On martingale optimality, BSDE and cross hedging energy risk*

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1 Basis risk: definition and examples

Basis = price of **hedged asset** - price of **hedging instrument**

problem of basis risk: uncertainties of processes describing the evolution of prices of asset and hedging instrument not identical, only highly correlated

Example 1: weather derivatives

hedged asset: heating oil sales, hedging instrument: HDD derivative

HDD derivative: contract paying a premium in case HDD above a critical threshold

Example 2: commodity markets

hedged asset: power spot price, hedging instrument: power futures

futures: contract to deliver amount of commodity at pre-fixed price

hedge spot price fluctuations on time slots not coinciding with futures delivery dates

2 a toy example

Aim: show problems with hedging basis risk, given very high correlation

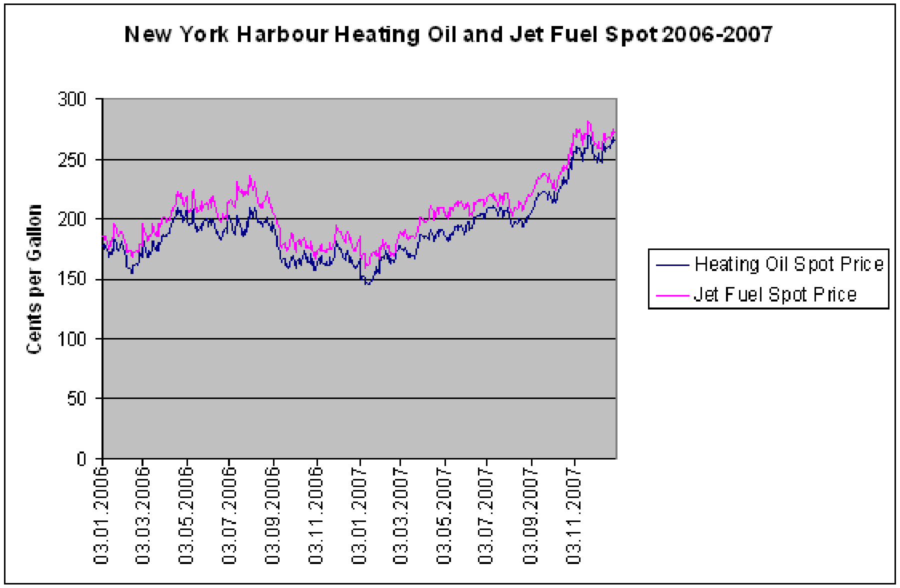


Abbildung 1: Daily Spot Prices

airline company, hedged asset: jet fuel spot price, hedging instrument: heating oil futures

diagram indicates high correlation between jet fuel spot price and heating oil spot price

2 Cross hedging principle: correlation

simplest caricature of hedging problem:

static situation: Y hedged asset, X hedging instrument, both standard Gaussian, possibly strongly correlated

$$\rho = E(XY) \quad \text{(correlation of} \quad X \quad \text{and} \quad Y)$$

decomposition of Y into part parallel to X and independent standard Gaussian part Z:

$$Z = \frac{1}{\sqrt{1-\rho^2}}[Y - \rho X]$$

then

$$\sqrt{1-\rho^2}E(XZ) = E(XY) - \rho E(X^2) = 0,$$

hence Z independent of X, and

$$Y = \rho X + Y - \rho X = \rho X + \sqrt{1-\rho^2}Z.$$

2 Cross hedging principle: mean variance

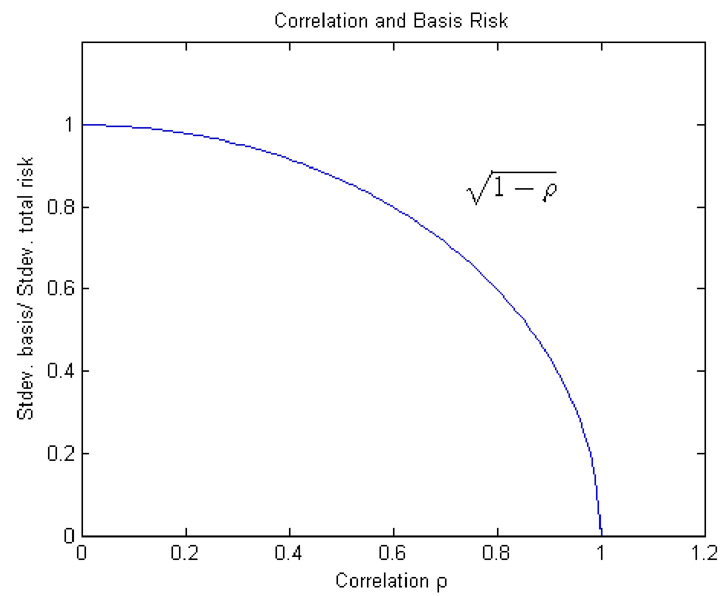
What quantity a of position X would agent hold to optimally hedge position Y?
quality of **hedging: minimize quadratic error**

