Mean-Field Backward Stochastic Differential Equations

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1 Introduction

$(\Omega, \mathcal{F}, P)$ - a complete probability space; $W$ - a $d$-dimensional Brownian motion over $(\Omega, \mathcal{F}, P)$; $T > 0$ : fixed time horizon.

Objective of the talk: Study of a “mean-field”-version of the “classical” BSDE.

Recall: “Classical” BSDE (E.Pardoux, S.Peng, 1990)

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t, \quad t \in [0, T], Y_T = \xi \in L^2(\mathcal{F}^W_T),$$

$f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ measurable, $\mathcal{F}^W$-progressively measurable, Lipschitz in $(y, z)$, uniformly in $(\omega, t) \in \Omega \times [0, T]$.

Remark: • If $f = 0$: Martingale Representation property:

$$Y_t = E[\xi | \mathcal{F}^W_t], \quad Z_t = E[D_t^W \xi | \mathcal{F}^W_t], \quad t \in [0, T];$$

• If $\xi$ and $f$ are deterministic:

$$dY_t = -f(t, Y_t, 0)dt, \quad t \in [0, T], Y_T = \xi.$$
Vast fields of application: stochastic control, PDEs, finance.

Our goal: Extension to “mean-field” type BSDEs.

Why?: Our motivation stems from:

1) Classical mean-field approaches in:
   - Statistical Mechanics and Physics (e.g., derivation of Boltzmann or Vlasov equations in the kinetic gas theory),
   - In Quantum Mechanics and Quantum Chemistry (e.g., the density functional models or also Hartree and Hartree-Fock type models);

What is a “mean-field” BSDE? Classical “mean-field” type SDE obtained by an average limit procedure: McKean-Vlasov SDE

\[ X_t = x_0 + \int_0^t E[b(x, X_s)]|_{x=X_s} \, ds + \int_0^t E[\sigma(x, X_s)]|_{x=X_s} \, dW_s, \quad t \in [0, T], \]

in general written in the form:

\[ X_t = x_0 + \int_0^t \int_{\mathbb{R}^d} b(X_s, x) P_{X_s}(dx) \, ds + \int_0^t \int_{\mathbb{R}^d} \sigma(X_s, x) P_{X_s}(dx) \, dW_s, \quad t \in [0, T]; \]

existence and uniqueness for \( b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times d} \) Lipschitz.

Mean-Field BSDE (MFBSDE) (driven by the solution $X$ of the McKean-Vlasov equation):

$$Y_t = E \left[ \Phi(x, X_T) \right]_{x=X_T} + \int_t^T E \left[ f(\lambda, \Lambda_s) \right]_{\lambda=\Lambda_s} ds - \int_t^T Z_s dW_s, \ t \in [0, T],$$

with $\Lambda = (X, Y, Z)$, or in the more classical form:

$$\left( Y_t = \int_{R^d} \Phi(X_T, x) P_{X_T} (dx) + \int_t^T \int_{R^M} f(\Lambda_s, \lambda) P_{\Lambda_s} (d\lambda) ds - \int_t^T Z_s dW_s \right),$$

where $f : R^M \times R^M \to R$ ($M := d + 1 + d$), $\Phi : R^d \times R^d \to R$ are Lipschitz. Existence and uniqueness of the solution $\Lambda = (X, Y, Z)$ is obvious.
Objectives of the talk:

- Discussion of general properties of MFBSDEs; their interpretation as generalized Feynman-Kac formula for associated nonlocal PDEs; based on:
  “Mean-Field Backward Stochastic Differential Equations and Related Partial Differential Equations”, accepted for publication in: Stochastic Processes and their Applications);

- Discussion of the MFBSDE as the limit over a sequence of “nearly classical” BSDEs; based on:
  “Mean-Field Backward Stochastic Differential Equations. A Limit Approach” (Bouamel Djehiche, Juan Li, Shige Peng, accepted for publication in: Annals of Probability).
2 MFBSDEs. Properties

Setting: Recall: \((\Omega, \mathcal{F}, P)\) - a complete probability space endowed with a \(d\)-dimensional Brownian motion \(W\);
\[(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)\] - (non-completed) tensorial product of \((\Omega, \mathcal{F}, P)\) with itself;
filtration: \(\bar{\mathcal{F}} = \{\bar{\mathcal{F}}_t = \mathcal{F} \otimes \mathcal{F}_t^W, \ 0 \leq t \leq T\}\);

- \(\xi \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)\) canonically extended to \(\bar{\Omega}\):
  \[\xi'(\omega', \omega) = \xi(\omega'), \ (\omega', \omega) \in \bar{\Omega} = \Omega \times \Omega.\]

- \(\theta \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}): \theta(., \omega) \in L^1(\Omega, \mathcal{F}, P), \ P(d\omega)\)-a.s.; its expectation:
  \[E'[\theta(., \omega)] = \int_{\Omega} \theta(\omega', \omega) P(d\omega').\]

Notice: \(E'[\theta] = E'[\theta(., \omega)] \in L^1(\Omega, \mathcal{F}, P)\), and
\[
\bar{E}[\theta](= \int_{\bar{\Omega}} \theta d\bar{P}) = \int_{\Omega} E'[\theta(., \omega)] P(d\omega)) = E[E'[\theta]].
\]
Data of our MFBSDE:

- \( f = f(\omega', \omega, t, y', z', y, z) : \bar{\Omega} \times [0, T] \times R \times R^d \times R \times R^d \to R \)
- \( \bar{\Omega} \)-progressively measurable; Lipschitz in \((y', z', y, z)\), uniformly in \((\omega', \omega, t)\); \( f(\cdot, 0, 0, 0, 0) \in L^2(dt d\bar{P}) \);
- \( \xi \in L^2(\Omega, \mathcal{F}, P) \).

