Optimal portfolios of a small investor in a limit order market – a shadow price approach

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Objectives

Solve Merton problem of small investor in a limit order market

The model should:

- Allow for choice between limit orders and market orders
- Incorporate trade-off between rebalancing portfolio quickly and trading at better prices (endogenous motivation to trade quickly)
- Incorporate trade-off between placing large limit orders to profit from spread and taking too much risk by the resulting large positions

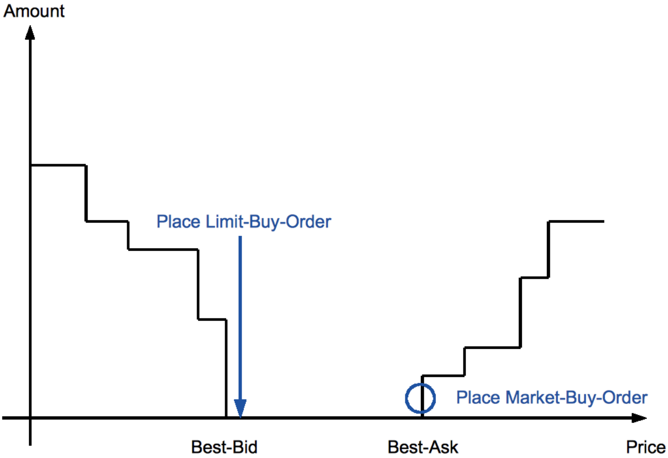
Overview

1. Brief introduction to limit order markets
2. Model of trading in a limit order market
3. Shadow price approach
4. Structure of the optimal strategy
5. Outline of solution

Basic terminology

- Market buy/sell order: Specifies **how many shares** to **buy/sell** at any available price. Results in immediate transaction
- Limit buy/sell order: Specifies not only **how many shares** to **buy/sell** but also the **maximal/minimal price per share** that is acceptable. (a limit buy order with limit high enough to warrant immediate and complete execution is in effect a market buy order)
- Order book: Aggregation of all unexecuted limit orders, usually in a computer system at the stock exchange
- Best-Bid Price: Highest limit price of all unexecuted limit buy orders
- Best-Ask Price Lowest limit price of all unexecuted limit sell orders

A look at the order book



A simple model of the limit order market

Assumption: The **small** investor cannot influence his trading opportunities

- Two assets: one bond B (=1 at all times), one risky asset S
- Best-bid price $\underline{S}$ modeled as GBM, i.e. $d\underline{S}_t = \underline{S}_t(\mu dt + \sigma dW_t)$
- Best-ask price proportional to best-bid: $\overline{S} = \underline{S}(1 + \lambda)$
- $\lambda \geq 0$ specifies the spread size
- Arrivals of market sell orders N^1 and market buy orders N^2 are modeled as independent time-homogenous Poisson processes with rates α_1 and α_2
- Order placement and cancelation is free

Some related models:

- Merton (1971): $\lambda = 0$, Davis/Norman (1990): $\alpha_1 = \alpha_2 = 0$
- Rogers/Zane (2002): $\lambda = 0$, trading only possible at Poisson times

Reduction of the dimension of the problem

Assuming investor is small and trading opportunities exogenously given as above leads to immense reduction of dimensionality:

- **Investor does not place limit buy order at limit price $> \underline{\mathbf{S}}$,** because every arriving exogenous market sell order will execute his limit order and he does not influence the arrival rates
- **Investor does not place limit buy order at limit price $< \underline{\mathbf{S}}$,** because $\underline{S}$ has continuous paths. He can just enter limit order at price $\underline{S}$ when $\underline{S}$ hits prespecified limit price $< \underline{S}$ at a predictable time
- **Investor will only place limit sell orders with limit price $\overline{\mathbf{S}}$,** by the same reasoning as above

$\Rightarrow$ Investor only has to decide on the **size** of the limit orders

Associated portfolio process

$$
\begin{aligned}
\varphi_t^0 &= \varphi_0^0 - \int_0^t \overline{S}_s \, dM_s^B - \int_0^{t-} L_s^B \underline{S}_s \, dN_s^1 \\
&\quad + \int_0^t \underline{S}_s \, dM_s^S + \int_0^{t-} L_s^S \overline{S}_s \, dN_s^2 - \int_0^t c_s \, ds \\
\varphi_t^1 &= \varphi_0^1 + M_t^B + \int_0^{t-} L_s^B \, dN_s^1 - M_t^S - \int_0^{t-} L_s^S \, dN_s^2
\end{aligned}
$$