$$E((Y - aX)^2) = E([(\rho - a)X + \sqrt{1-\rho^2}Z]^2) = (\rho - a)^2 + (1 - \rho^2) \quad \text{minimal,}$$

i.e.

$$a = \rho, \quad \text{Hedging error: } \sqrt{1-\rho^2}Z$$

ρ	$\sqrt{1-\rho^2}$	% uncertainty hedged
0.999	0.05	95
0.99	0.14	86
0.98	0.20	80
0.95	0.31	69
0.9	0.44	56



3 Mean variance hedging of jet fuel by heating oil

simple model for price processes (better: geometric BM) (J. C. Hull 2008)

jet fuel

$$J_t = J_0 + \mu t + \sigma Y_t, \qquad t \geq 0$$

heating oil

$$H_t = H_0 + \nu t + \beta X_t, \qquad t \geq 0$$

μ, ν, σ, $\beta \in \mathbf{R}$, X and Y correlated BM, to be estimated from data; $T = 1$ delivery date, observation days $0 = t_0 < t_1 < \ldots < t_N = 1$

ML estimator for σ

$$\widehat{\sigma} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{t_i - t_{i-1}} \left(J_{t_i} - J_{t_{i-1}} \right)^2 - \frac{1}{N} (J_1 - J_0)^2,$$

yields ML estimates $\widehat{\sigma} \approx 3,998$, $\widehat{\beta} \approx 3,835$;

ML estimator for correlation ρ between $\widehat{\sigma} Y$ and $\widehat{\beta} X$ requires estimate of quadratic variation of Y and X and yields $\widehat{\rho} = 0,897$

3 Mean variance hedging of jet fuel by heating oil

decomposition of the jet fuel price

$$J_t = J_0 + \mu t + 0.897\widehat{\sigma} X_t + 0.443\widehat{\sigma} Z_t,$$

Z BM independent of X.

airline aims at hedging increasing fuel prices by buying heating oil futures; suppose $K = E[H_1] = H_0 + \nu$ is price of heating oil futures at time 0; quantity of futures a the airline has to hold to **minimize quadratic error** determined by

$$E((J_1 - J_0) - a(H_1 - K))^2 = \mu^2 + (0.897\widehat{\sigma} - a\widehat{\beta})^2 + 0.321\widehat{\sigma}^2,$$

i.e. $a = 0.897\frac{\widehat{\sigma}}{\widehat{\beta}}$.

hedging error at time 1

$$I_1 = 0.443\widehat{\sigma} Z_1.$$

correlation between spot prices almost 0.9; only 56% of standard deviation of price change can be hedged!

4 Conclusions: mean variance vs. utility based hedging

- *Even if correlation very high, hedging error large!*
- correlation high: small change in correlation entails big change in percentage of basis risk relative to total risk
- correlation low: small change in correlation entails essentially no change in percentage of basis risk relative to total risk
- downside part of basis risk has to be properly respected
- *utility based* approach does this

5 Our approach of utility based hedging

Aims:

- present a **purely probabilistic** approach, combining martingale optimality and BSDE
- determine **utility indifference price**
- determine explicit **derivative hedge**, i.e. optimal cross hedging strategy
- clarify role of correlation in hedging
- describe reduction of risk by cross hedging

6 The financial market model

Index process, e.g. temperature, spot price

$$dR_t = \sigma(t, R_t)dW_t + b(t, R_t)dt,$$

$b : [0, T] \times \mathbf{R}^m \to \mathbf{R}^m$, $\sigma : [0, T] \times \mathbf{R}^m \to \mathbf{R}^{m\times d}$ deterministic functions, globally Lipschitz and of sublinear growth. R Markov process, $R_s^{t,r}$: start at t in r

Hedged asset: liability or derivative $F(R_T)$, $F : \mathbf{R}^m \to \mathbf{R}$ bounded

Hedging instrument: correlated financial market, k risky assets with price process:

$$\frac{dS_t^i}{S_t^i} = \beta_i(t, R_t)dW_t + \alpha_i(t, R_t)dt = \beta_i(t, R_t)[dW_t + \theta_t dt], \quad i = 1, \ldots, k,$$

$\alpha : [0, T] \times \mathbf{R}^m \to \mathbf{R}^k$, $\beta : [0, T] \times \mathbf{R}^m \to \mathbf{R}^{k\times d}, \theta = \beta^*[\beta\beta^*]^{-1}\alpha$.