Mean-Field BSDE:

\[
Y_t = \xi + \int_t^T E'[f(s, Y'_s, Z'_s, Y_s, Z_s)] ds - \int_t^T Z_s dW_s, \, t \in [0, T]
\]

Remark: recall that \( Y'_s(\omega', \omega) := Y_s(\omega') \); ditto for \( Z' \); so:

\[
E'[f(s, Y'_s, Z'_s, Y_s, Z_s)] = \int_\Omega f(\omega', \omega, s, Y_s(\omega'), Z_s(\omega'), Y_s(\omega), Z_s(\omega)) P(d\omega'),
\]

and if \( f \) doesn't depend on \((\omega', \omega)\):

\[
E'[f(s, Y'_s, Z'_s, Y_s, Z_s)] = \int_{R \times R^d} f(s, y, z, Y_s(\omega), Z_s(\omega)) P_{Y, Z}(dydz).
\]
Proposition: Existence and uniqueness of the solution \((Y, Z) \in S^2_{\mathcal{F}_T} (0, T) \times L^2_{\mathcal{F}_T} (0, T; \mathbb{R}^d)\).

Proof: “Ox trot” proof; indeed:

\[
\tilde{E} \left[ |E' [f (s, Y'_s, Z'_s, Y_s, Z_s)] - E' [f (s, U'_s, V'_s, U_s, V_s)]|^2 \right] \\
\leq CE \left[ |Y_s - U_s|^2 + |Z_s - V_s|^2 \right],
\]

for all \((Y, Z), (U, V) \in S^2_{\mathcal{F}_T} (0, T) \times L^2_{\mathcal{F}_T} (0, T; \mathbb{R}^d)\).

**Comparison principle**: What can we expect? Two examples:

Example 1 (\(f\) depending on \(z'\)): For \(d = 1\) (\(W\) 1-dim. BM), \(T = 1\),

\[
f (\omega', \omega, s, y', z', y, z) = -z', \quad \xi^i \in L^2 (\Omega, \mathcal{F}_1, P), \quad i = 1, 2;
\]

the MFBSDE takes the form:

\[
Y^i_t = \xi^i + \int_t^1 E' [-Z'^i_s] ds - \int_t^1 Z^i_s dW_s, \quad t \in [0, 1], \quad i = 1, 2.
\]

Let

\[
\xi_1 := -(W_1^+)^3, \quad \tilde{Y}^1_t := Y^1_t + \int_t^1 E[Z^1_s] ds.
\]
Then \((\tilde{Y}^1, Z^1)\): unique solution of the BSDE

\[
\tilde{Y}^1_t = \xi_1 - \int_t^1 Z^1_s dW_s, \ t \in [0, 1],
\]

and,

\[
\tilde{Y}^1_0 = E[\xi_1] = -E[(B^+_1)^3] = -\frac{2}{\sqrt{2\pi}},
\]

\[
E[Z^1_t] = E[D_t \xi] = E[-3(B^+_1)^2] = -\frac{3}{2}, \ t \in [0, 1],
\]

from where,

\[
Y^1_0 = \tilde{Y}^1_0 - \int_0^T E[Z^1_s] ds = -\frac{2}{\sqrt{2\pi}} + \frac{3}{2} > 0.
\]

On the other hand, for \(\xi_2 := 0\): \((Y^2, Z^2) = (0, 0)\), and \(Y^1_0 > Y^2_0\) although \(\xi_1 \leq \xi_2\), and \(P\{\xi_1 < \xi_2\} > 0\).
Example 2 (\(f\) strictly decreasing in \(y'\)): For \(d = 1\) (\(W\) 1-dim. BM), \(T = 2\),

\[ f(\omega', \omega, s, y', z', y, z) = -y', \quad \xi^i \in L^2(\Omega, \mathcal{F}_2^W, P), \quad i = 1, 2; \]
the MFBSDE takes the form:

\[
Y_t^i = \xi^i + \int_t^2 E'[\!-Y_s'^{ii}]ds - \int_t^2 Z_s^i dW_s, \quad t \in [0, 2], \quad i = 1, 2. 
\]

Let

\[
\xi_1 := (B_1)^2; \quad \text{then:} \]

\[
E[Y_t^1] = e^{-(2-t)}, \quad t \in [0, 2], \quad \text{and} 
\]

\[
Y_t^1 = (B_1)^2 - \int_t^2 e^{-(2-s)}ds = (B_1)^2 - (1 - e^{-(2-t)}), \quad Z_t^1 = 0, \quad t \in [1, 2]; 
\]
Consequently,

\[
Y_1^1 = (B_1)^2 - (1 - e^{-1}) < 0 \quad \text{on the set} \quad \{(B_1)^2 < 1 - e^{-1}\} \quad \text{of strictly positive probability. For} 
\]
\[
\xi_2 := 0 : \quad (Y^2, Z^2) = (0, 0). 
\]
Therefore, $P(Y_1^1 < Y_2^2) > 0$, although $\xi_1 > \xi_2$.

The both examples explain the formulation of the comparison theorem:

**Comparison Theorem:** Let $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$, $f_i = f_i(\bar{\omega}, t, y', z', y, z)$, $i = 1, 2$, be $\mathbf{F}$-progressively measurable, Lipschitz in $(y', z', y, z)$, and:

- One of both coefficients is independent of $z'$;
- One of both coefficients is nondecreasing in $y'$.

If $\xi_1 \leq \xi_2$, $P$-a.s., and $f_1 \leq f_2$, then:

$Y_1^1 \leq Y_2^2$, $t \in [0, T]$.

**Converse Comparison Theorem:** Under the assumptions of the Comparison Theorem and the additional hypothesis that, for some $t \in [0, T]$, $Y_t^1 = Y_t^2$, $P$-a.s.:

- $Y_s^1 = Y_s^2$, $s \in [t, T]$, $P$-a.s., and
- If $f_2$ is nondecreasing in $y'$ then $E'[f_1(s, Y_s^1', Z_s^1', Y_s^2, Z_s^2')] = E'[f_2(s, Y_s^2', Z_s^2', Y_s^2, Z_s^2')]$, dsd$P$-a.e. on $[t, T]$. 
3 PDE related to MFBSDE

\[ b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d} \] continuous, Lipschitz in \((y', z', y, z)\), uniformly in \(t \in [0, T]\);

\(x_0 \in \mathbb{R}^n;\)

SDE parameterized by \((t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)\):

\[ dX^{t, \zeta}_s = E'[b(s, (X^{0,x_0}_s)', X^{t, \zeta}_s)]ds + E'[\sigma(s, (X^{0,x_0}_s)', X^{t, \zeta}_s)]dW_s, s \in [t, T], \]

\(X^{t, \zeta}_t = \zeta.\)

Existence and uniqueness:  
- \((t, \zeta) := (0, x_0)\): SDE of McKean-Vlasov type;
- After having determined \(X^{0,x_0}\): SDE becomes classical.
$f(t, x', x, y', y, z)$ and $\Phi(x', x)$:

- real-valued, continuous, Lipschitz in $(x', x, y', y, z)$, uniformly w.r.t. $t \in [0, T]$;
- $y' \to f(t, x', x, y', y, z)$ nondecreasing.

