p_2p_3 and splits P into two polygons symmetric with respect to c. We denote the one that contains p_1p_2 by P_1. Now k is the number of pieces of P contained in P_1.

The dissection of P_1 is induced by those segments $x_i\hat{x}_{1-i}$, $x_i\hat{x}_{-i}$, $x'_i\hat{x}'_{1-i}$, $x'_i\hat{x}'_{-i}$ that intersect $\operatorname{int}(P_1)$. Among the edges of the triangles from Σ these are exactly the $x_i\hat{x}_{1-i}$, $x_i\hat{x}_{-i}$ satisfying $x_i \in x_0p_2 \setminus \{x_0, p_2\}$, that is, $1 \leq i \leq i_0 = \left\lceil \frac{d(x_0,p_2)}{d} \right\rceil - 1$.

Next we shall see that $x'_0\hat{x}'_0$ and $x'_1\hat{x}'_0$ are the only edges of triangles from Σ' that can intersect $\operatorname{int}(P_1)$. For this it suffices to show that $\hat{x}'_0 \in p_2p_3 \cap P_1$, because then all $\hat{x}'_i x'_{1-i}$, $\hat{x}'_i x'_{-i}$, $i \geq 1$, are separated from $\operatorname{int}(P_1)$ by $x_0\hat{x}_0$ and all $\hat{x}'_i x'_{-i}$, $\hat{x}'_i x'_{1-i}$, $i \leq -1$, are separated from $\operatorname{int}(P_1)$ by p_1p_2. We have

$$d(c, \hat{x}'_0) = \tfrac{d(x'_0, \hat{x}'_0)}{2} \leq \tfrac{d'}{2} = d(c, c_1) < d(c, p_2).$$

Hence $\hat{x}'_0$ belongs to the open half-line $\{p_2 + \mu(p_3 - p_2) : \mu > 0\}$. On the other hand, the slope $\tan\alpha$ of $x_0\hat{x}_0$ relative to the "horizontal" straight line l is larger than the slope $\tan(\alpha' - \delta)$ of $x'_0\hat{x}'_0$, since $\alpha' \leq \alpha$. This yields $\hat{x}'_0 \in p_2p_3 \cap P_1$. So the dissection of P_1 induced by the triangles from Σ' is realized by the two segments $c\hat{x}'_0$, $\hat{x}'_0 c_1$ and consists of the three polygons $P_1 \cap T'$, $P_1 \cap \hat{T}'_0$, $P_1 \cap \hat{T}'_{-1}$.

Now the full dissection of P_1 is completed by the segments $x_i\hat{x}_{1-i}$, $x_i\hat{x}_{-i}$ with $1 \leq i \leq i_0 = \left\lceil \frac{d(x_0,p_2)}{d} \right\rceil - 1$. First let $1 \leq i \leq i_0 - 1$, that is, $x_i \in x_0p_1 \setminus \{x_0, p_1\}$. Then each $x_i\hat{x}_{1-i}$ or $x_i\hat{x}_{-i}$ passes through $\operatorname{int}(P_1 \cap T')$ and intersects at most one of $\operatorname{int}(P_1 \cap \hat{T}'_0)$ and $\operatorname{int}(P_1 \cap \hat{T}'_{-1})$. Hence subdividing the previous dissection by such a segment increases the number of pieces by at most 2. After having used all these $2(i_0 - 1)$ segments we have at most $3 + 4(i_0 - 1) = 4i_0 - 1$ pieces.

We show now that the two remaining segments $x_{i_0}\hat{x}_{1-i_0}$, $x_{i_0}\hat{x}_{-i_0}$ together increase the total number of pieces of P_1 by at most 4. We have $x_{i_0} \in p_1p_2 \setminus \{p_2\}$. If $x_{i_0} \in p_1c_1 \setminus \{c_1\}$ we can argue as above. In the case $x_{i_0} = c_1$ each of $x_{i_0}\hat{x}_{1-i_0}$ and $x_{i_0}\hat{x}_{-i_0}$ either intersects both $\operatorname{int}(P \cap T')$ and $\operatorname{int}(P_1 \cap \hat{T}'_0)$, but misses $\operatorname{int}(P_1 \cap \hat{T}'_{-1})$, or is collinear with $c_1\hat{x}'_0$, or intersects only $\operatorname{int}(P_1 \cap \hat{T}'_{-1})$. This yields the claim. In the final case $x_{i_0} \in c_1p_2 \setminus \{c_1, p_2\}$ (which is displayed in Figure 2) we compute

$$|\angle p_1 c_1 \hat{x}'_0| + |\angle p_2 x_{i_0} \hat{x}_{-i_0}| = (\beta' + \delta) + \alpha \leq \pi$$

by (4). Thus $x_{i_0}\hat{x}_{-i_0}$ meets only $P_1 \cap \hat{T}'_{-1}$, but misses $\operatorname{int}(P \cap T')$ and $\operatorname{int}(P_1 \cap \hat{T}'_0)$. The segment $x_{i_0}\hat{x}_{1-i_0}$ may intersect all three open polygons $\operatorname{int}(P \cap T')$, $\operatorname{int}(P_1 \cap \hat{T}'_0)$, and $\operatorname{int}(P_1 \cap \hat{T}'_{-1})$. However, dissecting by both $x_{i_0}\hat{x}_{-i_0}$ and $x_{i_0}\hat{x}_{1-i_0}$ enlarges the number of pieces of P_1 by at most 4.

The last observation shows that the total number k of pieces in P_1 is bounded by $k \le (4i_0 - 1) + 4 = 4i_0 + 3$. Now the proof of Lemma 3 is completed by

$$\begin{array}{rcl}
\deg_{\mathrm{Isom}^+}(T,T') & \le & k \ \le \ 4i_0 + 3 \ = \ 4\big(\big\lceil \frac{d(x_0,p_2)}{d} \big\rceil - 1\big) + 3 \\
& \le & 4\big(\big\lceil \frac{1}{d}(d(x_0,c) + d(c,c_1) + d(c_1,p_2)) \big\rceil - 1\big) + 3 \\
& \le & 4\big(\big\lceil \frac{1}{d}\big(\frac{d}{2} + \frac{d'}{2} + \frac{d}{2}\big) \big\rceil - 1\big) + 3 \ = \ 4\big\lceil \frac{d'}{2d} \big\rceil + 3 \\
& = & 4\big\lceil \frac{1}{2} \max\big\{\frac{d}{d'}, \frac{d'}{d}\big\} \big\rceil + 3.
\end{array}$$

□

2.3. An improved estimate in terms of vertex numbers and diameters.

Theorem 2. *Let P_m and P'_n be convex polygons of the same area whose numbers of vertices are m and n and whose diameters are d and d', respectively. Then*

$$\deg_{\mathrm{Isom}^+}(P_m, P'_n) \le (m+n-5)\left(4\left\lceil \max\left\{\frac{\lfloor\frac{n}{2}\rfloor d}{d'}, \frac{\lfloor\frac{m}{2}\rfloor d'}{d}\right\}\right\rceil + 3\right).$$

In particular

$$\deg_{\mathrm{Isom}^+}(P_m, P'_n) < 2(m+n)^2 \max\left\{\frac{d}{d'}, \frac{d'}{d}\right\}.$$

The proof is prepared by two more lemmas.

