ON A CLASS OF RENEWAL QUEUEING AND RISK PROCESSES

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- PURPOSE:- In this paper, we investigate how queuing theory have been applied to derive results for a Sparre Andersen risk model in which the distribution of inter-claim time is *hyper Exponential.*
- Design/Methodology/Approach:- We exploit the duality results between the queueing theory and risk processes to derive expressions for *ultimate ruin probability* and *moments of time of ruin* in this renewal risk model.
- Practical implications:- The theme of this paper is to stress connection between queuing theory and risk process.
- Findings:-

  (1).This paper derives explicit expression for the Laplace transforms of the *idle/waiting time distributions* in GI/G/1 model and the ultimate ruin probability is obtained.

  (2). The relation between the *time of ruin and busy period* in M/G/1 queuing system is used to derive the expected time of ruin.

MODEL

- Let T_1, T_2 . . .be a sequence of i.i.d random variables.
- T_i has a Hyper Erlang distribution with

$$g(t) = \sum_{i=1}^{P} p_i \frac{(k_i\lambda_i)^{k_i} t^{k_i - 1} e^{-k_i\lambda_i t}}{(k_i - 1)!}, \ t \geq 0.$$

- With Laplace transform

$$\hat{g}(\theta) = \sum_{i=1}^{P} p_i \left(\frac{k_i\lambda_i}{k_i\lambda_i + \theta}\right)^{k_i}$$

- P, k_1, k_2 . . . are non-negative integers, λ_1, λ_2 . . . are positive numbers.

WHY Hyper Erlang?

- *Hyper Erlang* model is suitable for analytic analysis and general enough to capture the statistics of the random time variables of interest
- The *hyper Erlang* distribution can be used to approximate the distribution of any non-negative random variable
- Distributions such as the exponential model, the Erlang model and the hyper exponential are special cases of *hyper Erlang* distribution.

ULTIMATE RUIN?

- N(t) be the number of claim arrival upto time t.
- u ≥ 0 , initial reserve. Premiums flow at the rate c per unit time.
- The risk reserve process $R(t) = u + ct - \sum_{i=1}^{N(t)} X_i$
- The claim surplus process $S(t) = \sum_{i=1}^{N(t)} X_i - ct$
- Aggregate claim $\sum_{i=1}^{N(t)} X_i$ is comprised of a claim number process {N(t)} whose inter claim times are as *HE*

- The claim amounts X_1, X_2, . . ., independent of N(t), with distribution F(x) and mean m_1.
- Y=cT, the inter claim revenue random variable with A(y)=P{Y≤y} and Laplace transform , $\hat{a}(\theta) = \int_0^{\infty} e^{-\theta y} dA(y)$
- $$m_x(\theta) = \int_0^{\infty} e^{\theta x} dF(x)$$
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- The probability of ultimate ruin $\psi(u) = P\{\inf_{t \geq 0} R(t) < 0\}$
- The probability of ruin before time τ

$$\psi(u,\tau) = P\{\inf_{0 \leq t \leq \tau} R(t) < 0\}$$

- Time of ruin

$$\tau(u) = \inf\{t \geq 0 : R(t) < 0\} = \inf\{t \geq 0 : S(t) > u\}$$

- $M = \sup_{0 \leq t < \infty} \{S(t)\}$ maximum with infinite time probablity.

- Again the ultimate ruin probability

$$\psi(u) = P\{\tau(u) < \infty\} = P\{M > u\}$$

- Let r is a unique positive solution of the equation

$$\hat{a}(\theta) m_x(\theta) = 1$$

We call this solution as Lundberg's exponent.

$\eta = \frac{c-\rho}{\rho}$

Duality –Queuing & Ruin

- GI/G/1 - a single server renewal queuing discipline.
- $T_1, T_2, \ldots$ be the inter-arrival times with distribution G(t).
- $X_1, X_2, \ldots$ be the service times of customers with distribution F(x).
- The traffic intensity of the queue $\rho = \frac{E(X)}{E(T)}$
- In ruin,this ratio is the avg amount of claims/unit time.
- Ruin theory have the property $\frac{1}{t}\sum_{i=1}^{N(t)} X_i \to \rho, t \to \infty$
- Premium loading factor $\eta = \frac{c-\rho}{\rho}$
- Always try to ensure η > 0.

- If $\eta > 0$, then $M < \infty$ and $\psi(u) < 1$.
- Denote $Z_n = X_n - T_n$
- $S_n = Z_1 + Z_2 + \ldots + Z_n$, $S_0 = 0$
- Let $H(z) = P\{ Z_n \leq z \}$ and $E(z)$ exists.
- Basic process underlying the queuing is the random walk $\{ S_n \}$.