W $d-$dimensional Brownian motion, correlation expressed by β and σ

7 The optimal investment problem

(N. El Karoui, R. Rouge '00; J. Sekine '02; J. Cvitanic, J. Karatzas '92, Kramkov, Schachermayer '99,...)

investment strategy λ : value of portfolio fraction invested in risky assets

wealth gain on $[0, s]$ (here $\beta_t = \beta(t, \cdot)$ etc.)

$$G_s^\lambda = \sum_{i=1}^{k} \int_0^s \lambda_u^i \frac{dS_u^i}{S_u^i} = \int_0^s \lambda_u \beta_u [dW_u + \theta_u du],$$

utility function: $U(x) = -e^{-\eta x}$ ($0 < \eta$ risk aversion); maximal expected utility from terminal wealth without and with derivative:

$$V^0(v) = \sup_{\lambda \in \tilde{C}} EU(v + G_T^\lambda), \quad V^F(v) = \sup_{\lambda \in \tilde{C}} EU(v + G_T^\lambda - F(R_T))$$

utility indifference $V^F(v^F) = V^0(v^0)$, λ^0 resp. λ^F optimal strategies
utility indifference price $\Delta_v = v^F - v^0 = p = p(r) = p(t, r)$
derivative hedge $\Delta_\lambda = \lambda^F - \lambda^0$

8 Optimization under non-convex constraints

interpretation as maximization problem with convex constraints

$$\tilde{C} \subset \mathbf{R}^k \text{ convex}, \quad \lambda \in \tilde{C}$$

$$\pi_t = \lambda_t \beta_t \in C_t = \tilde{C} \beta_t$$

$$C_t \text{ convex}$$

Aim: construct solution combining **martingale optimality** with **BSDE**, even for non-convex constraints

(N. El Karoui, R. Rouge '00 for convex constraints)

$$\tilde{C} \subset \mathbf{R}^k \text{ closed}, \quad \lambda \in \tilde{C}$$

8 Optimization under non-convex constraints

$F = F(R_T)$ hedged asset

First formulation:

Find

$$V(v) = \sup_{\lambda \in \tilde{C}} E(U(G_T^{\lambda} - F)) = \sup_{\lambda \in \tilde{C}} E(U(v + \int_0^T \lambda_s \beta_s [dW_s + \theta_s ds] - F)).$$

For simplicity:

$$\begin{aligned} \pi &= \lambda\beta, \\ C &= \tilde{C}\beta. \end{aligned}$$

$$G_t^{\pi} = v + \int_0^t \pi_s [dW_s + \theta_s ds], \quad t \in [0, T]$$

Second formulation:

Find

$$V(v) = \sup_{\pi \in C} E(U(G_T^{\pi} - F)) = \sup_{\pi \in C} E(-\exp(-\eta(v + \int_0^T \pi_s [dW_s + \theta_s ds] - F))).$$

9 Martingale optimality and BSDE

Idea: Construct family of processes $Q^{(\pi)}$ such that

form 1

$$
\begin{aligned}
Q_0^{(\pi)} &= \text{constant},\\
Q_T^{(\pi)} &= -\exp(-\eta(G_T^\pi - F)),\\
Q^{(\pi)} & \quad \text{supermartingale}, \quad \pi \in C,\\
Q^{(\pi^*)} & \quad \text{martingale, for (exactly) one} \quad \pi^* \in C.
\end{aligned}
$$

Then

$$
\begin{aligned}
E(-\exp(-\eta[G_T^\pi - F])) &= E(Q_T^{(\pi)})\\
&\leq E(Q_0^\pi)\\
&= V(v)\\
&= E(Q_0^{(\pi^*)})\\
&= E(-\exp(-\eta[G_T^{(\pi^*)} - F])).
\end{aligned}
$$

Hence π^* optimal strategy.