Lemma 4. *Let the area $\lambda(T)$ of a triangle T be represented as a sum $\lambda(T) = \lambda_1 + \ldots + \lambda_k$ of k positive real numbers. Then one can dissect T into k triangles $T_1, \ldots, T_k$ such that*

$$\lambda(T_i) = \lambda_i \quad \text{and} \quad \mathrm{diam}(T_i) > \frac{\mathrm{diam}(T)}{2} \quad \text{for } 1 \le i \le k.$$

Proof. Let $T = \triangle x_1x_2x_3$ and suppose that $d(x_1,x_2) \ge d(x_2,x_3) \ge d(x_1,x_3)$. We fix points $y_1, \ldots, y_{k-1}$ of the edge x_1x_3 such that $d(x_1,y_1) = \frac{\lambda_1}{\lambda(T)}\, d(x_1,x_3)$, $d(y_{i-1},y_i) = \frac{\lambda_i}{\lambda(T)}\, d(x_1,x_3)$ for $2 \le i \le k-1$, and $d(y_{k-1},x_3) = \frac{\lambda_k}{\lambda(T)}\, d(x_1,x_3)$. Then T splits into $T_1 = \triangle x_1x_2y_1$, $T_i = \triangle y_{i-1}x_2y_i$ for $2 \le i \le k-1$, and $T_k = \triangle y_{k-1}x_2x_3$. The areas of these triangles are proportional to the lengths of their edges contained in x_1x_3. Hence $\lambda(T_i) = \lambda_i$, $1 \le i \le k$. An estimate of their diameters can be obtained by the aid of the orthogonal projection π onto the long edge x_1x_2, namely

$$\mathrm{diam}(T_i) \ge d(y_{i-1},x_2) > d(\pi(y_{i-1}),x_2) > d(\pi(x_3),x_2) \ge \frac{d(x_1,x_2)}{2} = \frac{\mathrm{diam}(T)}{2}$$

for $2 \le i \le k$. For T_1 we even have $\mathrm{diam}(T_1) = d(x_1,x_2) = \mathrm{diam}(T)$. □

Lemma 5. *Let P_m and P'_n be as in Theorem 2. Then there exist dissections of P_m into $m+n-5$ triangles $T_1, \ldots, T_{m+n-5}$ and of P'_n into $m+n-5$ triangles $T'_1, \ldots, T'_{m+n-5}$ such that, for $1 \le i \le m+n-5$,*

$$\lambda(T_i) = \lambda(T'_i), \quad \frac{d}{2\lfloor\frac{m}{2}\rfloor} < \mathrm{diam}(T_i) \le d, \quad \text{and} \quad \frac{d'}{2\lfloor\frac{n}{2}\rfloor} < \mathrm{diam}(T'_i) \le d'.$$

Proof. By Lemma 1, there exist dissections of P_m into triangles $S_1, \ldots, S_{m-2}$ with $\mathrm{diam}(S_i) \ge \frac{d}{\lfloor\frac{m}{2}\rfloor}$ and of P'_n into triangles $S'_1, \ldots, S'_{n-2}$ with $\mathrm{diam}(S'_j) \ge \frac{d'}{\lfloor\frac{n}{2}\rfloor}$. Let $\mu_i = \sum_{l=1}^{i} \lambda(S_l)$, $0 \le i \le m-2$. Then the intervals $I_i = [\mu_{i-1}, \mu_i]$, $1 \le i \le m-2$, have the lengths $\lambda(S_i)$ and together constitute a dissection of $[0, \lambda(P_m)]$. Similarly, let $\nu_j = \sum_{l=1}^{j} \lambda(S'_l)$, $0 \le j \le n-2$. The intervals $J_j = [\nu_{j-1}, \nu_j]$, $1 \le j \le n-2$, have the lengths $\lambda(S'_j)$ and form a dissection of $[0, \lambda(P'_n)] = [0, \lambda(P_m)]$. The numbers $\mu_1, \ldots, \mu_{m-3}, \nu_1, \ldots, \nu_{n-3}$ cut $[0, \lambda(P_m)]$ into at most $m+n-5$ subintervals each being completely covered by some I_i and by some J_j. Hence there exists a dissection of $[0, \lambda(P_m)]$ into closed intervals $K_1, \ldots, K_{m+n-5}$ of positive lengths which refines both subdivisions $\{I_1, \ldots, I_{m-2}\}$ and $\{J_1, \ldots, J_{n-2}\}$ simultaneously.

The interval I_i of length $\lambda(S_i)$ splits into suitable $K_{i_1}, \dots, K_{i_l}$ of the lengths $\lambda_{i_1}, \dots, \lambda_{i_l}$. By Lemma 4, S_i can be decomposed into triangles $T_{i_1}, \dots, T_{i_l}$ such that

$$\lambda(T_{i_r}) = \lambda_{i_r} \quad \text{and} \quad \operatorname{diam}(T_{i_r}) > \tfrac{\operatorname{diam}(S_i)}{2} \geq \tfrac{d}{2\lfloor \frac{m}{2} \rfloor} \quad \text{for } 1 \leq r \leq l.$$

This gives the dissection of P_m into $T_1, \dots, T_{m+n-5}$. In the same way we can dissect P'_n into triangles $T'_1, \dots, T'_{m+n-5}$ with

$$\lambda(T'_i) = \lambda_i \quad \text{and} \quad \operatorname{diam}(T'_i) > \tfrac{d'}{2\lfloor \frac{n}{2} \rfloor} \quad \text{for } 1 \leq i \leq m+n-5$$

by partitioning $S'_1, \dots, S'_{n-2}$. In particular $\lambda(T_i) = \lambda_i = \lambda(T'_i)$. This completes the proof. □

Proof of Theorem 2. We dissect P_m and P'_n by Lemma 5. Then, for every $i \in \{1, \dots, m+n-5\}$, we apply Lemma 3 to T_i and T'_i. This gives dissections of P_m and P'_n proving the first estimate. The second one can be shown as follows.

$$\begin{aligned}
\deg_{\mathrm{Isom}^+}(P_m, P'_n) &< (m+n-5)\Big(4\Big(\max\big\{\tfrac{\lfloor \frac{n}{2} \rfloor d}{d'}, \tfrac{\lfloor \frac{m}{2} \rfloor d'}{d}\big\} + 1\Big) + 3\Big) \\
&\leq (m+n-5)\Big(4\max\big\{\tfrac{\frac{n}{2} d}{d'}, \tfrac{\frac{m}{2} d'}{d}\big\} + 7\Big) \\
&= (m+n-5)\big(2\max\big\{\tfrac{nd}{d'}, \tfrac{md'}{d}\big\} + 7\big) \\
&\leq (m+n-5)\big(2(m+n-3)\max\big\{\tfrac{d}{d'}, \tfrac{d'}{d}\big\} + 7\big) \\
&\leq (m+n-5)\big(2(m+n-3) + 7\big)\max\big\{\tfrac{d}{d'}, \tfrac{d'}{d}\big\} \\
&= 2(m+n-5)\big(m+n+\tfrac{1}{2}\big)\max\big\{\tfrac{d}{d'}, \tfrac{d'}{d}\big\} \\
&< 2(m+n)^2 \max\big\{\tfrac{d}{d'}, \tfrac{d'}{d}\big\}.
\end{aligned}$$

□

2.4. **An estimate in terms of vertex numbers, diameters, and radii of inscribed circles.**

Theorem 3. *Let P_m be a convex m-gon of diameter d containing a circle of radius r and let P'_n be a convex n-gon of the same area having the diameter d' and containing a circle of radius r'. Then*

$$\deg_{\mathrm{Isom}^+}(P_m, P'_n) \leq (m+n-1)\left(4\left\lceil \tfrac{1}{2}\max\big\{\tfrac{d}{r'}, \tfrac{d'}{r}\big\}\right\rceil + 3\right).$$

In particular

$$\deg_{\mathrm{Isom}^+}(P_m, P'_n) < (m+n)\big(2\max\big\{\tfrac{d}{r'}, \tfrac{d'}{r}\big\} + 7\big).$$

Again the problem is reduced to piecewise congruences of triangles.

Lemma 6. *Let P_m and P'_n be as in Theorem 3. Then there exist dissections of P_m into $k \leq m+n-1$ triangles $T_1, \dots, T_k$ and of P'_n into k triangles $T'_1, \dots, T'_k$ such that, for $1 \leq i \leq k$,*

$$\lambda(T_i) = \lambda(T'_i), \qquad r < \operatorname{diam}(T_i) \leq d, \qquad \textit{and} \qquad r' < \operatorname{diam}(T'_i) \leq d'.$$

Proof. Let $x_0, \dots, x_{m-1}$ and $x'_0, \dots, x'_{n-1}$ be the vertices of P_m and P'_n, respectively, ordered counterclockwise along the boundaries. Let c and c' be the midpoints of the inscribed circles of the respective polygons. We define a bijection p of the half-open interval $[0, \lambda(P_m))$ onto $\operatorname{bd}(P_m)$ such that the counterclockwise arc from x_0 to $p(\lambda)$ along $\operatorname{bd}(P_m)$ together with the segments cx_0 and $cp(\lambda)$ bounds a polygon of area λ. Similarly, we introduce $p' : [0, \lambda(P_m)) = [0, \lambda(P'_n)) \to \operatorname{bd}(P'_n)$.