- W_n – waiting time of the n^{th} customer
- I_n- idle period, just terminates upon the arrival of the n^{th} customer.
- Then $W_{n+1} = (W_n + Z_{n+1})^+$, $I_{n+1} = (W_n + Z_{n+1})^-$.
- Let $M_n = \max\{ 0, S_1 , S_2 , . . .\}$.
- M_n – the max. aggregate loss in ruin theory.

- Famous known result $W_n \overset{D}{=} M_n$

- $\rho < 1$, W_n converges to a r v W, M_n converges to M as $n \to \infty$, $P\{W > u\} = \psi(u) = P\{M > u\}$.

- Survival probability $\delta(u) = P\{M \le u\}$.

- Laplace transform of M

$$\hat{M}(\theta) = \theta\hat{\delta}(\theta)$$

- Define the R.V. N = min{ n > 0: S_n > 0 } and
  $\overline{N}$ = min{n > 0: S_n < 0 }

- Define the Ladder height distribution in (0, ∞)

$$G_n^{+}(x) = P\{N = n, S_N \le x\}$$

and $G_n^{-}(x) = P\{\overline{N} = n, S_{\overline{N}} \le x\}$ in (-∞, 0)

- We express the distribution of H in terms of the ladder height distributions G_n^{+} and G_n^{-}

- Transforms of first ascending and descending ladder epochs

$$\chi(\gamma,\omega) = E\{\gamma^{N} e^{i\omega S_{N}}\}, \ \bar{\chi}(\gamma,\omega) = E\{\gamma^{\bar{N}} e^{i\omega S_{\bar{N}}}\}$$

- Wiener-Hopf factorization

$$1-\gamma E\{e^{i\omega Z_n}\} = [1-E\{\gamma^{N} e^{i\omega S_N}\}][1-E\{\gamma^{\bar{N}} e^{i\omega S_{\bar{N}}}\}]$$

- I_n – Total idle period, W_n – Waiting time.

$$(W_n^d, I_n^d) \stackrel{D}{=} (I_n, W_n)$$

- $$(M_n - S_n, M_n) \stackrel{D}{=} (I_n, W_n)$$

Main Results

- Queue interarrival times T_i have distribution G(t), service time X_i have density HE_r
- $Z_n = X_n - T_n$, α = E(Z). $\Phi_1(\omega)$ and $\Phi_2(\omega)$ be the Ch. fn. of T_n and X_n.
- $$\phi(\omega) = \phi_1(-\omega)\phi_2(\omega)$$
$$= \sum_{m=1}^{P} p_m \left(\frac{k_m \lambda_m}{k_m \lambda_m - i\omega}\right)^{k_m} \phi_1(-\omega)$$
- $$\rho = -\frac{\sum_{i=1}^{P} \frac{p_i}{\lambda_i}}{\hat{f}(0)}$$

Result-1

- For the random walk induced by the above ch. fn.

- $$\chi(\gamma,\omega) = 1 - \prod_{j=1}^{N} [1 - \frac{k_m\lambda_m - k_1\lambda_1(1-\xi_j)}{k_m\lambda_m - i\omega}]$$

- $$\bar{\chi}(\gamma,\omega) = \frac{\prod_{m=1}^{P}(k_m\lambda_m - i\omega)^{k_m} - \gamma\sum_{m=1}^{P} p_m (k_m\lambda_m)^{k_m} \prod_{\substack{j=1 \\ j\neq m}}^{P} (k_j\lambda_j - i\omega)^{k_j} \phi_1(-\omega)}{\prod_{j=1}^{N}(k_1\lambda_1(1-\xi_j) - i\omega)}$$

- $\xi_j = \xi_j(\gamma),\ j = 1,2,\ldots N$ are the roots of the equation

$$\gamma\sum_{m=1}^{P} \frac{p_m}{(1 - \frac{k_1\lambda_1}{k_m\lambda_m}(1-\xi))^{k_m}} \hat{g}(k_1\lambda_1(1-\xi)) = 1$$

of which $\xi_j < 1$ j=1,2, . . .P.