BSDE associated to our forward SDE:

$$dY_{t, \xi}^s = -E'[f(s, (X^0_s, x_0)', X^t_s, \xi, (Y^0_s, x_0)', Y^t_s, \xi, Z^t_s, \xi)]ds + Z^t_s, \xi dW_s, \ s \in [t, T],$$

$$Y^t_T = E'[\Phi((X^0_T, x_0)', X^t_T, \xi)].$$

Existence and uniqueness: ● $(t, \xi) := (0, x_0)$: Mean-Field BSDE;
- After having determined $(Y^{0, x_0}, Z^{0, x_0})$: BSDE becomes classical.
From standard estimates: \( \exists C \in R_+ \) only depending on the Lipschitz and the growth constants of \( b, \sigma, f \) and \( \Phi \), s.t., for all \( t \in [0, T] \), and \( \zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; R^n) \):

- \( E \left[ \sup_{s \in [t, T]} \left( |X_t^s, \zeta|^2 + |Y_t^s, \zeta|^2 \right) + \int_t^T |Z_t^s, \zeta|^2 ds | \mathcal{F}_t \right] \leq C(1 + |\zeta|^2), \text{ a.s.} \)

- \( E \left[ \sup_{s \in [t, T]} \left( |X_t^s, \zeta - X_t^s, \zeta'||^2 + |Y_t^s, \zeta - Y_t^s, \zeta'||^2 \right) + \int_t^T |Z_t^s, \zeta - Z_t^s, \zeta'|^2 ds | \mathcal{F}_t \right] \leq C|\zeta - \zeta'|^2, \text{ a.s.} \)

In particular,

- \( |Y_t^t, \zeta| \leq C(1 + |\zeta|), \text{ a.s.} \)
- \( |Y_t^t, \zeta - Y_t^t, \zeta'| \leq C|\zeta - \zeta'|, \text{ a.s.} \)
Introduction of the deterministic function $u : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$:

$$u(t, x) = Y_{t^\xi}^t|_{\xi=x}, \ (t, x) \in [0, T] \times \mathbb{R}^n.$$

From the estimates for $Y_{t^\xi}^t$:

- $|u(t, x) - u(t, y)| \leq C|x - y|$, for all $x, y \in \mathbb{R}^n$;
- $|u(t, x)| \leq C(1 + |x|)$, for all $x \in \mathbb{R}^n$.

Program for the study of $u$: 1) DPP (dynamic programming principle); byproduct: $u(., x)$ is $1/2$-Hölder in $t$; 2) $u$ as continuous viscosity solution of the associated nonlocal PDE. For this: extension of S.Peng’s idea of backward stochastic semigroups to MFBSDEs:
Backward stochastic semigroup: Given $0 \leq t < t + \delta \leq T$, $x \in \mathbb{R}^n$ and $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; R)$, we put:

$$G^{t,x}_{s,t+\delta}[\eta] := \tilde{Y}^{t,x}_s, s \in [t, t + \delta],$$

where

$$d\tilde{Y}^{t,x}_s = -E'[f(s, (X^{0,x_0}_s)' , X^{t,x}_s', (Y^{0,x_0}_s)' , Y^{t,x}_s, \tilde{Z}^{t,x}_s)] ds - \tilde{Z}^{t,x}_s dW_s, \quad s \in [t, t + \delta],$$

$$\tilde{Y}^{t,x}_{t+\delta} = \eta;$$

$X^{t,x}$, $Y^{t,x}$ - the solutions introduced above.

Obviously, for the solution $(Y^{t,x}, Z^{t,x})$:

$$G^{t,x}_{t,T}[\Phi(X^{t,x}_T)] = G^{t,x}_{t,t+\delta}[Y^{t,x}_{t+\delta}], 0 \leq t < t + \delta \leq T.$$ 

Remark: S. Peng’s definition of backward semigroup associated with $X^{t,x}$; now: definition of backward semigroup associated with $(X^{t,x}, Y^{t,x})$ (but not with $Z^{t,x}$) $\rightarrow$ simplification of the approach.
DPP: for all $0 \leq t < t + \delta \leq T$, $x \in \mathbb{R}^n$, 

$$u(t, x) = Y_{t,x}^t = G_{t,T}^{t,x} [\Phi(X_T^{t,x})] = G_{t,t+\delta}^{t,x} [Y_{t+\delta}^{t,x}] = G_{t,t+\delta}^{t,x} [u(t + \delta, X_{t+\delta}^{t,x})].$$

For the latter relation we have used:

$$Y_{t+\delta}^{t,x} = Y_{t+\delta, X_{t+\delta}^{t,x}}^{t,x}.$$

Special case: $f$ independent of $(y, z)$:

$$G_{s,t+\delta}^{t,x} [\eta] = E[\eta + \int_s^{t+\delta} E'[f(r, (X_r^{0,x_0})', X_r^{t,x}, (Y_r^{0,x_0})')] dr | \mathcal{F}_s], s \in [t, t + \delta].$$

From DDP it can be derived that:

**Proposition.** $u,(.,x)$ is $1/2$-Hölder in $t$, i.e., for all $x \in \mathbb{R}^n$, $t, t' \in [0, T]$,

$$|u(t, x) - u(t', x)| \leq C (1 + |x|)|t - t'|^{\frac{1}{2}}.$$

**Corollary.** $u \in C_{\ell}([0, T] \times \mathbb{R}^n)$. 
Related PDE:
\[
\frac{\partial}{\partial t} u(t,x) + Au(t,x) + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t,x), Du(t,x)E[\sigma(t, X_t^{0,x_0}, x)]] = 0, \\
(t,x) \in [0,T) \times \mathbb{R}^n,
\]
\[
u(T,x) = E[\Phi(X_T^{0,x_0}, x)], x \in \mathbb{R}^n,
\]
with
\[
Au(t,x) := \frac{1}{2} tr(E[\sigma(t, X_t^{0,x_0}, x)]E[\sigma(t, X_t^{0,x_0}, x)]^T D^2 u(t,x)) + Du(t,x).E[b(t, X_t^{0,x_0}, x)]
\]