Now let $\{0 = p^{-1}(x_0), \dots, p^{-1}(x_{m-1})\} \cup \{0 = (p')^{-1}(x'_0), \dots, (p')^{-1}(x'_{n-1})\} = \{\lambda_0, \dots, \lambda_{k-1}\}$ be ordered such that $0 = \lambda_0 < \dots < \lambda_{k-1}$. Of course, $k \le m+n-1$. We define

$$T_i = \begin{cases} \operatorname{conv}\{c, p(\lambda_{i-1}), p(\lambda_i)\}, & 1 \le i \le k-1, \\ \operatorname{conv}\{c, p(\lambda_{k-1}), x_0\}, & i = k, \end{cases}$$

and

$$T'_i = \begin{cases} \operatorname{conv}\{c', p'(\lambda_{i-1}), p'(\lambda_i)\}, & 1 \le i \le k-1, \\ \operatorname{conv}\{c', p'(\lambda_{k-1}), x'_0\}, & i = k, \end{cases}$$

this way obtaining dissections of P_m into $T_1, \dots, T_k$ and of P'_n into $T'_1, \dots, T'_k$. Then $\lambda(T_i) = \lambda_i - \lambda_{i-1} = \lambda(T'_i)$ for $1 \le i \le k-1$ and $\lambda(T_k) = \lambda(P_m) - \lambda_{k-1} = \lambda(T'_k)$. Since every T_i contains the centre c as well as at least one vertex outside the inscribed circle of radius r, the lower estimate is obvious. The upper one is trivial. The triangles T'_i behave analogously. $\square$

Now Theorem 3 can be inferred from Lemma 6 as Theorem 2 has been proved by Lemma 5.

2.5. An estimate for regular polygons.

Theorem 4. *Let P^r_m and P^r_n be regular polygons of the same area having m and n vertices, respectively. Then*

$$\deg_{\mathrm{Isom}^+}(P^r_m, P^r_n) \le 7(m+n-1).$$

Theorem 4 is an immediate consequence of the following claim and of Lemma 3.

Lemma 7. *Let P^r_m and P^r_n be regular polygons of area 1 having m and n vertices, respectively. Then there exist dissections of P^r_m into $k \le m+n-1$ triangles $T_1, \dots, T_k$ and of P^r_n into k triangles $T'_1, \dots, T'_k$ such that, for $1 \le i \le k$,*

$$\lambda(T_i) = \lambda(T'_i) \quad \text{and} \quad \tfrac{1}{2} < \tfrac{\operatorname{diam}(T_i)}{\operatorname{diam}(T'_i)} < 2.$$

Proof. Simple trigonometric calculations show that the radius r_m of the largest inscribed circle, the radius R_m of the smallest circumscribed circle, and the edge length e_m of P^r_m are

$$r_m = \frac{1}{\sqrt{m \tan \frac{\pi}{m}}}, \qquad R_m = \sqrt{\frac{1+\tan^2 \frac{\pi}{m}}{m \tan \frac{\pi}{m}}}, \qquad e_m = 2\sqrt{\frac{1}{m} \tan \frac{\pi}{m}}.$$

We assume $3 \le m < n$ without loss of generality.

Case 1. $n = 4$. Then $m = 3$. We cut P^r_3 along an axis of symmetry into T_1, T_2 and P^r_4 along a diagonal into T'_1, T'_2. Then $\lambda(T_i) = \lambda(T'_i) = \frac{1}{2}$ and $\frac{\operatorname{diam}(T_i)}{\operatorname{diam}(T'_i)} = \frac{e_3}{2R_4} = 2^{\frac{1}{2}} 3^{-\frac{1}{4}} = 1.07...$ for $i = 1, 2$.

Case 2. $n \ge 5$. Now we define dissections of P^r_m into $T_1, \dots, T_k$, $k \le m+n-1$, and of P^r_n into $T'_1, \dots, T'_k$ as we did in the proof of Lemma 6. We obtain in particular

$$\lambda(T_i) = \lambda(T'_i), \qquad r_m < \operatorname{diam}(T_i), \qquad \text{and} \qquad r_n < \operatorname{diam}(T'_i)$$

for $1 \le i \le k$. For the remaining estimate of $\frac{\operatorname{diam}(T_i)}{\operatorname{diam}(T'_i)}$ let i be fixed.

Case 2.1. $m = 3$. The triangle T'_i is of the form $\triangle c' y'_1 y'_2$, c' being the centre of P^r_n and y'_1, y'_2 lying on a common edge of P^r_n. We can estimate $\operatorname{diam}(T'_i) = \max\{|c'y'_1|, |c'y'_2|, |y'_1 y'_2|\}$ by

$$r_5 \le r_n < \operatorname{diam}(T'_i) \le \max\{R_n, R_n, e_n\} \le \max\{R_5, e_5\} = e_5. \tag{5}$$

Similarly, we obtain $T_i = \triangle c y_1 y_2$, where c is the centre of P^r_3 and y_1, y_2 lie on a common edge of P^r_3. The respective triangle T'_i, which has the same area as T_i, is contained in one of the n pairwise congruent triangles defined as the convex hulls of c' and an edge of P^r_n. Hence $\lambda(T_i) = \lambda(T'_i) \le \frac{1}{n} \le \frac{1}{5}$. Since r_3 is the height of

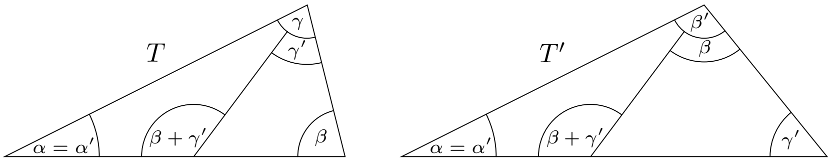

FIGURE 4. Proof of Lemma 8

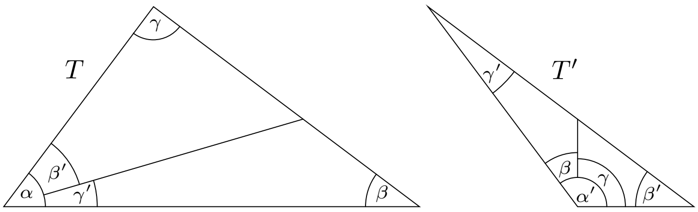

FIGURE 5. Proof of Lemma 9

T_i over the edge y_1y_2, we obtain $\frac{|y_1y_2|\,r_3}{2} = \lambda(T_i) \le \frac{1}{5}$ and hence $|y_1y_2| \le \frac{2}{5r_3}$. Thus $\mathrm{diam}(T_i)$ satisfies the estimate

$$r_3 < \mathrm{diam}(T_i) = \max\{|cy_1|, |cy_2|, |y_1y_2|\} \le \max\left\{R_3, R_3, \tfrac{2}{5r_3}\right\} = \tfrac{2}{5r_3}.$$

Combining this with (5) we obtain the required inequalities, namely

$$0.57... = \tfrac{r_3}{e_5} < \tfrac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_i')} < \tfrac{\frac{2}{5r_3}}{r_5} = 1.73...$$

Case 2.2. $m \ge 4$. Arguments similar to those for showing (5) give

$$\begin{aligned} r_4 \le r_m &< \mathrm{diam}(T_i) \le \max\{R_m, R_m, e_m\} \le \max\{R_4, e_4\} = e_4, \\ r_4 < r_n &< \mathrm{diam}(T_i') \le \max\{R_n, R_n, e_n\} < \max\{R_4, e_4\} = e_4. \end{aligned}$$

Consequently, $\frac{1}{2} = \frac{r_4}{e_4} < \frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_i')} < \frac{e_4}{r_4} = 2$. This completes the proof. ☐

3. CONGRUENCE BY DISSECTION WITH RESPECT TO SIMILARITIES

3.1. Triangles T, T' satisfying $\deg_{\mathrm{Sim}}(T, T') \le 2$.

Lemma 8. *If the size α of an angle of a triangle T coincides with the size α' of an angle of a triangle T', then* $\deg_{\mathrm{Sim}}(T, T') \le 2$.