Result-2

- For the queue GI/ $HE_r(k_i, \lambda_i)$/ 1

- $$\lim_{n\to\infty} E\{e^{i\omega W_n}\} = \prod_{m=1}^{P}\left(1-\frac{i\omega}{k_m\lambda_m}\right)^{k_m}\prod_{j=1}^{N}\frac{k_1\lambda_1}{k_1\lambda_1-\frac{i\omega}{1-\xi_j}}, \ \rho<1$$

- $$\lim_{n\to\infty} E\{e^{i\omega I_n}\} = \frac{\prod_{m=1}^{P}\left(\frac{k_m\lambda_m}{k_1\lambda_1}\right)^{k_m}\times(k_1\lambda_1)\alpha\, i\omega\prod_{j=1}^{N-1}\left(k_1\lambda_1+\frac{i\omega}{1-\xi_j}\right)}{\prod_{m=1}^{P}(k_m\lambda_m+i\omega)^{k_m}-\sum_{m=1}^{P}p_m(k_m\lambda_m)^{k_m}\prod_{\substack{j=1\\ j\neq m}}^{P}(k_j\lambda_j+i\omega)^{k_j}\phi_1(\omega)}, \ \rho>1$$

Result-3

- For the dual queue $HE_r(k_i, \lambda_i)/G/1$

- $$\lim_{n\to\infty} E\{e^{i\omega W_n}\} = \frac{\prod_{m=1}^{P}\left(\frac{k_m\lambda_m}{k_1\lambda_1}\right)^{k_m} k_1\lambda_1 \sum_{m=1}^{P}\frac{p_m}{\lambda_m}(1-\rho_1)\, i\omega \prod_{j=1}^{N-1}\left(k_1\lambda_1+\frac{i\omega}{1-\xi_j}\right)}{\prod_{m=1}^{P}(k_m\lambda_m+i\omega)^{k_m} - \sum_{m=1}^{P} p_m (k_m\lambda_m)^{k_m} \prod_{\substack{j=1\\ j\neq m}}^{P}(k_j\lambda_j+i\omega)^{k_j}\, \phi_1(\omega_1)}$$

- $$\lim_{n\to\infty} E\{e^{i\omega I_n}\} = \prod_{m=1}^{P}\left(1-\frac{i\omega}{k_m\lambda_m}\right)^{k_m} \prod_{j=1}^{N}\left(\frac{k_1\lambda_1}{k_1\lambda_1-\frac{i\omega}{1-\xi_j}}\right)$$

Applications to Risk

- Renewal risk model of the form $(HE_r(k_i, \lambda_i), G, c)$ then

$$\hat{\delta}(s) = \frac{\prod_{m=1}^{P}\left(c\frac{k_m\lambda_m}{k_1\lambda_1}\right)^{k_m}\left(k_1\lambda_1\sum_{j=1}^{P}\frac{p_j}{\lambda_j} - m_1\frac{k_1\lambda_1}{c}\right)\prod_{j=1}^{N-1}\left(\frac{k_1\lambda_1}{c} - \frac{s}{1-\xi_j}\right)}{\sum_{m=1}^{P}p_m(k_m\lambda_m)^{k_m}\prod_{\substack{j=1\\ j\neq m}}^{P}(k_j\lambda_j - cs)^{k_m}\hat{f}(s) - \prod_{m=1}^{P}(k_m\lambda_m - cs)^{k_m}}$$

- where ξ_j are the solutions of the equation

$$\sum_{m=1}^{P}\frac{p_m}{\left(1-\frac{k_1\lambda_1}{k_m\lambda_m}(1-\xi)\right)^{k_m}}\hat{f}\left(\frac{k_1\lambda_1}{c}(1-\xi)\right) = 1$$

- with $\xi_j < 1$, $j = 1 . . . P$.

- Dual risk model of the form **(G, HEr(ki, λi), c),** where the interarrival of claims follow any arbitrary distribution and claim size distribution is *hyper Erlang(k_i , λ_i), then*

$$\hat{\delta}(s) = \prod_{m=1}^{P} \frac{(k_m \lambda_m + s)^{k_m}}{s} \prod_{j=1}^{N} \left(\frac{k_1 \lambda_1}{k_1 \lambda_1 + \frac{s}{1-\xi_j}} \right)$$

where ξ_j are the solutions of the equation

$$\sum_{m=1}^{P} \frac{p_m}{\left(1 - \frac{k_1 \lambda_1}{k_m \lambda_m}(1-\xi)\right)^{k_m}} \hat{g}(ck_1 \lambda_1 (1-\xi)) = 1$$

under the condition that $\xi_j < 1, j = 1. . . P.$

Some Explicit Results

- Risk processes with claim amount distribution is Erlang (n, β) and inter-occurrence of claims are *hyper Erlang*(k_i , λ_i), then

$$\hat{\delta}(s) = \frac{(\beta + s)^n}{s} \prod_{i=1}^{n} \left(\frac{1-\xi_i}{\beta(1-\xi_i)+s} \right)$$