9 Martingale optimality and BSDE

Introduction of BSDE into problem

Find generator f of BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad Y_T = F,$$

such that with

$$Q_t^{(\pi)} = -\exp(-\eta[G_t^\pi - Y_t]), \quad t \in [0,T],$$

we have

$$\begin{aligned} Q_0^{(\pi)} &= -\exp(-\eta(v - Y_0)) \\ &= \text{constant}, \end{aligned}$$

(fulfilled)

form 2

$$Q_T^{(\pi)} = -\exp(-\eta(G_T^\pi - F))$$

(fulfilled)

$Q^{(\pi)}$ supermartingale, $\pi \in C$,

$Q^{(\pi^*)}$ martingale, for (exactly) one $\pi^* \in C$.

This gives solution of valuation problem.

10 Construction of generator of BSDE

How to determine f:

Suppose f generator of BSDE. Then

$$\begin{aligned}
Q_t^{(\pi)} &= -\exp(-\eta[G_t^\pi - Y_t]) \\
&= -\exp(-\eta[v - Y_0]) \cdot \exp(-\eta[\int_0^t (\pi_s - Z_s)dW_s - \int_0^t [f(s, Z_s) - \pi_s\theta_s]ds]) \\
&= \exp(-\eta[v - Y_0]) \cdot \exp(-\eta \int_0^t (\pi_s - Z_s)dW_s - \frac{\eta^2}{2}\int_0^t (\pi_s - Z_s)^2 ds) \\
&\qquad \cdot - \exp(\int_0^t [\eta f(s, Z_s) - \eta\pi_s\theta_s + \frac{\eta^2}{2}(\pi_s - Z_s)^2]ds) \\
&= M_t^{(\pi)} \cdot A_t^{(\pi)},
\end{aligned}$$

with $M^{(\pi)}$ nonnegative martingale. $Q^{(\pi)}$ satisfies **(form 2)** iff for

$$q(\cdot, \pi, z) = f(\cdot, z) - \pi\theta + \frac{\eta}{2}(\pi - z)^2, \quad \pi \in C, z \in \mathbb{R},$$

we have

10 Construction of generator of BSDE

form 3
$$\begin{aligned} q(\cdot,\pi,z) &\geq 0, \quad \pi \in C \quad \text{(supermartingale cond.)} \\ q(\cdot,\pi^*,z) &= 0, \quad \text{for (exactly) one} \quad \pi^* \in C \quad \text{(martingale cond.)}. \end{aligned}$$

Now

$$\begin{aligned} q(\cdot,\pi,z) &= f(\cdot,z) - \pi\theta + \frac{\eta}{2}(\pi - z)^2 \\ &= f(\cdot,z) + \frac{\eta}{2}(\pi - z)^2 - (\pi - z)\cdot\theta + \frac{1}{2\eta}\theta^2 \quad - z\theta - \frac{1}{2\eta}\theta^2 \\ &= f(\cdot,z) + \frac{\eta}{2}[\pi - (z + \frac{1}{\eta}\theta)]^2 \quad - z\theta - \frac{1}{2\eta}\theta^2. \end{aligned}$$

Under **non-convex constraint** $\pi \in C$:

$$[\pi - (z + \frac{1}{\eta}\theta)]^2 \geq d^2(C, z + \frac{1}{\eta}\theta).$$

with **equality** for at least one possible choice of p^* due to **closedness** of C.
Hence **(form 3)** is solved by the choice

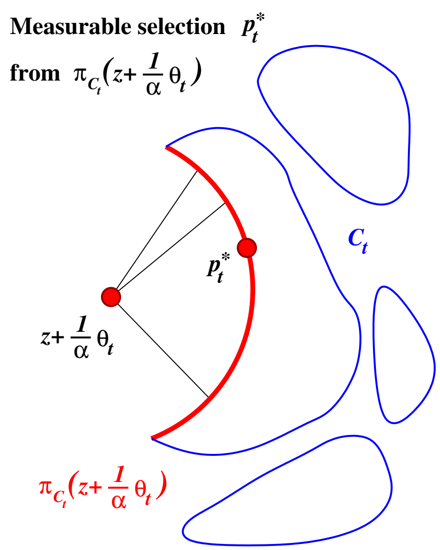
10 Construction of generator of BSDE

form 4

$$f(\cdot, z) = -\frac{\eta}{2} d^2(C, z + \frac{1}{\eta}\theta) + z \cdot \theta + \frac{1}{2\eta}\theta^2 \quad \text{(supermartingale)}$$

$$\pi^* \quad \text{such that} \quad d(C, z + \frac{1}{\eta}\theta) = d(\pi^*, z + \frac{1}{\eta}\theta) \quad \text{(martingale)}.$$

Problem: Let
$\Pi_C(v) = \{\pi \in \mathbb{R}^d : d(C, v) = d(\pi, v)\}$. Find measurable selection π_t^* from $\Pi_{C_t}(Z_t + \frac{1}{\eta}\theta_t)$. Solved by classical **measurable selection method**.