Proof. Let α, β, γ and $\alpha' = \alpha, \beta', \gamma'$ be the sizes of the angles of T and T', respectively. We can assume that $\beta < \beta'$ and $\gamma' < \gamma$, because $\beta + \gamma = \beta' + \gamma'$ (and the case $\beta = \beta', \gamma = \gamma'$ is trivial). Corresponding dissections are shown in Figure 4. ☐

Lemma 9. *If the size α of an angle of a triangle T and the size α' of an angle of a triangle T' satisfy $\alpha + \alpha' = \pi$, then* $\deg_{\mathrm{Sim}^+}(T, T') \le 2$.

Proof. We obtain $\alpha = \beta' + \gamma'$ and $\alpha' = \beta + \gamma$. The dissections are illustrated in Figure 5. ☐

Corollary 2. *Two triangles T, T' satisfy* $\deg_{\mathrm{Sim}}(T, T') \le 2$ *if and only if T has an angle of size α and T' has an angle of size α' such that $\alpha = \alpha'$ or $\alpha + \alpha' = \pi$.*

Proof. The sufficiency is shown by Lemmas 8 and 9.

The proof of the necessity is based on considerations of the angles of the pieces of dissection. A point x is called a vertex of a general polygon P if $\{x\}$ is the intersection of at least two distinct maximal line segments belonging to the boundary of P. (x need not be an endpoint of these segments.) The size of the angle of P at x can be defined by $2\pi \lim_{\delta\downarrow 0} \frac{\lambda(B(x,\delta)\cap P)}{\lambda(B(x,\delta))}$, $B(x, \delta)$ denoting the circular disc with centre x and radius δ. We collect the sizes of all angles of P in a multiset $\mathbf{A}(P)$. That is, if different angles have the same size then this size appears several times in $\mathbf{A}(P)$.

Let us assume that a congruence by dissection of T and T' is realized by the dissections $T = P_1 \cup P_2$ and $T' = P_1' \cup P_2'$, where P_1 is similar to P_1' and P_2 is similar to P_2'. Clearly, $\mathbf{A}(P_i) = \mathbf{A}(P_i')$, $i = 1, 2$.

If there are $\alpha_1 \in \mathbf{A}(P_1)$, $\alpha_2 \in \mathbf{A}(P_2)$ such that $\alpha_1 + \alpha_2 = 2\pi$ then one of these sizes is greater than or equal to π. Hence it represents an angle at a vertex in the interior of T and the other size represents the angle at the same vertex of the complementary piece of dissection. We eliminate α_1 from $\mathbf{A}(P_1)$ and α_2 from $\mathbf{A}(P_2)$. This procedure is continued until the remaining sets $\mathbf{A}_1(P_1)$ and $\mathbf{A}_1(P_2)$ do not contain any more pair $(\alpha_1, \alpha_2) \in \mathbf{A}_1(P_1) \times \mathbf{A}_1(P_2)$ such that $\alpha_1 + \alpha_2 = 2\pi$. Then $\mathbf{A}_1(P_i)$ consists of all sizes of angles of P_i at vertices lying on the boundary of T.

Now suppose that there exists $(\alpha_1, \alpha_2) \in \mathbf{A}_1(P_1) \times \mathbf{A}_1(P_2)$ such that $\alpha_1 + \alpha_2 = \pi$. It is impossible that both sizes belong to angles of P_1 and P_2 at vertices of T, because in that case they would represent distinct parts of at most two angles of T and their sum had to be less than π. So at least one of the sizes α_1, α_2 represents a vertex in the relative interior of an edge of T. Hence the other size can be understood as the size of the angle at the same vertex of the complementary piece of dissection. As above, we eliminate α_1 from $\mathbf{A}_1(P_1)$ and α_2 from $\mathbf{A}_1(P_2)$ and continue until there exist no more pairs of that kind. The resulting sets $\mathbf{A}_2(P_i)$, $i = 1, 2$, contain exactly the sizes of those angles of P_i that appear at the vertices of T. In particular $\mathbf{A}_2(P_1) = \{\delta_1, \ldots, \delta_{k_1}\}$ and $\mathbf{A}_2(P_2) = \{\eta_1, \ldots, \eta_{k_2}\}$ with $0 \le k_i \le 3$, $i = 1, 2$, and $k_1 + k_2 \ge 3$, because every size in $\mathbf{A}(T)$ is the sum of at most one element of $\mathbf{A}_2(P_1)$ and at most one element of $\mathbf{A}_2(P_2)$, depending on whether the corresponding vertex of T belongs to P_1 and/or to P_2.

Note that the above reductions of $\mathbf{A}(P_1)\big(= \mathbf{A}(P_1')\big)$ and of $\mathbf{A}(P_2)\big(= \mathbf{A}(P_2')\big)$ did not depend on T, but only on the two multisets themselves. Hence the same procedure shows that $\mathbf{A}_2(P_1)\big(= \mathbf{A}_2(P_1')\big)$ and $\mathbf{A}_2(P_2)\big(= \mathbf{A}_2(P_2')\big)$ represent the angles of P_1' and P_2' at the vertices of T', too.

Case 1. There is $i \in \{1, 2\}$ such that $k_i = 3$. Then P_i and P_i' contain all vertices of T and T', respectively. The similarity of P_i and P_i' implies the similarity of $T = \operatorname{conv}(P_i)$ and $T' = \operatorname{conv}(P_i')$. This yields the claim.

Case 2. $k_1 + k_2 = 3$. Then $\mathbf{A}(T) = \mathbf{A}_2(P_1) \cup \mathbf{A}_2(P_2) = \mathbf{A}(T')$. The sizes of the angles of T agree with those of T'.

Case 3. $k_1, k_2 < 3$ and $k_1 + k_2 > 3$. Now $k_1 = k_2 = 2$ and, without loss of generality, $\mathbf{A}(T) = \{\delta_1 + \eta_1, \delta_2, \eta_2\}$. The multiset $\mathbf{A}(T')$ must be one of $\{\delta_1 + \eta_1, \delta_2, \eta_2\}$, $\{\delta_1 + \eta_2, \delta_2, \eta_1\}$, $\{\delta_2 + \eta_1, \delta_1, \eta_2\}$, and $\{\delta_2 + \eta_2, \delta_1, \eta_1\}$. In the first three situations T and T' have at least one angle of the same size. Finally, in the last case we obtain $(\delta_1 + \eta_1) + (\delta_2 + \eta_2) = (\delta_1 + \eta_1) + \delta_2 + \eta_2 = \pi$. This completes the proof. □

3.2. A collection of lemmas concerning Sim^+.

Lemma 10. $\deg_{\mathrm{Sim}^+}(3, 3) \le 3$.

Proof. The corresponding dissections are illustrated in Figure 6. The construction is possible if γ and γ' represent the largest angles in the respective triangles. It is closely related to [7, Hilfssatz 3], but does not use improper similarities. □

Lemma 11. $\deg_{\mathrm{Sim}^+}(3, 4) \le 4$.