φ^0: Amount held in bond
φ^1: Number of shares held of risky asset

N^1: Arrival of exogenous market sell orders, **Poisson** process
N^2: Arrival of exogenous market buy orders, **Poisson** process

M^B: accumulated market buy orders, **non-decreasing predictable** process
M^S: accumulated market sell orders, **non-decreasing predictable** process

L^B: size of limit buy order at best-bid, **non-negative predictable** process
L^S: size of limit sell order at best-ask, **non-negative predictable** process
c: consumption rate, **optional** process

Merton problem of the small investor

Optimization problem:

Given initial wealth $(\varphi_0^0, \varphi_0^1)$ choose a **strategy** $\hat{\mathfrak{S}} = (M^B, M^S, L^B, L^S, c)$ that maximizes the expected utility from consumption, i.e. such that

$$\mathcal{J}(\hat{\mathfrak{S}}) := \mathbf{E}\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right) = \sup_{\mathfrak{S}} \mathcal{J}(\mathfrak{S}) =: V(\varphi_0^0, \varphi_0^1)$$

Admissible strategies:

To make sure that $V(\varphi_0^0, \varphi_0^1) < \infty$, we have to exclude consumption without caring about wealth. We demand

$$\varphi_t^0 + \varphi_t^1 1_{\{\varphi_t^1 \geq 0\}} \underline{S}_t + \varphi_t^1 1_{\{\varphi_t^1 < 0\}} \overline{S}_t \geq 0 \quad \forall t \geq 0$$

i.e. if the position in the risky asset is closed by using market orders, this must result in a positive holding in the money market account

Shadow price approach: Basic idea

Consider Merton problem in **fictitious market**: Same two assets, similar investment-consumption problem, but market frictionless:

Fictitious market

- Strategies: (ψ^0, ψ^1, c)
- No spread: only one price process $\widetilde{S}$ for risky asset
- Results characterizing the optimal strategies known (Goll/Kallsen 2000)

Limit order market

- Strategies: (M^B, M^S, L^B, L^S, c)
- Spread and exogenous market orders have to be taken into account
- Goal: Find and verify optimal strategy in this market

Idea: Use results on optimal strategies in frictionless markets and suspected form of optimal strategy in limit order market to construct price process $\widetilde{S}$ for fictitious market s.t. solution to Merton problem in fictitious market is also solution to original Merton problem in limit order market

Shadow price approach

Definition

A real-valued semimartingale $\widetilde{S}$ is called **shadow price process** of the risky asset if it satisfies

- $\forall t \geq 0$:

$$\begin{aligned} &\underline{S}_t \leq \widetilde{S}_t \leq \overline{S}_t, \\ &\widetilde{S}_t = \underline{S}_t, \quad \text{if } \Delta N^1_t > 0 \qquad \text{arrival of exogenous market sell order} \\ &\widetilde{S}_t = \overline{S}_t, \quad \text{if } \Delta N^2_t > 0 \qquad \text{arrival of exogenous market buy order} \end{aligned}$$

- $\exists$ an admissible $\mathfrak{S} = (M^B, M^S, L^B, L^S, c)$ s.t. the portfolio process (φ^0, φ^1) of $\mathfrak{S}$ together with c is an optimal strategy in the frictionless market with $\widetilde{S}$ as the price process of the risky asset

Shadow price approach: One solution for both markets

Showing $\widetilde{S}$ is a shadow price also solves the original Merton problem:

1. Any strategy in limit order market can be carried out in fictitious market where all trading is at most as expensive. Thus, at least the same consumption may be financed by trading the strategy in the fictitious market.
   $\Rightarrow V_{\text{LOM}} \leq V_{\text{FM}}$
2. If $\widetilde{S}$ is a shadow price process, there exists a strategy $\mathfrak{S} = (M^B, M^S, L^B, L^S, c)$ in the limit order market s.t. expected utility from consumption equals maximal expected utility from consumption in the frictionless market, i.e.
   $\exists\ \mathfrak{S} : \mathcal{J}(\mathfrak{S}) = V_{\text{FM}}$

$\Rightarrow \mathcal{J}(\mathfrak{S}) = V_{\text{LOM}}$

Kallsen/Muhle-Karbe (2008) showed existence of a shadow price process for Merton problem in case of proportional transaction costs