- Taking inverse Laplace transform

$$\phi(u) = 1 - \sum_{i=1}^{n} \xi_i^n e^{-\beta(1-\xi_i)u} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(1-\xi_j)}{\xi_i - \xi_j}$$

where ξ_j are the solutions of the equation

$$\xi^n = \sum_{i=1}^{P} p_i \left(\frac{k_i \lambda_i}{k_i \lambda_i + c\beta(1-\xi)} \right)^{k_i}$$

with $|\xi_j| < 1$, j = 1, 2, . . .P.

Time of Ruin

- N(t) denotes the number of claim arrivals upto the time of ruin, assume c =1, N(t) will be

$$N(t) = \inf\{n : u + \sum_{i=1}^{n} T_i - \sum_{i=1}^{n} X_i < 0\}$$

- Again $\tau(u) = \sum_{i=1}^{N(t)} T_i$

- The overshoot above the level u of the random walk $\{S_n\}$ be

$$Y^+(u) = \begin{cases} S(\tau(u)) - u & if \ \tau(u) < \infty \\ \infty & if \ \tau(u) = \infty \end{cases}$$

- Assume the inter-arrival of claims are *Hyper Erlang*(k_i, λ_i) with claim size distribution $1/\beta$. The risk process of the type $(HEr(k_i, \lambda_i), M, 1)$

- Dual queueing $M/HEr(k_i, \lambda_i)/1$

- T_i be the inter-arrival time, and X_j be the service time of the customer in the busy period.

- $V(u)$ be the duration of busy period, *I(u)* the busy period that follows the busy period.

- Duality arguments between queueing and risk process give V(u) is distributed as $\tau(u)+u$
- $Y^{+}(u)$ and *I(u) are identically distributed*
- *But in* queue, u is not a factor, V(0) = V and *I*(0) = *I* are the *busy period* and *idle period* of the regular queueing setup.
- $Y^{+}(0)$ is distributed as *I* and *τ(0) = τ is distibuted as V.*
- *V(u) is distributed as* $\tau(u)+u$ *in M/HEr*(k_i , λ_i)/1 queueing system:
- IMPORTANT: the service time of the first customer starts in T_1+u and the service times of all customers are distributed as H*yper Erlang*

TWO CASES ARISE:

- $\beta \sum_{i=1}^{P} \frac{p_i}{\lambda_i} < 1$, ruin occurs with probability 1.
- Busy period is finite.
- The Laplace transform, where $\hat{V}(\theta)$ is the Laplace transform of the busy period in M/HEr(k_i , λ_i)/1 queueing system.

$$\hat{V}_u(\theta) = e^{-u(\theta+\beta(1-\theta)\hat{V}(\theta))}\hat{V}(\theta)$$

- There fore, $E\{V(u)\} = \dfrac{u + \sum_{i=1}^{P} \frac{p_i}{\lambda_i}}{1 - \beta \sum_{i=1}^{P} \frac{p_i}{\lambda_i}}$

- $V(u) = \tau(u) + u$ gives $E\{\tau(u)\} = \dfrac{\sum_{i=1}^{P} \frac{p_i}{\lambda_i}(\beta u + 1}{1 - \beta \sum_{i=1}^{P} \frac{p_i}{\lambda_i}}$

CASE II

- $\beta \sum_{i=1}^{P} \frac{p_i}{\lambda_i} > 1$ the probability of ruin is *less than* 1
- To get an expression for time of ruin in this case, obtain the Esscher transform of G and F
- Transformed risk process has inter-arrival time distribution $g_\theta(t) = \dfrac{e^{-\theta t} g(t)}{\hat{g}(\theta)}$

$f_\theta(x) = (\beta-\theta)e^{-(\beta-\theta)x}$
$g_r(t) = \frac{e^{-rt}g(t)}{\hat{g}(r)}$