Proof. Let a triangle T and a convex quadrilateral P_4 be given. T is determined up to a proper similarity by the sizes α, β, γ of its angles, counted in counterclockwise order. We assume that $\alpha, \beta < \frac{\pi}{2}$. Let v_1, v_2, v_3, v_4 be the vertices of P_4, in the

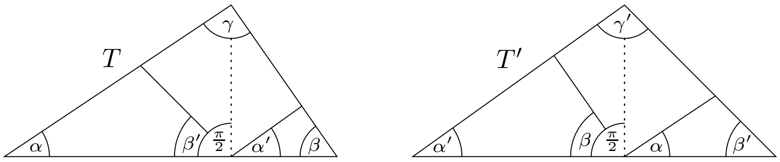

FIGURE 6. Proof of Lemma 10

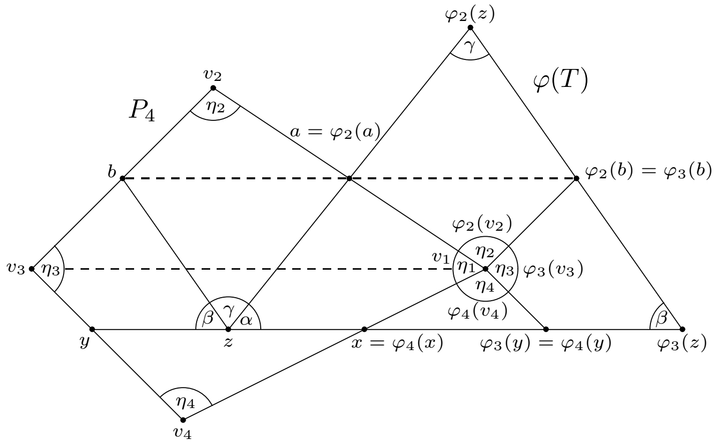

FIGURE 7. Proof of Lemma 11

same counterclockwise order, and let $\eta_1, \eta_2, \eta_3, \eta_4$ be the sizes of the corresponding angles (see Figure 7). We assume without loss of generality that the orthogonal projection of v_2 onto $l(v_1, v_3)$ belongs to the relative interior of v_1v_3. (This is the case in particular if $\eta_2 \geq \frac{\pi}{2}$. At least one angle of that size must exist.)

For $\lambda, \mu \in (0, 1)$, we define $x = x(\lambda) = v_1 + \lambda(v_4 - v_1)$, $y = y(\lambda) = v_3 + \lambda(v_4 - v_3)$, and $z = z(\lambda, \mu) = x + \mu(y - x)$. Moreover, we determine $a = a(\lambda, \mu) \in l(v_2, v_1)$ and $b = b(\lambda, \mu) \in l(v_2, v_3)$ such that both a and b are on the same side of $l(x, y)$ as v_2 and such that $|\angle xza| = \alpha$ and $|\angle yzb| = \beta$. This is possible, because $\angle v_1v_3v_2$ and $\angle v_3v_1v_2$ are acute and $\alpha, \beta < \frac{\pi}{2}$ according to the above assumptions.

Now we fix λ, μ such that ab is parallel to v_1v_3 (and hence to xy as well). Indeed, in the limit case $\lambda_0 = 0$, that is $x = v_1$, $y = v_3$, we find such μ_0 by the intermediate value theorem. Then, by continuity, for every sufficiently small $\lambda > 0$, we find a corresponding $\mu = \mu(\lambda) \in (0, 1)$. (It will be important for the proof of Lemma 12 that $\lambda > 0$ can be chosen arbitrarily small.)

P_4 splits into three quadrilaterals $Q_1 = \operatorname{conv}\{v_1, a, z, x\}$, $Q_2 = \operatorname{conv}\{v_2, b, z, a\}$, $Q_3 = \operatorname{conv}\{v_3, y, z, b\}$, and the triangle $T_4 = \triangle v_4xy$. Let $\varphi_2, \varphi_3, \varphi_4 \in \mathrm{Sim}^+$ be defined by $\varphi_2(p) = a - \frac{d(v_1,a)}{d(v_2,a)}(p - a)$, $\varphi_3(p) = p + (v_1 - v_3)$, and $\varphi_4(p) = x - \frac{d(v_1,x)}{d(v_4,x)}(p - x)$. It can easily be checked that $Q_1, \varphi_2(Q_2), \varphi_3(Q_3)$, and $\varphi_4(T_4)$ form a dissection of a triangle $\varphi(T)$, which is the image of the original triangle T under some proper similarity φ. This proves the lemma. $\square$

Lemma 12. *For every triangle T and every convex pentagon P_5, there exists a triangle $\hat{T} \subseteq T$ such that* $\deg_{\mathrm{Sim}^+}(\operatorname{cl}(T \setminus \hat{T}), P_5) \leq 4$.

Proof. Let $w_1, \ldots, w_5$ denote the vertices of P_5 according to their counterclockwise order and let $\delta_1, \ldots, \delta_5$ be the sizes of the corresponding angles. We count the indices modulo 5. The asserted congruence by dissection is prepared by two claims (Steps 1 and 2).

Step 1. There exists $i \in \{1, \ldots, 5\}$ such that $\delta_{i-1} + \delta_i, \delta_i + \delta_{i+1}, \delta_{i+1} + \delta_{i+2} > \pi$.

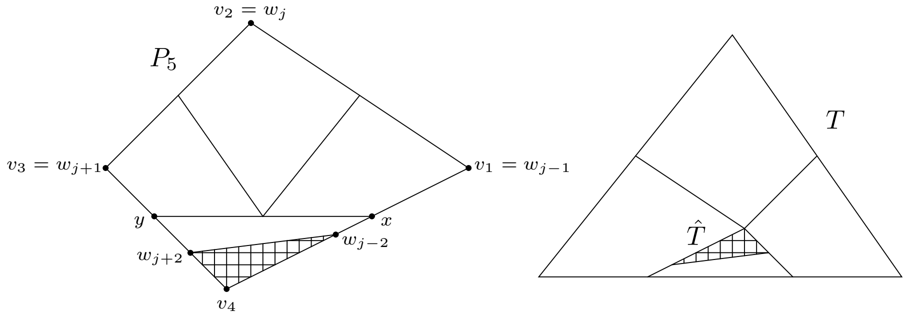

Figure 8. Proof of Lemma 12

Note that, for every k, $\delta_{k-2} + \delta_{k-1} > \pi$ or $\delta_{k+1} + \delta_{k+2} > \pi$, because $\delta_k < \pi$ and the sum of all five angles is 3π.

A first application of this to any fixed index k gives l such that $\delta_l + \delta_{l+1} > \pi$. A second application to $k = l + 3$ shows that $\delta_{l-1} + \delta_l > \pi$ or $\delta_{l+1} + \delta_{l+2} > \pi$. Say, $\delta_{l+1} + \delta_{l+2} > \pi$ without loss of generality. A last application to $k = l + 1$ yields $\delta_{l-1} + \delta_l > \pi$ or $\delta_{l+2} + \delta_{l+3} > \pi$. Hence the claim of Step 1 is satisfied for $i = l$ or for $i = l + 1$.

Step 2. For every index i, there exists $j \in \{i + 2, i + 3, i + 4\}$ such that the orthogonal projection of w_j onto $l(w_{j-1}, w_{j+1})$ belongs to the relative interior of the diagonal $w_{j-1}w_{j+1}$.

If $|\angle w_{i+2}w_{i+1}w_{i+3}|, |\angle w_{i+2}w_{i+3}w_{i+1}| < \frac{\pi}{2}$ we can choose $j = i + 2$. In the case $|\angle w_{i+4}w_iw_{i+3}|, |\angle w_{i+4}w_{i+3}w_i| < \frac{\pi}{2}$ the claim is satisfied for $j = i + 4$. If none of the two situations appears we estimate $|\angle w_{i+2}w_{i+1}w_{i+3}| + |\angle w_{i+2}w_{i+3}w_{i+1}| \geq \frac{\pi}{2}$ and $|\angle w_{i+4}w_iw_{i+3}| + |\angle w_{i+4}w_{i+3}w_i| \geq \frac{\pi}{2}$. Now consideration of the angles of $\triangle w_{i+1}w_{i+2}w_{i+3}$ and $\triangle w_{i+3}w_{i+4}w_i$ shows

$$\begin{aligned}
|\angle w_{i+3}w_{i+2}w_{i+4}| &< |\angle w_{i+3}w_{i+2}w_{i+1}| \\
&= \pi - (|\angle w_{i+2}w_{i+1}w_{i+3}| + |\angle w_{i+2}w_{i+3}w_{i+1}|) \leq \frac{\pi}{2}, \\
|\angle w_{i+3}w_{i+4}w_{i+2}| &< |\angle w_{i+3}w_{i+4}w_i| \\
&= \pi - (|\angle w_{i+4}w_iw_{i+3}| + |\angle w_{i+4}w_{i+3}w_i|) \leq \frac{\pi}{2}.
\end{aligned}$$

This yields the claim for $j = i + 3$.