Structure of the solution

The optimally controlled portfolio has the structure: (Kühn, S. 2009)

$\exists\ \pi_{\text{min}}, \pi_{\text{max}} \in \mathbb{R}$ with $\pi_{\text{min}} < \pi_{\text{max}}$ such that

- Proportion of wealth invested in the risky asset kept in the intervall $[\pi_{\text{min}}, \pi_{\text{max}}]$ by using **market orders**, i.e.

$$\pi_{\text{min}} \leq \frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} \leq \pi_{\text{max}} \qquad \forall t > 0$$

- At all times two (permanently adjusted) **limit orders** are kept in the order book such that

$$\frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} = \pi_{\text{max}}, \quad \text{after limit buy order is executed with limit } \underline{S}_t$$

$$\frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} = \pi_{\text{min}}, \quad \text{after limit sell order is executed with limit } \overline{S}_t$$

- Optimal **consumption** proportional to wealth measured w.r.t. the shadow price, i.e. $c_t = \delta \widetilde{V}_t$

Structure of the solution: Wealth cone

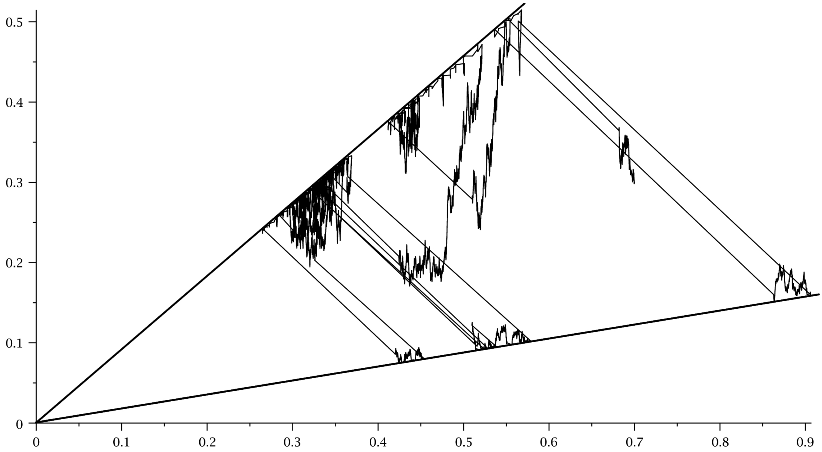


Figure: Simulated path of optimal $(\varphi^0, \varphi^1 \underline{S}_t)$

Structure of the solution: Trading strategy

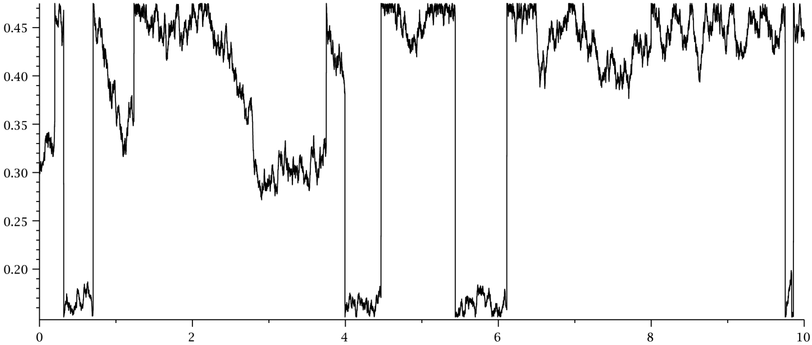


Figure: Simulated path of optimal proportion $\pi_t = \frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t}$

Outline of solving the problem

- Ansatz: $\widetilde{S} = Se^{C}$ with $\overline{C} := \ln(1+\lambda)$ and C is $[0, \overline{C}]$-valued process

$$dC_t = \widetilde{\mu}(C_{t-})dt + \widetilde{\sigma}(C_{t-})dW_t - C_{t-}dN_t^1 + (\overline{C} - C_{t-})dN_t^2,$$

  where $\widetilde{\mu}$ and $\widetilde{\sigma}$ are unknown functions
- Assume $d\varphi^1 = 0$ and calculate $d\widetilde{\pi}$ twice, once directly from $d\widetilde{S}$ and once using Itô's formula as an unknown function of C, i.e. assuming $\widetilde{\pi} = f(C)$
- Compare the factors of the two representations of $d\widetilde{\pi}$ to see that both $\widetilde{\mu}$ and $\widetilde{\sigma}$ can be expressed as transformations of one unknown function f