11 Main result

Thm 1
(Y, Z) unique solution of BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s)ds, \quad t \in [0, T],$$

with

$$f(t, Z_t) = -\frac{\eta}{2} d^2(C_t, Z_t + \frac{1}{\eta}\theta_t) + Z_t \cdot \theta_t + \frac{1}{2\eta}\theta_t^2.$$

Then value function of utility optimization problem under constraint $\pi \in \mathcal{A}$ given by

$$V(v) = -\exp(-\eta[v - Y_0]).$$

There exists an (non-unique) optimal trading strategy $\pi^* \in \mathcal{A}$ such that

$$\pi_t^* \in \Pi_{C_t}(Z_t + \frac{1}{\eta}\theta_t), \quad t \in [0, T].$$

Proof:

- existence, uniqueness for BSDE with quadratic non-linearity in z (M. Kobylanski '00)
- measurable selection theorem for $\Pi_{C_t}(Z_t + \frac{1}{\eta}\theta_t)$
- BMO properties of the martingales $\int Z_s dW_s, \int \pi_s^* dW_s$ for uniform integrability of exponentials (regularity of coefficients)•

12 Calculation of derivative hedge

generalization to $[t,T]$ instead of $[0,T]$, cond. on $R_t = r$:

$(Y^{t,r}, Z^{t,r})$,$\pi^{t,r}$ (without F) resp. $(\hat{Y}^{t,r}, \hat{Z}^{t,r})$,$\hat{\pi}^{t,r}$ (with F) instead of (Y, Z),π yields

$$V^0(t,v,r) = -\exp(-\eta(v - Y_t^{t,r})), \quad V^F(t,v,r) = -\exp(-\eta(v - \hat{Y}_t^{t,r})),$$

instead of $V(v) = -\exp(v - Y_0)$.

due to linearity of $C(t,r)$ projections unique and linear, hence

$$\pi_s^{t,r} = \Pi_{C(t,r)}[Z_s^{t,r} + \frac{1}{\eta}\theta(s, R_s^{t,r})], \quad \hat{\pi}_s^{t,r} = \Pi_{C(t,r)}[\hat{Z}_s^{t,r} + \frac{1}{\eta}\theta(s, R_s^{t,r})],$$

and so

$$(\Delta_\lambda\, \beta)(s, R_s^{t,r}) = \Pi_{C(t,r)}[\hat{Z}_s^{t,r} - Z_s^{t,r}].$$

13 Markov property and its consequences

Markov property of R implies (Kobylanski '00, El Karoui, Peng, Quenez '97):

Thm 2
There are measurable (deterministic) functions u and $\hat{u}$ such that

$$Y_s^{t,r} = u(s, R_s^{t,r}), \quad \hat{Y}_s^{t,r} = \hat{u}(s, R_s^{t,r}).$$

There are measurable (deterministic) functions v and $\hat{v}$ such that

$$Z_s^{t,r} = v\sigma(s, R_s^{t,r}), \quad \hat{Z}^{t,r} = \hat{v}\sigma(s, R_s^{t,r}).$$

Corollary 1

$$p(t,r) := Y_t^{t,r} - \widehat{Y}_t^{t,r} = u(t,r) - \hat{u}(t,r)$$

is the **indifference price**, i.e. $V^F(t, v - p(t,r), r) = V^0(t, v, r)$.

p depends only on R, not on S

Aim: Explicit description of Δ_λ

14 Differentiability

Thm 3 (**Parameter Differentiability**) smoothness conditions on F, f
There exists a version of $(\widehat{Y}_s^{t,r}, \widehat{Z}_s^{t,r})$ such that a.s.
- $\widehat{Y}_s^{t,r}$ is continuous in s and cont. differentiable in r (classical sense)
- $\widehat{Z}^{t,r}$ is differentiable in a weak sense (norm topology)
- $(\nabla_r \widehat{Y}^{t,r}, \nabla_r \widehat{Z}^{t,r})$ solves the BSDE

$$\begin{aligned}\nabla_r \widehat{Y}_t^r &= \nabla_r F(R_s^{t,r}) \nabla_r R_s^{t,r} - \int_t^T \nabla_r \widehat{Z}_s^{t,r} dW_s \\ &\quad + \int_t^T \Big[\nabla_r f(s, R_s^{t,r}, \widehat{Z}_s^{t,r}) \nabla_r R_s^{t,r} \\ &\qquad + \nabla_z f(s, R_s^{t,r}, \widehat{Z}_s^{t,r}) \nabla_r \widehat{Z}_s^{t,r} \Big] ds.\end{aligned}$$