Step 3. Now the proof of Lemma 12 is close to that of Lemma 11 (see Figure 8). We choose i, j according to Steps 1 and 2. Since $\delta_{j-2} + \delta_{j+2} > \pi$, the straight lines $l(w_{j-1}, w_{j-2})$ and $l(w_{j+1}, w_{j+2})$ meet at a point v_4 such that P_5 is obtained from the quadrilateral $P_4 = \mathrm{conv}\{w_{j-1}, w_j, w_{j+1}, v_4\}$ by cutting off the triangle $\triangle v_4w_{j-2}w_{j+2}$ along $w_{j-2}w_{j+2}$. Now the procedure of the proof of Lemma 11 is applied to P_4 and T, more precisely to $v_1 = w_{j-1}, v_2 = w_j, v_3 = w_{j+1}$, and v_4. This is possible, because the orthogonal projection of v_2 onto $l(v_1, v_3)$ is in the relative interior of v_1v_3 by Step 2. As we have seen in that proof, we can choose $\lambda > 0$ arbitrarily small. This way we obtain x, y sufficiently close to v_1 and v_3, respectively, such that $x \in w_{j-1}w_{j-2} \setminus \{w_{j-2}\}$ and $y \in w_{j+1}w_{j+2} \setminus \{w_{j+2}\}$. Hence the piece $T_4 = \triangle v_4xy$ of P_4 splits into a quadrilateral Q_4 and $\triangle v_4w_{j-2}w_{j+2}$. This way we obtain dissections of P_5 into $Q_1, \ldots, Q_4$ and of T into images of $Q_1, \ldots, Q_4$, and of $\triangle v_4w_{j-2}w_{j+2}$ under suitable maps from Sim^+. The image of $\triangle v_4w_{j-2}w_{j+2}$ is the desired triangle $\hat{T}$. $\square$

Lemma 13. $\deg_{\mathrm{Sim}^+}(3, n + 3) \leq \deg_{\mathrm{Sim}^+}(3, n) + 4$ *for every* $n \geq 3$.

Proof. Let T be a triangle and P_{n+3} a convex $(n + 3)$-gon. We split P_{n+3} into a convex pentagon $P_{5,1}$ and a convex n-gon $P_{n,2}$. By Lemma 12, there exists a

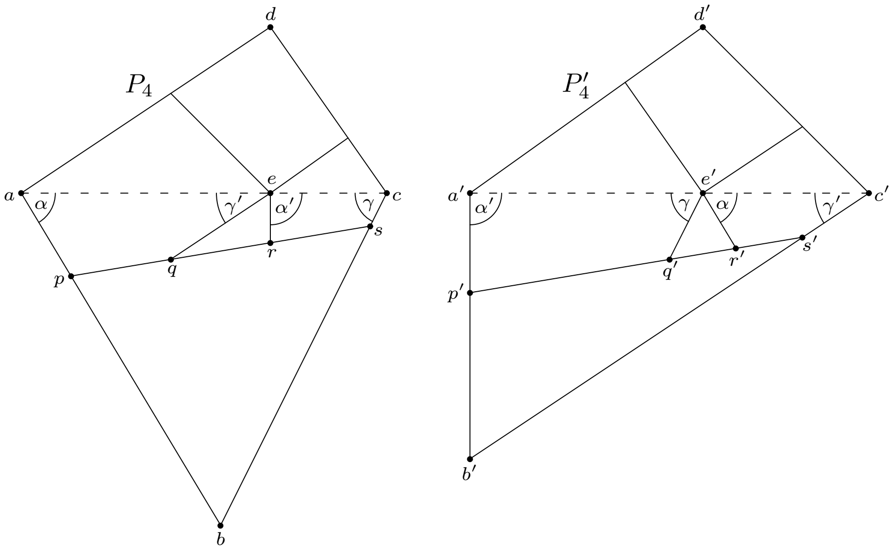

FIGURE 9. Proof of Lemma 16; dissections of P_4 and P'_4

triangle $\hat{T} \subseteq T$ such that $\deg_{\mathrm{Sim}^+}(\mathrm{cl}(T \setminus \hat{T}), P_{5,1}) \leq 4$. Hence

$$\deg_{\mathrm{Sim}^+}(T, P_{n+3}) \leq \deg_{\mathrm{Sim}^+}(\mathrm{cl}(T \setminus \hat{T}), P_{5,1}) + \deg_{\mathrm{Sim}^+}(\hat{T}, P_{n,2}) \leq 4 + \deg_{\mathrm{Sim}^+}(3, n).$$

□

Lemma 14. *Given a triangle T and a real number $0 < \delta < \frac{2\pi}{3}$, there exists a dissection of T into two triangles T_1, T_2 such that T_1 has an angle of size δ.*

Proof. Let a, b, c be the vertices of T and let $\alpha \leq \beta \leq \gamma$ be the sizes of the corresponding angles. Of course, $\alpha \leq \frac{\pi}{3} \leq \gamma$. For every $x \in ab \setminus \{a, b\}$, T is dissected into $T_{1,x} = \triangle axc$ and $T_{2,x} = \triangle bxc$. When x runs through $ab \setminus \{a, b\}$, $\angle acx$ attains all sizes in $(0, \gamma)$ and $\angle axc$ all sizes in $(\beta, \pi - \alpha) \supseteq \left(\gamma, \frac{2\pi}{3}\right)$. The remaining size γ is obtained by dissecting T along a straight line through a or b. □

Lemma 15. $\deg_{\mathrm{Sim}^+}(3, 5) \leq 6$.

Proof. Let a triangle T and a convex pentagon P_5 be given. We cut P_5 along a diagonal into a triangle T_0 containing the largest angle of P_5 and into a quadrilateral Q. So T_0 has an angle of size $\eta \geq \frac{3\pi}{5}$. By Lemma 14, T splits into two triangles T_1, T_2 such that T_1 has an angle of size $\pi - \eta$. Now Lemmas 9 and 11 yield

$$\deg_{\mathrm{Sim}^+}(T, P_5) \leq \deg_{\mathrm{Sim}^+}(T_1, T_0) + \deg_{\mathrm{Sim}^+}(T_2, Q) \leq 2 + 4 = 6.$$

□

Lemmas 10, 11, 13, and 15 yield the following estimate for $\deg_{\mathrm{Sim}^+}(3, n)$.

Corollary 3. $\deg_{\mathrm{Sim}^+}(3, n) \leq \left\lfloor \frac{4n-2}{3} \right\rfloor$ *for every* $n \geq 3$.

Lemma 16. $\deg_{\mathrm{Sim}^+}(4, 4) \leq 5$.

Proof. Let two convex quadrilaterals P_4, P'_4 be given. Following the counterclockwise order, we denote their vertices by a, b, c, d and a', b', c', d', respectively. We can assume that the angles at d and d' are not acute. Then the orthogonal projections e and e' of d and d' onto $l(a, c)$ and $l(a', c')$ belong to the relative interiors of ac and $a'c'$, respectively. For the sake of convenience we assume ac and $a'c'$ to be collinear. Figure 9 illustrates dissections of P_4 and P'_4, each consisting of one quadrilateral, two pentagons, and two triangles. The dissections of $\triangle acd$ and $\triangle a'c'd'$ are con-

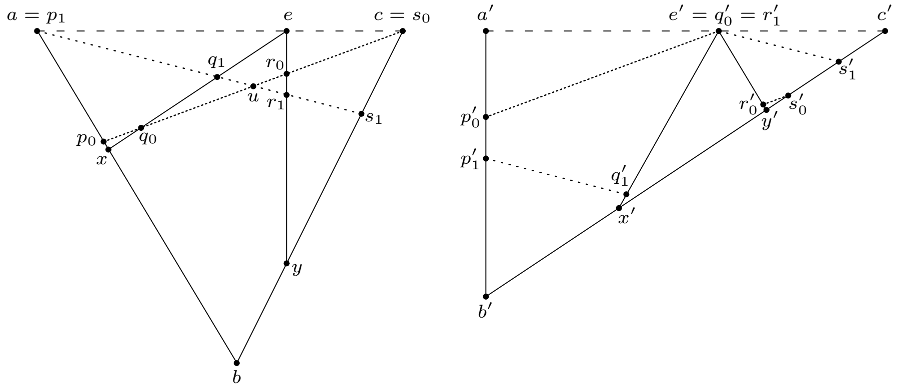

Figure 10. Proof of Lemma 16; the choice of p, s, p', s'

structed in the same way as in Figure 6. The choice of the segments ps and $p's'$ has to be explained.