Notice: Since
\[
E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t,x), Du(t,x)E[\sigma(t, X_t^{0,x_0}, x)]]]
\]
\[
= \int_{\mathbb{R}^n} f(t, x', x, u(t, x'), u(t,x), Du(t,x).E[\sigma(t, X_t^{0,x_0}, x)])P_{X_t^{0,x_0}}(dx'),
\]
the above equation is indeed nonlocal.
Objective: Characterization of \( u(t, x) = Y_t^{t,x} \) as viscosity solution of the above nonlocal PDE

\[
\frac{\partial}{\partial t} u(t, x) + A u(t, x) + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), D u(t, x) E[\sigma(t, X_t^{0,x_0}, x)]]] = 0, \\
u(T, x) = E[\Phi(X_T^{0,x_0}, x)], \ x \in \mathbb{R}^n,
\]

Recall: \( u \in C_\ell([0, T] \times \mathbb{R}^n) \).

**Definition.** \( u \in C_p([0, T] \times \mathbb{R}^n) \) is called

- **viscosity subsolution** if
  1) \( u(T, x) \leq E[\Phi(X_T^{0,x_0}, x)], \ x \in \mathbb{R}^n, \) and
  2) for all \( \phi \in C_{\ell,b}^3([0, T] \times \mathbb{R}^n) \) and \( (t, x) \in [0, T) \times \mathbb{R}^n \) such that \( u - \phi \) attains a local maximum at \( (t, x) \):

\[
\frac{\partial}{\partial t} \phi(t, x) + D \phi(t, x). E[b(t, X_t^{0,x_0}, x)] + \frac{1}{2} \text{tr}(E[\sigma(t, X_t^{0,x_0}, x)] E[\sigma(t, X_t^{0,x_0}, x)]^T D^2 \phi(t, x)) + E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), u(t, x), D \phi(t, x). E[\sigma(t, X_t^{0,x_0}, x)]]] \geq 0;
\]
• viscosity supersolution if
1) \( u(T, x) \geq E[\Phi(X_T^{0,x_0}, x)], x \in \mathbb{R}^n \), and
2) for all \( \varphi \in C^3_b([0,T] \times \mathbb{R}^n) \) and \((t,x) \in [0,T] \times \mathbb{R}^n\) such that \( u - \varphi \) attains a local minimum at \((t,x):(t,x)\):

\[
\frac{\partial}{\partial t} \varphi(t,x) + D\varphi(t,x).E[b(t,X_t^{0,x_0},x)]
\]
\[
+ \frac{1}{2} tr(E[\sigma(t,X_t^{0,x_0},x)]E[\sigma(t,X_t^{0,x_0},x)]^T D^2 \varphi(t,x))
\]
\[
+ E[f(t,X_t^{0,x_0},x,u(t,X_t^{0,x_0}),u(t,x),D\varphi(t,x).E[\sigma(t,X_t^{0,x_0},x)])] \leq 0;
\]

• viscosity solution if it is both a viscosity sub- and a supersolution.

Theorem. The function \( u(t, x) = Y_t^{t,x} \) is a viscosity solution of the above nonlocal PDE; uniqueness in \( C_p([0,T] \times \mathbb{R}^n) \).

Remarks. 1) Proof of the existence: by adaptation of S.Peng’s BSDE method;
2) Barles, Buckdahn, Pardoux: optimal class for uniqueness for (local) PDE:

\[ \Theta = \{ \varphi \in C([0, T] \times \mathbb{R}^n) : \exists \tilde{A} > 0 \text{ such that} \] 

\[ \lim_{|x| \to \infty} |\varphi(t, x)| \exp\{-\tilde{A} (\log |x|)^2\} = 0 \text{ uniformly in } t \in [0, T] \}. \]

However, \( u \in \Theta \), the coefficient \( E[f(t, X_t^{0,x_0}, x, u(t, X_t^{0,x_0}), y, z)] \) may be not well defined.
4 Approximation of MFBSDEs

Objective: an approximation for MFBSDEs governed by a McKean-Vlasov SDE. A first approach: For initial data $(0, x_0)$, approximation of the system:

$$X_t = x_0 + \int_0^t E'[b(X'_s, X_s)] ds + \int_0^t E'[^\sigma(X'_s, X_s)] dW_s,$$

$$Y_t = E'[\Phi(X'_T, X_T)] + \int_t^T E'[f(\Lambda', \Lambda_s)] ds - \int_t^T Z_s dW_s, t \in [0, T],$$

by

$$X_t^N = x_0 + \frac{1}{N} \sum_{j=1}^N \int_0^t b(X_{s}^{N-1,j}, X_s^N) ds + \frac{1}{N} \sum_{j=1}^N \int_0^t ^\sigma(X_{s}^{N-1,j}, X_s^N) dW_s,$$

$$Y_t^N = \frac{1}{N} \sum_{j=1}^N \Phi(X_{T}^{N-1,j}, X_T^N) + \frac{1}{N} \sum_{j=1}^N \int_0^t f(\Lambda_{s}^{N-1,j}, \Lambda_s^N) ds - \int_t^T Z_s^N dW_s,$$

where $\Lambda_{N,j} = (X_{N,j}, Y_{N,j}, Z_{N,j}), 1 \leq i \leq N$, i.i.d., with the same law as $\Lambda_N = (X_N, Y_N, Z_N)$ and independent of the BM $W$. 
Setting: We choose a special \((\Omega, \mathcal{F}, P)\):

- \(I\) - a countable set (to be specified later; \(0 \in I\))
- \(\Omega = C_0([0, T]; \mathbb{R}^d)^I\) is endowed with the product topology generated by the uniform convergence on its components \(C_0([0, T]; \mathbb{R}^d)\);
- \(\mathcal{B}(\Omega)\) - Borel \(\sigma\)-field over \(\Omega\);

\[
B = (W^i)_{i \in I} - \text{coordinate process:}
\]

\[
W^i_t(\omega) = \omega^i_t, \quad t \in [0, T], \omega \in \Omega, i \in I;
\]