Outline of solving the problem

- Itô's formula yields semimartingale characteristics of $\widetilde{S}$, Goll/Kallsen (2000) then implies that optimal $\widetilde{\pi}$ in fictitious market has to satisfy

$$
\begin{aligned}
\mu + \frac{\widetilde{\sigma}(C_{t-})^2}{2} &+ \sigma\widetilde{\sigma}(C_{t-}) + \widetilde{\mu}(C_{t-}) - \widetilde{\pi}_t(\sigma + \widetilde{\sigma}(C_{t-}))^2 \\
&+ \alpha_1(e^{-C_{t-}} - 1)\left(\frac{1}{1 + \widetilde{\pi}_t(e^{-C_{t-}} - 1)}\right) \\
&+ \alpha_2(e^{\overline{C} - C_{t-}} - 1)\left(\frac{1}{1 + \widetilde{\pi}_t(e^{\overline{C} - C_{t-}} - 1)}\right) = 0
\end{aligned}
$$

- Plug in expressions for $\widetilde{\mu}$ and $\widetilde{\sigma}$ to get second order ODE for f
- We know that $f(0) = \overline{\pi}, f(\overline{C}) = \underline{\pi}$ has to be satisfied. But since we deal with second order ODE, we also need boundary condition for f'

Outline of solving the problem

Obtaining the missing boundary condition:

- $\widetilde{\pi}$ depends on Brownian fluctuations of S, thus needs local time to keep it within an intervall
- Hence, smooth function f has to possess exploding first derivative as C has no singular component
- To avoid this, turn problem around by considering inverse $C = g(\widetilde{\pi}) := f^{-1}(\widetilde{\pi})$
- Now, by letting $g'(\underline{\pi}) = g'(\overline{\pi}) = 0$ it is possible to have local time in $\widetilde{\pi}$ but not in C (which would imply arbitrage)

Outline of solving the problem: The free boundary problem

$$\begin{aligned} g''(y) &= \left(-\frac{2}{\sigma^2}B(y,g(y)) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}}\right) \\ &+ \left(\frac{6}{\sigma^2}B(y,g(y)) + \frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1+e^y)\right) g'(y) \\ &+ \left(-\frac{6}{\sigma^2}B(y,g(y)) - \frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^y)\right)(g'(y))^2 \\ &+ \left(\frac{2}{\sigma^2}B(y,g(y)) - \frac{2\delta}{\sigma^2}(1+e^y)\right)(g'(y))^3. \end{aligned}$$

with

$$B(y,g(y)) := \alpha_1(e^{-g(y)}-1)\left(\frac{1}{1+\frac{e^{\underline{C}-g(y)}-1}{1+e^{-y}}}\right) + \alpha_2(e^{\overline{C}-g(y)}-1)\left(\frac{1}{1+\frac{e^{\overline{C}-g(y)}-1}{1+e^{-y}}}\right)$$

Boundary conditions:

$$g(\underline{\pi}) = \overline{C}, \qquad g(\overline{\pi}) = 0, \qquad g'(\underline{\pi}) = g'(\overline{\pi}) = 0,$$

where $\underline{\pi}$ and $\overline{\pi}$ have to be chosen.

Verification

Given the free boundary problem (FBP) from the previous slide proceed as follows to prove existence of a shadow price:

1. Show a solution to the FBP exists
2. Use solution to FBP to construct $\widetilde{\pi}$ as a solution to a Skorohod problem
3. Derive resulting dynamics of $\widetilde{S} = \underline{S} \exp(g(\widetilde{\pi}))$
4. Solve Merton problem for the resulting $\widetilde{S}$ and show that the optimal strategy can indeed be executed in the limit order market at the same prices

Some observations and remarks

When tackling the Merton problem for proportional transaction costs, the approach of Kallsen/Muhle-Karbe leads to an easier FBP than the approach of Davis/Norman. Expecting further complications in the FBP due to the arriving exogenous market orders, this was a good reason to use the shadow price approach

Myopic nature of logarithmic utility in frictionless markets very helpful. Allows to use local characterization of optimal behavior in the fictitious market to construct shadow price

In markets with bid-ask spread there is no a priori correct price to value holdings in the risky asset. Optimal consumption proportional to wealth measured according to shadow price hints that shadow price is correct price for internal valuation of wealth

The End

Thank you!