We fix $x, y \in ab \cup bc$ and $x', y' \in a'b' \cup b'c'$ such that $|\angle aex| = |\angle e'c'b'|$, $|\angle cey| = |\angle e'a'b'|$, $|\angle a'e'x'| = |\angle ecb|$, and $|\angle c'e'y'| = |\angle eab|$ (see Figures 9 and 10). The transformations $\varphi_1, \varphi_2 \in \mathrm{Sim}^+$ mapping the two triangular pieces of $\triangle acd$ onto the respective ones of $\triangle a'c'd'$ are defined by $\varphi_1(a) = e'$, $\varphi_1(e) = c'$ and $\varphi_2(e) = a'$, $\varphi_2(c) = e'$. We pick a point $p_0 \in ab \setminus \{a, b\}$ relatively close to a. The segment $p_0c = p_0s_0$ intersects ex in q_0 and ey in r_0. We define two segments $p_0'q_0'$ and $r_0's_0'$ parallel to p_0s_0 by $p_0' = \varphi_2(r_0)$, $q_0' = \varphi_2(s_0) = e'$, $r_0' = \varphi_1(p_0)$, and $s_0' = \varphi_1(q_0)$ (see Figure 10). Similarly, we fix some $s_1 \in bc \setminus \{c, b\}$ close to c, obtain intersections q_1 and r_1 of $as_1 = p_1s_1$ with ex and ey, respectively, and define segments $p_1'q_1'$ and $r_1's_1'$ parallel to p_1s_1 by $p_1' = \varphi_2(r_1)$, $q_1' = \varphi_2(s_1)$, $r_1' = \varphi_1(p_1) = e'$, and $s_1' = \varphi_1(q_1)$. (If p_0 and s_1 are sufficiently close to a and c, respectively, then $p_0'q_0', r_0's_0', p_1'q_1', r_1's_1' \subseteq \triangle a'b'c'$.) Let $\{u\} = p_0s_0 \cap p_1s_1$. We consider the family of segments ps passing through u such that p ranges in p_0p_1 and s in s_0s_1. Corresponding points q, r, p', q', r', s' are defined as above. By the intermediate value theorem there is a choice of ps such that p', q', r', s' are collinear. This position is used for the dissections given in Figure 9.

$\triangle bsp$ and $\triangle e'q'r'$ are congruent with respect to Sim^+, because their edges are pairwise parallel. The same argument applies to $\triangle eqr$ and $\triangle b's'p'$. The similarities of the quadrilaterals $apqe$ and $e'r's'c'$ under φ_1 and of $ersc$ and $a'p'q'e'$ under φ_2 show that these quadrilaterals can be combined with the corresponding triangular parts of $\triangle acd$ and $\triangle a'c'd'$ this way constituting two pairs of respectively similar pentagons. This gives the required congruence by dissection of P_4 and P_4'. $\square$

3.3. **Main result on Sim^+.**

Theorem 5. *For arbitrary* $3 \le m \le n$,

$$\deg_{\mathrm{Sim}^+}(m, n) \le \begin{cases} \left\lfloor \frac{5m-9}{2} \right\rfloor = \left\lfloor \frac{7m+8n-27}{6} \right\rfloor & \text{if } n = m, \\ \left\lfloor \frac{7m+8n-24}{6} \right\rfloor & \text{if } n > m. \end{cases}$$

Proof. Let a convex m-gon P_m and a convex n-gon P_n' be given.

Case 1. $n = m$ *and* $m = 2k + 1$ *is odd* ($k \ge 1$). Both P_m and P_m' are dissected into $k - 1$ quadrilaterals and one triangle, $P_m = Q_1 \cup \ldots \cup Q_{k-1} \cup T$, $P_m' = Q_1' \cup \ldots \cup Q_{k-1}' \cup T'$. Hence, by Lemmas 16 and 10,

$$\deg_{\mathrm{Sim}^+}(P_m, P_m') \le \sum_{i=1}^{k-1} \deg_{\mathrm{Sim}^+}(Q_i, Q_i') + \deg_{\mathrm{Sim}^+}(T, T') \le 5(k-1) + 3 = \tfrac{5m-9}{2}.$$

Case 2. $n = m$ and $m = 2k$ is even ($k \geq 2$). Both P_m and P'_m are dissected into $k-1$ quadrilaterals, $P_m = Q_1 \cup \ldots \cup Q_{k-1}$, $P'_m = Q'_1 \cup \ldots \cup Q'_{k-1}$. By Lemma 16,

$$\deg_{\mathrm{Sim}^+}(P_m, P'_m) \leq 5(k-1) = \left\lfloor 5(k-1) + \tfrac{1}{2} \right\rfloor = \left\lfloor \tfrac{5m-9}{2} \right\rfloor .$$

Case 3. $n > m$ and $m = 2k+1$ is odd ($k \geq 1$). P_m is dissected into $k-1$ quadrilaterals $Q_1, \ldots, Q_{k-1}$ and one triangle T. P'_n is cut into $k-1$ quadrilaterals $Q'_1, \ldots, Q'_{k-1}$ and one $(n-2(k-1))$-gon P'_{n-2k+2}. By Lemma 16 and Corollary 3,

$$\begin{aligned} \deg_{\mathrm{Sim}^+}(P_m, P'_n) &\leq \textstyle\sum_{i=1}^{k-1} \deg_{\mathrm{Sim}^+}(Q_i, Q'_i) + \deg_{\mathrm{Sim}^+}(T, P'_{n-2k+2}) \\ &\leq 5(k-1) + \left\lfloor \tfrac{4(n-2k+2)-2}{3} \right\rfloor = \left\lfloor \tfrac{7k+4n-9}{3} \right\rfloor \\ &= \left\lfloor \tfrac{7k+4n-9}{3} + \tfrac{1}{6} \right\rfloor = \left\lfloor \tfrac{7m+8n-24}{6} \right\rfloor. \end{aligned}$$

Case 4. $n > m$ and $m = 2k$ is even ($k \geq 2$). P_m is dissected into $k-2$ quadrilaterals $Q_1, \ldots, Q_{k-2}$ and two triangles T_1, T_2. P'_n is cut into $k-1$ quadrilaterals $Q'_1, \ldots, Q'_{k-1}$ and one $(n-2(k-1))$-gon P'_{n-2k+2}. By Lemmas 16 and 11 and Corollary 3,

$$\begin{aligned} &\deg_{\mathrm{Sim}^+}(P_m, P'_n) \\ &\quad\leq \textstyle\sum_{i=1}^{k-2} \deg_{\mathrm{Sim}^+}(Q_i, Q'_i) + \deg_{\mathrm{Sim}^+}(T_1, Q'_{k-1}) + \deg_{\mathrm{Sim}^+}(T_2, P'_{n-2k+2}) \\ &\quad\leq 5(k-2) + 4 + \left\lfloor \tfrac{4(n-2k+2)-2}{3} \right\rfloor = \left\lfloor \tfrac{7k+4n-12}{3} \right\rfloor = \left\lfloor \tfrac{7m+8n-24}{6} \right\rfloor. \end{aligned}$$