\(P\) - Wiener measure over \((\Omega, \mathcal{B}(\Omega))\): the unique probability measure w.r.t. which the coordinates \(W^i, i \in I\), form a family of independent \(d\)-dimensional BMs;

- \(\mathcal{F} := \mathcal{B}(\Omega) \vee \mathcal{N}_P\).
- \(W := W^0\) - our driving BM; \(\mathcal{F} = \mathcal{F}^W \vee \mathcal{G}\) - filtration, where

\[
\mathcal{G} = \sigma\{W^i_t, t \in [0, T], i \in I \setminus \{0\}\} \vee \mathcal{N}_P.
\]

Observe: \((W, \mathcal{F})\) has the Martingale Representation Property.
Specification the countable index set $I$:

$I := \{ i \mid i \in \{1,2,3,\ldots\}^k, k \geq 1 \} \cup \{0\}$ - set of all finite sequences of naturals; for $i = (i_1, \ldots, i_k)$, $i' = (i'_1, \ldots, i'_{k'}) \in I$:

$$i \oplus i' = (i_1, \ldots, i_k, i'_1, \ldots, i'_{k'}) \in I \text{ (with } 0 \oplus i := i).$$

We also shall introduce a family of shift operators $\Theta^k : \Omega \to \Omega$, $k \geq 0$, over $\Omega$:

$$\Theta^k(\omega) = (\omega^{(k)} \oplus j)_{j \in I}, \quad \omega = (\omega^j)_{j \in I} \in \Omega, k \geq 0.$$

Observe: • $\Theta^k(\omega) \in \Omega$, i.e., $\Theta^k : \Omega \to \Omega$.

Illustration:

<table>
<thead>
<tr>
<th>$\omega^{(1)}$</th>
<th>$\omega^{(2)}$</th>
<th>$\omega^{(3)}$</th>
<th>$\omega^{(4)}$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega^{(1,1)}$, $\omega^{(1,2)}$, $\omega^{(1,3)}$, $\ldots$</td>
<td>$\omega^{(2,1)}$, $\omega^{(2,2)}$, $\ldots$</td>
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<tr>
<td>$\omega^{(1,1,1)}$, $\omega^{(1,1,2)}$, $\ldots$, $\omega^{(1,2,1)}$, $\ldots$</td>
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<td>$\omega^{(1,1,1,1)}$, $\omega^{(1,1,1,2)}$, $\ldots$, $\omega^{(1,1,2,1)}$, $\ldots$</td>
<td>$\ldots$</td>
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</tbody>
</table>
\( P_{\Theta^k} = P \); this allows to interpret \( \Theta^k \) as operator over \( L^0(\Omega, \mathcal{F}, P; R^\ell) \).

- For all \( \xi \in L^0(\Omega, \mathcal{F}, P) \), the r.v.s \( \Theta^k(\xi) \), \( k \geq 1 \), are i.i.d., of the same law as \( \xi \) and independent of \( W \).

This framework allows to write the approximation of

\[
X_t = x_0 + \int_0^t E'[b(X_s', X_s)] \, ds + \int_0^t E'[\sigma(X_s', X_s)] \, dW_s,
\]

\[
Y_t = E'[\Phi(X_T', X_T)] + \int_t^T E'[f(\Lambda', \Lambda_s)] \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T],
\]

in the form

\[
X^N_t = x_0 + \frac{1}{N} \sum_{j=1}^N \left( \int_0^t b(\Theta^j(X^{N-1}_s), X^N_s) \, ds + \int_0^t \sigma(\Theta^j(X^{N-1}_s), X^N_s) \, dW_s \right),
\]

\[
Y^N_t = \frac{1}{N} \sum_{j=1}^N \left( \Phi(\Theta^j(X^{N-1}_T), X^N_T) + \int_0^t f(\Theta^j(\Lambda^{N-1}_s), \Lambda^N_s) \, ds \right) - \int_t^T Z^N_s \, dW_s.
\]
Proposition. Under the Lipschitz assumption and that of boundedness on the coefficients, for \((X^0, Y^0, Z^0) = (x_0, 0, 0)\) the equations

\[
X^N_t = x_0 + \frac{1}{N} \sum_{j=1}^{N} \left( \int_0^t b(\Theta^j(X^{N-1}_s), X^N_s) \, ds + \int_0^t \sigma(\Theta^j(X^{N-1}_s), X^N_s) \, dW_s \right),
\]

\[
Y^N_t = \frac{1}{N} \sum_{j=1}^{N} \left( \Phi(\Theta^j(X^{N-1}_T), X^N_T) + \int_0^t f(\Theta^j(\Lambda^{N-1}_s), \Lambda^N_s) \, ds \right) - \int_t^T Z^N_s \, dW_s,
\]

possess a unique \(F\)-adapted solution \(\Lambda^N = (X^N, Y^N, Z^N)\). Moreover,

\[
E \left[ \sup_{t \in [0,T]} (|X^N_t - X_t|^2 + |Y^N_t - Y_t|^2) + \int_0^T |Z^N_t - Z_t|^2 \, dt \right] \leq \frac{C}{N}, \quad N \geq 1.
\]

Proof. i) Clear; ii) follows from the following estimate combined with SDE standard estimates (analogous for the BSDE):
\[ E \left[ \left| \frac{1}{N} \sum_{k=1}^{N} \sigma(\Theta^k(X_t), X_t) - E'[\sigma(X'_t, X_t)] \right|^2 \right] \]

(Independence between \( X_t \in \mathcal{F}_t^W \) and \( \Theta^k(X_t), k \geq 1 \))

\[
= \int_{\mathbb{R}^d} E \left[ \left| \frac{1}{N} \sum_{k=1}^{N} \sigma(\Theta^k(X_t), x) - E[\sigma(X_t, x)] \right|^2 \right] P_X(dx)
\]

\[
= \sum_{1 \leq i, j \leq d} \int_{\mathbb{R}^d} \text{Var} \left( \frac{1}{N} \sum_{k=1}^{N} (\sigma_{i,j}(\Theta^k(X_t), x)) \right) P_X(dx)
\]

\[
= \frac{1}{N} \sum_{1 \leq i, j \leq d} \int_{\mathbb{R}^d} \text{Var} (\sigma_{i,j}(X_t, x)) P_X(dx) \leq \frac{C}{N}
\]

(Recall: \( \Theta^k(X_t), k \geq 1 \), i.i.d., of the same law as \( X_t \)).
5 Central Limit Theorem for MFBSDEs

We have seen that, if the sequence $X^N, N \geq 1$, is defined as approximation of the McKean-Vlasov SDE, the speed of the convergence of $(X^N, Y^N, Z^N) \rightarrow (X, Y, Z)$ is of order $1/\sqrt{N}$, i.e., the sequence
$$\sqrt{N} (X^N - X, Y^N - Y, Z^N - Z), N \geq 1,$$
is bounded in $S^2_F \times S^2_F \times L^2_F$.