Proof uses norm inequalities, and inverse Hölder inequalities, based on BMO properties of the stochastic integral processes of $\hat{Z}^{t,r}$

Thm 4 (Malliavin Differentiability)

$$D_\vartheta \widehat{Y}_s^{t,r} = \nabla_r \widehat{u}(s, R_s^{t,r}) D_\vartheta R_s^{t,r}$$

and

$$\widehat{Z}_s^{t,r} = D_s \widehat{Y}_s^{t,r} = \nabla_r \widehat{u}(s, R_s^{t,r}) \sigma(s, R_s^{t,r})$$

15 Explicit description of derivative hedge

Properties of the BSDEs $\Longrightarrow$

Thm 5
The **indifference price** $p(t,r) = Y_t^{t,r} - \widehat{Y}_t^{t,r}$ is differentiable in r.

Thm 6
The **derivative hedge** Δ_λ at time t depends only on R_t, and

$$\begin{aligned} \Delta_\lambda(t,r)\beta(t,r) &= \Pi_{C(t,r)}[\widehat{Z}_t^{t,r} - Z_t^{t,r}] \\ &= \Pi_{C(t,r)}[\nabla_r(\widehat{Y}_t^{t,r} - Y_t^{t,r})\sigma(t,r)] \\ &= -\Pi_{C(t,r)}[\nabla_r p(t,r)\sigma(t,r)]. \end{aligned}$$

Remarks:

- complete case: $\Delta_\lambda =$ 'delta hedge'
- where is the risk aversion η?

16 Example: Heating degree days

- common underlying of weather derivatives
- $T_i =$ average of the maximum and the minimum temperature on day i at a specific location
- $HDD_i = \max(0, 18 - T_i)$

Cumulative heating degree days

$$cHDD_t = \sum_{i=1}^{30} HDD_{t-i}$$

Derivatives:

- Option: $(cHDD - K)^+$
- Swap: $b(cHDD - K)$

16 Example: Heating degree days

$cHDD$:

- statistical analysis shows: cHDDs are log-normally distributed (M. Davis '01)
- cHDD can be modeled as a *geometric Brownian motion*

$$dX_t = \mu X_t dt + \nu X_t dW_t$$

(moving average)

Other indices: cooling degree days

$$CDD_i = \min\left(0, 18 - T_i\right)$$

16 Example: Heating degree days

- $R =$ cHDDs (geometric Brownian Motion)
- $d = 2$
- 1-dim market + index: $k = m = 1$
- index volatility: $\sigma = \begin{pmatrix} \alpha & 0 \end{pmatrix}$
- price volatility: $\beta = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$ with $\alpha, \beta_1, \beta_2 \in \mathbf{R} \setminus \{0\}$

Then

$$\Delta_\lambda(t,r) = -\alpha \frac{\partial p(t,r)}{\partial r} \frac{\beta_1}{\beta_1^2 + \beta_2^2}.$$

17 Example: Heating degree days; diversification pressure

derivative hedge:

$$\Delta_\lambda(t,r) = -\alpha \frac{\partial p(t,r)}{\partial r} \frac{\beta_1}{\beta_1^2 + \beta_2^2}.$$

Call option: $F(R_T) = (R_T - K)^+$

$\Longrightarrow \frac{\partial p(t,r)}{\partial r} > 0$

Comparison of the optimal strategies:

- $\beta_1 \alpha < 0$ (negative correlation)
  $\Longrightarrow \quad F(R_T)$ diversifies portfolio $\Longrightarrow \quad \Delta_\lambda > 0$
  $\Longrightarrow \quad \widehat{\pi} > \pi$

- $\beta_1 \alpha > 0$ (positive correlation)
  $\Longrightarrow \quad F(R_T)$ amplifies portfolio $\Longrightarrow \quad \Delta_\lambda < 0$
  $\Longrightarrow \quad \widehat{\pi} < \pi$