□

The two bounds $b_{(2)}(m,n) = 3(n-2)$ from (2) and $b_{\mathrm{Thm5}}(m,n)$ from Theorem 5 agree for $m = n = 3$, where the value $b_{(2)}(3,3) = b_{\mathrm{Thm5}}(3,3) = 3$ is optimal by Corollary 2. For all other $3 \leq m \leq n$, $b_{\mathrm{Thm5}}(m,n) < b_{(2)}(m,n)$. In particular, if $((m_i, n_i))_{i=1}^{\infty}$ is a sequence such that $3 \leq m_i \leq n_i$ and $\lim_{i\to\infty} n_i = \infty$ then

$$\limsup_{i\to\infty} \frac{b_{\mathrm{Thm5}}(m_i,n_i)}{b_{(2)}(m_i,n_i)} = \frac{4}{9} + \frac{7}{18} \limsup_{i\to\infty} \frac{m_i}{n_i} \in \left[\frac{4}{9}, \frac{5}{6}\right].$$

3.4. **Congruence by dissection with respect to general similarities.**

Lemma 17. *$\deg_{\mathrm{Sim}}(3,n) \leq n+3$ for every $n \geq 3$.*

Proof. Let a triangle $T = \triangle abc$ and a convex n-gon P_n be given. We assume the angle of T at c to be not smaller than those at a and b. The vertices of P_n, ordered according to the same orientation as a, b, c, are denoted $v_1, \ldots, v_n$. The following assumptions are possible without loss of generality: $P_n \subseteq T \setminus (ac \cup bc)$. The orthogonal projection of v_1 onto $l(v_2, v_n)$ is in the relative interior of $v_2 v_n$. $v_2 v_n$ is parallel to ab. $v_1 \in \mathrm{int}(T)$ and cv_1 is perpendicular to ab. P_n meets ab either in a single vertex v_k or in an edge $v_k v_{k+1}$ where $k \geq 2$. The size of P_n compared with that of T is small (that is, P_n is contained in a sufficiently small neighbourhood of the intersection of ab with the perpendicular straight line through c). Figure 11 illustrates the situation.

Let $\{l_2\} = l(c, v_1) \cap l(a, v_2)$ and $\{r_n\} = l(c, v_1) \cap l(b, v_n)$. By the above assumptions, $l_2, r_n \in \triangle v_1 v_2 v_n \setminus (v_1 v_2 \cup v_1 v_n)$. For $3 \leq i \leq k$, we define $l_i \in l(a, v_{i-1}) \cap P_n$ such that the angles $\angle v_{i-1} a v_i$ and $\angle v_{i-1} v_i l_i$ are of the same size. (This is possible if P_n is sufficiently small compared with T, since then the angles $\angle v_{i-1} a v_i$ are small.) Similarly, for $k \leq j \leq n-1$ in the case $T \cap \mathrm{bd}(P_n) = \{v_k\}$ and for $k+1 \leq j \leq n-1$ if $T \cap \mathrm{bd}(P_n) = v_k v_{k+1}$, we pick $r_j \in l(b, v_{j+1}) \cap P_n$ such that $\angle v_{j+1} b v_j$ and $\angle v_{j+1} v_j r_j$ are congruent.

Now P_n is dissected into the triangles $\triangle v_1 v_2 l_2$, $\triangle v_1 v_n r_n$, $\triangle v_{i-1} v_i l_i$ with $3 \leq i \leq k$, $\triangle v_{j+1} v_j r_j$ with $k \leq j \leq n-1$, and into a remaining polygon Q. (Note that the triangles do not overlap if P_n is sufficiently small compared with T. In the case $T \cap \mathrm{bd}(P_n) = v_k v_{k+1}$ the triangle $\triangle v_{k+1} v_k r_k$ does not appear. Q vanishes if P_n is a triangle, since then $v_2, v_3 \in ab$. These particular cases give rise to dissections into a smaller number of pieces.) The triangle T splits into $\triangle cal_2$, $\triangle cbr_n$, $\triangle av_i l_i$

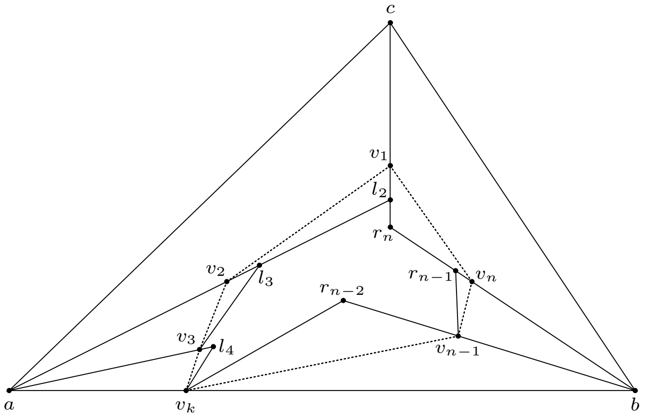

Figure 11. Proof of Lemma 17

for $3 \le i \le k$, $\triangle b v_j r_j$ with $k \le j \le n-1$, and into Q. (As above, $\triangle b v_k r_k$ and Q vanish if the respective parts of P_n do not appear.) Hence

$$\begin{aligned}\deg_{\mathrm{Sim}}(T, P_n) \;\le\; & \deg_{\mathrm{Sim}}(\triangle c a l_2, \triangle v_1 v_2 l_2) + \deg_{\mathrm{Sim}}(\triangle c b r_n, \triangle v_1 v_n r_n) + \deg_{\mathrm{Sim}}(Q, Q) \\ & + \textstyle\sum_{i=3}^{k} \deg_{\mathrm{Sim}}(\triangle a v_i l_i, \triangle v_{i-1} v_i l_i) + \sum_{j=k}^{n-1} \deg_{\mathrm{Sim}}(\triangle b v_j r_j, \triangle v_{j+1} v_j r_j).\end{aligned}$$

Lemma 8 applies to $\triangle c a l_2$ and $\triangle v_1 v_2 l_2$, because they have a common angle. The same is satisfied for $\triangle c b r_n$ and $\triangle v_1 v_n r_n$. For $3 \le i \le k$, $\triangle a v_i l_i$ and $\triangle v_{i-1} v_i l_i$ are similar, since they have a common angle at l_i and $\angle l_i a v_i (= \angle v_{i-1} a v_i)$ is of the same size as $\angle v_{i-1} v_i l_i$. For $k \le j \le n-1$, $\triangle b v_j r_j$ and $\triangle v_{j+1} v_j r_j$ are similar by analogous arguments. Thus the above estimate can be continued to

$$\deg_{\mathrm{Sim}}(T, P_n) \le 2 + 2 + 1 + (k-2) + \big((n-1) - (k-1)\big) = n + 3.$$

This completes the proof. ☐

Theorem 6. *For arbitrary* $3 \le m \le n$*,*

$$\deg_{\mathrm{Sim}}(m, n) \le \begin{cases} n + 3 & \text{if } m = 3, \\ m + n + \lfloor \frac{m}{3} \rfloor & \text{if } 4 \le m \le 11, \\ m + n + 4 & \text{if } m \ge 12. \end{cases}$$

Proof. Lemma 17 gives the estimate for $m = 3$. Now let $4 \le m \le n$ and let a convex m-gon P_m and a convex n-gon P'_n be given. We split P_m into a triangle T and an $(m-1)$-gon P_{m-1}. P'_n is cut into an $(n-1)$-gon P'_{n-1} and a triangle T'. By Lemma 17 and Corollary 3,

$$\begin{aligned}\deg_{\mathrm{Sim}}(P_m, P'_n) &\le \deg_{\mathrm{Sim}}(T, P'_{n-1}) + \deg_{\mathrm{Sim}}(P_{m-1}, T') \\ &\le \big((n-1) + 3\big) + \min\left\{(m-1) + 3, \left\lfloor \tfrac{4(m-1)-2}{3} \right\rfloor\right\} \\ &= m + n + \min\left\{4, \left\lfloor \tfrac{m}{3} \right\rfloor\right\}.\end{aligned}$$

This proves the theorem. ☐

The bound from Theorem 6 is better than that from Theorem 5 (and hence improves (2)) if $m + n$ is sufficiently large. Theorem 5 is stronger for small m, n.

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