**Problem**: What is the limit behavior of $\sqrt{N} (X^N - X, Y^N - Y, Z^N - Z)$ as $N \rightarrow +\infty$?

In order to answer to this question: $(X^N, Y^N, Z^N)$ is considered as solution of the system

$$X^N_t = x_0 + \frac{1}{N} \sum_{j=1}^{N} \left( \int_0^t b(\Theta^j(X^N_s), X^N_s)ds + \int_0^t \sigma(\Theta^j(X^N_s), X^N_s)dW_s \right),$$

$$Y^N_t = \frac{1}{N} \sum_{j=1}^{N} \left( \Phi(\Theta^j(X^N_T), X^N_T) + \int_0^t f(\Theta^j(\Lambda^N_s), \Lambda^N_s)ds \right) - \int_t^T Z^N_s dW_s.$$
**Limit behavior of** $\sqrt{N}(X^N - X, Y^N - Y, Z^N - Z)$

**Assumptions:** $b : R^d \times R^d \to R^d$, $\sigma : R^d \times R^d \to R^{d \times d}$, $\Phi : R^d \times R^d \to R$ and $f : R^M \times R^M \to R$ ($M = d + 1 + d$) are bounded, continuously differentiable functions with bounded first order derivatives; a restriction on $f$: $f((x', y', z'), (x, y, z)) = f((x', y'), (x, y, z))$, for all $(x', y', z'), (x, y, z) \in R^d \times R \times R^d$.

A technical result which plays a crucial role:

**Lemma** There is some constant $C$ such that, $dt dP$-a.e.,

$$|Z_t| \leq C, \quad \text{and} \quad |Z_t^N| \leq C, \quad \text{for all } N \geq 1.$$

**Proof:** uses standard arguments from the Malliavin calculus.

**An immediate consequence:** For all $m \geq 1$ there is some constant $C_m$ such that, for all $t, t' \in [0, T]$ and all $N \geq 1$,

$$E[|Y_t^N - Y_{t'}^N|^{2m}] \leq C_m |t - t'|^m.$$
Notation: Let \( \xi = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}) \)
\[
= \{((\xi^{(1,i)}_t(x))_{1 \leq i \leq d}, (\xi^{(2,i,j)}_t(x))_{1 \leq i, j \leq d}, \xi^{(3)}(x), \xi^{(4)}_t(x, y, z)), (t, x, y, z) \\
\in [0, T] \times R^d \times R \times R^d \}
\]
be a \((d + d \times d + 2)\)-dimensional continuous zero-mean Gaussian field independent of \( W \) and s.t., with the notation \([\xi] = \xi - E[\xi] (\xi \in L^1(\Omega, \mathcal{F}, P))\),

\[
E \left[ \begin{pmatrix}
\xi^{(1)}_t(x) \\
\xi^{(2)}_t(x) \\
\xi^{(3)}(x) \\
\xi^{(4)}_t(x, y, z)
\end{pmatrix} \otimes \begin{pmatrix}
\xi^{(1)}_t(x') \\
\xi^{(2)}_t(x') \\
\xi^{(3)}(x') \\
\xi^{(4)}_t(x', y', z')
\end{pmatrix} \right] = \\
E \left[ \begin{pmatrix}
[b(x, X_t)] \\
[\sigma(x, X_t)] \\
[\Phi(x, X_T)] \\
[f((x, y, z), \Lambda_t)]
\end{pmatrix} \otimes \begin{pmatrix}
[b(x', X_{t'})] \\
[\sigma(x', X_{t'})] \\
[\Phi(x', X_T)] \\
[f((x', y', z'), \Lambda_{t'})]
\end{pmatrix} \right] ,
\]

\((t, x, y, z), (t', x', y', z') \in [0, T] \times R^d \times R \times R^d\)
(Recall: • for \( a = (a^i)_{1 \leq i \leq m}, b = (b^i)_{1 \leq i \leq m} \in \mathbb{R}^m, a \otimes b = (a^i b^j)_{1 \leq i,j \leq m} \in \mathbb{R}^{m \times m} \); • \( \xi_t^{(2)}(x), \sigma(x,x') \) regarded as \( d^2 \)-dim. vectors).

Let \( \overline{F} = F^W \lor \overline{F}_0 \), where \( \overline{F}_0 = \sigma\{\xi_s, s \in [0,T]\} ((W, \overline{F}) \text{ BM with MRP}). \)

Let \( \overline{\Lambda} = (\overline{X}, \overline{Y}, \overline{Z}) \) be the unique \( \overline{F} \)-progress. meas. solution of:

\[
\overline{X}_t = \int_0^t \xi_s^{(1)}(X_s) ds + \int_0^t \xi_s^{(2)}(X_s)dW_s + \\
\quad \quad + \int_0^t \left( E' \left[ (\nabla_{x'} b) (X'_s, X_s) \right] \overline{X}_s + E' \left[ (\nabla_x b) (X'_s, X_s) \overline{X}'_s \right] \right) ds \\
\quad \quad + \int_0^t \left( E' \left[ (\nabla_{x'} \sigma) (X'_s, X_s) \right] \overline{X}_s + E' \left[ (\nabla_x \sigma) (X'_s, X_s) \overline{X}'_s \right] \right) dW_s,
\]

\[
\overline{Y}_t = \left\{ \xi^{(3)}(X_T) + E'[\nabla_{x'} \Phi(X'_T, X_T)] \overline{X}_T + E[\nabla_x \Phi(X'_T, X_T) \overline{X}'_T] \right\}
\]

\[
+ \int_t^T \left( \xi^{(4)}_s (\Lambda_s) + E'[\nabla_{\lambda'} f (\Lambda'_s, \Lambda_s)] \overline{\Lambda}_s + \\
\quad \quad + E[\nabla_{\lambda} f (\Lambda'_s, \Lambda_s) \overline{\Lambda}'_s] \right) ds - \int_t^T \overline{Z}_s dW_s.
\]
Theorem: The sequence \( \left( \sqrt{N}(X^N - X, Y^N - Y, Z^N - Z) \right)_{N \geq 1} \) converges in law over \( C([0, T]; R^d) \times C([0, T]; R) \times L^2([0, T]; R^d) \) to \( \Lambda = (X, Y, Z) \).