- Claim size distribution $f_\theta(x) = \frac{e^{\theta x} f(x)}{m_x(\theta)}$

- With Laplace transforms $\hat{g}_\theta(s) = \frac{\hat{g}(\theta+s)}{\hat{g}(s)}$

  and $f_\theta(x) = (\beta-\theta)e^{-(\beta-\theta)x}$

- r is the Lundberg exponent $g_r(t) = \frac{e^{-rt}g(t)}{\hat{g}(r)}$

- Consider the risk process with distribution in which inter-claim distribution

$$g_r(t) = \frac{e^{-rt}g(t)}{\hat{g}(r)}$$

$\beta\sum_{i=1}^{P}\frac{p_i}{\lambda_i} > 1$

- With Laplace transform

$$\hat{g}_r(\theta) = \frac{\sum_{i=1}^{P} p_i \left(\frac{k_i\lambda_i}{k_i\lambda_i + \theta + r}\right)_i^{k}}{\sum_{i=1}^{P} p_i \left(\frac{k_i\lambda_i}{k_i\lambda_i + r}\right)^{k_i}}$$

- Claim size distribution is <u>*exp(β-r)*</u>

- Assume $\beta\sum_{i=1}^{P}\frac{p_i}{\lambda_i} > 1$

- Laplace transform of the time of ruin $\tau(u)$

$$\hat{\tau}_u(\theta) = (1-\frac{r}{\beta})e^{-u(\beta-(\beta-r)\hat{V}_r(\theta))}\hat{V}_r(\theta)$$

$$\text{where } \hat{V}_r(\theta) = \frac{\sum_{i=1}^{P} p_i\left(\frac{k_i\lambda_i}{k_i\lambda_i+\theta+\beta-\hat{V}_r(\theta)(\beta-r)}\right)^{k_i}}{\sum_{i=1}^{P} p_i\left(\frac{k_i\lambda_i}{k_i\lambda_i+r}\right)^{k_i}}$$

- δ_1 , the first moment of the busy period in M/HEr/1 queue with arrival rate β-r

$$\delta_1 = \frac{d}{d\theta}\hat{V}_r(\theta)\,|_{\theta=0}$$

$$= \frac{\sum_{i=1}^{P} \frac{p_i}{\lambda_i} \left(\frac{k_i\lambda_i}{k_i\lambda_i + r}\right)^{k_i+1}}{\sum_{i=1}^{P} p_i \left(\frac{k_i\lambda_i}{k_i\lambda_i + r}\right)^{k_i} - (\beta - r) \sum_{i=1}^{P} \frac{p_i}{\lambda_i} \left(\frac{k_i\lambda_i}{k_i\lambda_i + r}\right)^{k_i+1}}$$

- Taking the first derivative of (7.10) w.r.t θ at θ = 0

$$E\{\tau(u):\tau<\infty\}=\delta_1(1+u(\beta-r))(1-\frac{r}{\beta})e^{-ru}$$

- Then

$$E\{\tau(u)\,|\,\tau<\infty\}=\delta_1(1+u(\beta-r))$$

EXAMPLES

- If the inter-arrival of claims is Er (n, λ) and claim size is β

$$E\{\tau(u) \mid \tau < \infty\} = \frac{n(1+u(\beta - r))}{(\lambda + r) - (\beta - r)}$$

- If *hyper exponential* (p_i , λ_i)

$$E\{\tau(u) \mid \tau < \infty\} = \frac{\sum_{i=1}^{P} p_i \frac{\lambda_i}{(\lambda_i + r)^2}}{\sum_{i=1}^{P} p_i (\frac{\lambda_i}{\lambda_i + r}) - (\beta - r) \sum_{i=1}^{P} p_i \frac{\lambda_i}{(\lambda_i + r)^2}}$$

CONCLUSION

Several results of queuing theory dam/storage processes can be effectively applied to risk theory by slightly changing the arguments used. Now it is widely accepted that the modeling ideas used in queuing theory has relevance in risk theory also. *Our work is an attempt in this direction.*

- Asmussen, S (2000), "Ruin Probabilities", World Scientific, Singapore, NJ, London, Hong Kong.
- Borovkov, A.A.,(1976)"Stochastic Processes in Queueing Theory", Springer-Verlag, New York, Heidelberg, Berlin
- Cheng,Y.,Qihe,T.(2003), "The moments of the surplus before ruin and deficit at ruin in the Erlang(2) risk process," *North American Actuarial Journal*, Vol. I7, No. 1, pp. 1-12.
- Dickson,D.C.M., Hipp,C.(1998),"Ruin probabilities for Erlang(2) risk processes". Insurance: Mathematics and Economics 22, 251-262.
- Frostig, E.(2004) "Upper bounds on the expected time of ruin and on the expected recovery time", Adv. Appl. Prob., 36, 377-397.
- Prabhu, N.U.(1998),"Stochastic Storage Processes, Queues, Insurance Risk and Dams"(second edn.) Springer, New York, Heidelberg, Berlin.


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