Proof of pathwise continuity of \( \xi \) with the help of Kolmogorov's Continuity Criterion for multi-parameter processes; crucial for this: \( f((x, y, z), (x', y', z')) \) is independent of \( z' \); \( Z^N \) is bounded, uniformly w.r.t. \( N \geq 1 \), which allows to make estimates for the BSDEs like for forward SDEs; standard (forward) SDE estimates for \( X^N, Y^N \).
Proof of the theorem:

\[
\sqrt{N}(Y_t^N - Y_t) = \xi_{3,N}^3(X_T) + \sqrt{N} \left( E'[\Phi(X_T^{N'}, X_T^N)] - E'[\Phi(X_T^I, X_T)] \right) \\
+ \int_t^T \left\{ \xi_{4,N}^4(\Lambda_s^N) + \sqrt{N} \left( E'[f((\Lambda_s^N)', \Lambda_s^N)] - E'[f(\Lambda_s^I, \Lambda_s)] \right) \right\} ds \\
- \int_t^T \sqrt{N}(Z_s^N - Z_s)dW_s, \quad t \in [0, T],
\]

where

\[
\xi_{3,N}^3(x) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \left( \Phi(\Theta^k(X_T^N), x) - E[\Phi(X_T^N, x)] \right), \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]

\[
\xi_{4,N}^4(\lambda) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \left( f(\Theta^k(\Lambda_t^N), \lambda) - E[f(\Lambda_t^N, \lambda)] \right), \quad (t, \lambda) \in [0, T] \times \mathbb{R}^M.
\]

- \( \xi_{i,N}, i = 1, 2 \), defined analogously to \( \xi_{3,N}, \xi_{4,N} \), but with respect to \( \sigma, b \).
- Remark: \( \xi_{i,N}, 1 \leq i \leq 4 \), are independent of \( W \), their paths are jointly continuous in all parameters.
Proposition The laws of the stochastic fields \( \xi^N = (\xi^{1,N}, \xi^{2,N}, \xi^{3,N}, \xi^{4,N}), \ N \geq 1 \), converge weakly on
\[
C := C([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \times C([0, T] \times \mathbb{R}^d; \mathbb{R}^{d \times d}) \times C(\mathbb{R}^d; \mathbb{R}) \times C([0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d; \mathbb{R})
\]
to the law of \( \xi = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}) \).

Proof follows from the following both lemmas.
The sequence of the laws of the stochastic processes $\xi^N = (\xi_1^N, \xi_2^N, \xi_3^N, \xi_4^N), N \geq 1$, is tight on $C$. In addition to the estimates established in Lemma 4.3 for $\xi_i^N, N \geq 1, i = 1, 2$, we have for all $m \geq 1$ the existence of some constant $C_m$ such that, for all $(t, x, y, z), (t', x', y', z') \in [0, T] \times R^M, N \geq 1$,

i) $E \left[ |\xi_3^N(x)|^{2m} + |\xi_4^N(x, y, z)|^{2m} \right] \leq C_m$;

ii) $E \left[ |\xi_3^N(x) - \xi_3^N(x')|^{2m} + |\xi_4^N(x, y, z) - \xi_4^N(x', y', z')|^{2m} \right]$

$$\leq C_m(|t - t'|^m + |x - x'|^{2m} + |y - y'|^{2m} + |z - z'|^{2m})$$;

iii) $E \left[ \sup_{(x, y, z) \in R^M} \left( \frac{|\xi_3^N(x)| + |\xi_4^N(x, y, z)|}{1 + |x| + |y| + |z|} \right)^{2m} \right] \leq C_m$. 
Lemma The finite-dimensional laws of the stochastic fields $\xi^N = (\xi_{1,N}, \xi_{2,N}, \xi_{3,N}, \xi_{4,N})$, $N \geq 1$, converge weakly to the corresponding finite-dimensional laws of $\xi = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)})$.

Since $\xi^N$, $N \geq 1$, are independent of $W$ and converge in law to $\xi$, Skorohod’s Representation Theorem yields that on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ there exist copies $\tilde{\xi}^N$, $N \geq 1$, and $\tilde{\xi}$ of $\xi^N$, $N \geq 1$, and $\xi$, as well as a $d$-dimensional BM $\tilde{W}$, s.t.

i) $\tilde{P}_\xi = P_\xi$, $\tilde{P}_{\xi^N} = P_{\xi^N}$, $N \geq 1$;

ii) $\tilde{\xi}^N = (\tilde{\xi}_{1,N}, \tilde{\xi}_{2,N}, \tilde{\xi}_{3,N}, \tilde{\xi}_{4,N}) \rightarrow \tilde{\xi} = (\tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}, \tilde{\xi}^{(3)}, \tilde{\xi}^{(4)})$, uniformly on the compacts of $[0, T] \times R^M$, $\tilde{P}$-a.s.;

iii) $\tilde{W}$ is independent of $\tilde{\xi}$ and $\tilde{\xi}^N$, for all $N \geq 1$.

Filtration on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$: $\tilde{\mathcal{F}} = \mathcal{F}^{\tilde{W}} \lor \tilde{\mathcal{F}}_0$, where $\tilde{\mathcal{F}}_0 = \sigma\{\tilde{\xi}_s, \tilde{\xi}^N_s, s \in [0, T], N \geq 1\} \lor \mathcal{N}_{\tilde{P}}$. 
Remark: $(\tilde{W}, \tilde{F})$ is a BM and has the Martingale Representation Property.

Redefinition of $X^N, X$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$: description of $X^N$ as solution of an SDE governed only by the stoch. processes $(\xi^{1,N}, \xi^{2,N})$ and $W$; definition of $\tilde{X}^N$ by the same equation on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, with $(\xi^{1,N}, \xi^{2,N}, W)$ replaced by $(\tilde{\xi}^{1,N}, \tilde{\xi}^{2,N}, \tilde{W})$;

In the same spirit redefinition of $X$: $\tilde{X}$ - unique solution of the McKean-Vlasov SDE driven by $\tilde{W}$.

Remark: • $\tilde{X}$ is $\mathcal{F}\tilde{W}$-adapted.
• The coincidence of the laws $(\xi^{N,1}, \xi^{N,2}, \tilde{W})$ and $(\xi^{N,1}, \xi^{N,2}, W)$ implies that

$$\tilde{P}(\tilde{X}, \tilde{X}^N, \tilde{\xi}^{N,1}, \tilde{\xi}^{N,2}, \tilde{W}) = P(X, X^N, \xi^{N,W}), \ N \geq 1.$$ 

Redefinition of $(Y^N, Z^N)$ and $(Y, Z)$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$: redefinition $(\tilde{Y}^N, \tilde{Z}^N), (\tilde{Y}, \tilde{Z})$ of $(Y^N, Z^N)$ and $(Y, Z)$, resp., as unique solution of the BSDEs
\[ \tilde{Y}_t^N = \left( \frac{1}{\sqrt{N}} \tilde{\xi}_3^N (\tilde{X}_T^N) + \tilde{E}'[\Phi((\tilde{X}_T^N)', \tilde{X}_T^N)] \right) + \int_t^T \left( \frac{1}{\sqrt{N}} \tilde{\xi}_4^N (\tilde{\Lambda}_s^N) + \tilde{E}'[f((\tilde{\Lambda}_s^N)', \tilde{\Lambda}_s^N)] \right) ds - \int_t^T \tilde{Z}_s^N d\tilde{W}_s, \]

and
\[ \tilde{Y}_t = \tilde{E}'[\Phi((\tilde{X}_T)', \tilde{X}_T)] + \int_t^T \tilde{E}'[f((\tilde{\Lambda}_s)', \tilde{\Lambda}_s)] ds - \int_t^T \tilde{Z}_s d\tilde{W}_s, \]

resp., where \( \tilde{\Lambda}^N = (\tilde{X}^N, \tilde{Y}^N, \tilde{Z}^N), \tilde{\Lambda} = (\tilde{X}, \tilde{Y}, \tilde{Z}) \).

(Recall that the coefficients \( \xi_3^N(.) \) and \( \xi_4^N(.) \) are Lipschitz, uniformly with respect to \((\omega, s) \in \Omega \times [0, T]\), and so are \( \tilde{\xi}_3^N(.) \) and \( \tilde{\xi}_4^N(.) \).)

Obviously,
- \( \tilde{P}_{(\tilde{\xi}^N, \tilde{W}, \tilde{\Lambda}^N, \tilde{\Lambda})} = P_{(\xi^N, W, \Lambda^N, \Lambda)}, \) for all \( N \geq 1; \)
- \( \tilde{P}_{(\tilde{\xi}, \tilde{W}, \tilde{\Lambda})} = P_{(\xi, W, \Lambda)}. \)
The proof of the theorem now follows from

**Proposition** Let \( \hat{X} \) be the unique solution of the SDE

\[
\hat{X}_t = \int_0^t \left( \xi_1^1(\hat{X}_s) + \tilde{E}'[\nabla_x b(\hat{X}'_s, \hat{X}_s)] \hat{X}_s + \tilde{E}'[\nabla_x b(\hat{X}'_s, \hat{X}_s)] \hat{X}'_s \right) ds
\]

\[
+ \int_0^t \left( \xi_2^2(\hat{X}_s) + \tilde{E}'[\nabla_x \sigma(\hat{X}'_s, \hat{X}_s)] \hat{X}_s + \tilde{E}'[\nabla_x \sigma(\hat{X}'_s, \hat{X}_s)] \hat{X}'_s \right) d\tilde{W}_s,
\]

and \((\hat{Y}, \hat{Z})\) that of the BSDE

\[
\hat{Y}_t = \left\{ \xi_3^3(\hat{X}_T) + \tilde{E}'[\nabla_x \Phi(\hat{X}'_T, \hat{X}_T)] \hat{X}_T + \tilde{E}'[\nabla_x \Phi(\hat{X}'_T, \hat{X}_T)] \hat{X}'_T \right\}
\]

\[
+ \int_t^T \left( \xi_4^4(\hat{\Lambda}_s) + \tilde{E}'[\nabla_x f(\hat{\Lambda}'_s, \hat{\Lambda}_s)] \hat{\Lambda}_s + \tilde{E}'[\nabla_x f(\hat{\Lambda}'_s, \hat{\Lambda}_s)] \hat{\Lambda}'_s \right) ds
\]

\[- \int_t^T \hat{Z}_s d\hat{W}_s, \quad t \in [0, T],
\]

where \( \hat{\Lambda} = (\hat{X}, \hat{Y}, \hat{Z}) \).
Then
\[
\tilde{E}\left[ \sup_{s \in [0,t]} \left( |\sqrt{N}(\tilde{X}^N_s - \tilde{X}_s) - \tilde{X}_s|^2 + |\sqrt{N}(\tilde{Y}^N_s - \tilde{Y}_s) - \tilde{Y}_s|^2 \right) + \\
+ \int_0^T |\sqrt{N}(\tilde{Z}^N_s - \tilde{Z}_s) - \tilde{Z}_s|^2 ds \right] \longrightarrow 0, \text{ as } N \rightarrow +\infty.
\]

Consequence of the proposition: From
\[
\tilde{P}(\tilde{\Lambda}^N, \tilde{\Lambda}) = P(\Lambda^N, \Lambda), \tilde{P}(\tilde{\Lambda}, \tilde{\Lambda}) = P(\Lambda, \bar{\Lambda})
\]
the main result follows:
\[
\sqrt{N}(X^N - X, Y^N - Y, Z^N - Z) \Rightarrow (\bar{X}, \bar{Y}, \bar{Z})
\]
in law over \( C([0, T]; R^d) \times C([0, T]; R) \times L^2([0, T]; R^d) \).