Interbank Lending
Modelling the Development of Interbank Markets during a Liquidity Shock

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Motivation: decoupling of interest rates
The model

We study a model with

- three dates $t = 0, t_1, t_2$
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- a government bond with certain return at $t = t_2$
- costless storage
Management of funds

We assume that each bank has one unit of the households’ funds under management at $t = 0$ and offers claims that can be withdrawn either at $t = t_1$ or $t = t_2$ worth $c_1$ and $c_2$ respectively.
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Thus banks have the following random cashflows:

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>$t_1$</th>
<th>$t_2$</th>
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</thead>
<tbody>
<tr>
<td>deposit contracts</td>
<td>1</td>
<td>$-\Lambda c_1$</td>
<td>$-(1 - \Lambda)c_2$</td>
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External investment possibilities

- The payoff of the illiquid asset is modelled by an $\mathbb{R}_+$-valued process $(S_t)_{0 \leq t \leq t_2}$ and the price for the bond is denoted by $P_0$ in $t = 0$ and $P_2$ in $t = t_2$. 
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Then we have the following assets and financial claims:

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<tr>
<td>$t$</td>
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</tr>
<tr>
<td>cash</td>
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</tr>
<tr>
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<td>$0$</td>
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</tr>
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<tr>
<td>unsecured ib debt</td>
<td>(-\gamma_+)</td>
<td>(\gamma_+ \hat{p}(1 + r))</td>
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The optimization problem in $t_1$

We consider the investment decision from $t = 0$, $(\alpha, \beta_0, \delta_0)$ as given. Let $\lambda$, $r$ and $P_1$ be fixed, then the optimization problem in $t_1$ is

$$\max_{\beta_1, \gamma_+, \gamma_-, \delta_1} E[u(V_2(\beta_1, \gamma_+, \gamma_-, \delta_1))]$$

Constraints are

$$\beta_1 + \gamma_+ + \delta_1 = \delta_0 + \beta_0 \frac{P_1}{P_0} + \gamma_- - \lambda c_1,$$

$$\beta_1, \gamma_+, \gamma_-, \delta_1 \geq 0,$$

where

$$V_2(\beta_1, \gamma_+, \gamma_-, \delta_1) = \alpha \frac{S_{t_2}}{S_0} + \beta_1 \frac{P_2}{P_1} + \gamma_+(1+r)\hat{p} - \gamma_-(1+r) + \delta_1 - (1-\lambda)c_2.$$
Optimal policies in $t_1$

We have to optimize pointwise for every possible liquidity demand and every price pair on the interbank market, thus we derive an optimal supply and demand function, i.e. a function

$$(\beta_1, \gamma_+, \gamma_-, \delta_1) : \text{supp}(\Lambda) \times (-1, \infty) \times (0, \infty) \rightarrow \mathbb{R}^4_{\geq 0}$$

$$(\lambda, r, P_1) \mapsto (\beta_1^\lambda(r, P_1), \gamma^\lambda_+(r, P_1), \gamma^\lambda_-(r, P_1), \delta_1^\lambda(r, P_1))$$

which (pointwise for every $(\lambda, r, P_1)$) solves our optimization problem.
Since expectation is monotone and the utility function is increasing the objective function

\[ E[u(\alpha \frac{S_{t2}}{S_0} + \beta_1^\lambda \frac{P_2}{P_1} + \gamma^\lambda_+(1 + r)\hat{p} - \gamma^\lambda_-(1 + r) + \delta^\lambda_1 - (1 - \lambda)c_2)] \]

becomes

\[ \beta_1^\lambda \frac{P_2}{P_1} + \gamma^\lambda_+(1 + r)\hat{p} - \gamma^\lambda_-(1 + r) + \delta^\lambda_1, \]

which is an affine function (as well as the constraint functions). Therefore Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality.
In equilibrium, aggregate demand equals aggregate supply. Aggregate net demand functions are

$$E[(\beta_1^\Lambda(r, P_1) - \beta_0 \frac{P_1}{P_0})^+], \quad E[\gamma_+^\Lambda(r, P_1)]$$

net supply is

$$E[(\beta_0 \frac{P_1}{P_0} - \beta_1^\Lambda(r, P_1))^+], \quad E[\gamma_-^\Lambda(r, P_1)].$$
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$$E[\gamma^-(r, P_1)].$$

Thus an equilibrium price pair \((r^*, P_1^*)\) satisfies

$$E[\gamma^+(r^*, P_1^*)] = E[\gamma^-(r^*, P_1^*)],$$

$$E[\beta^1(r^*, P_1^*)] = \beta_0 \frac{P_1^*}{P_0}.$$
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For $\delta_0 > E(\Lambda)c_1$, there is secured interbank lending. An unsecured interbank market develops iff

$$\beta_0 \frac{P_2}{P_0} + \delta_0 < c_1 \text{ess sup}(\Lambda).$$

The equilibrium prices are given by

$$r = \frac{1}{\hat{\rho}} - 1, \quad P_1 = P_2.$$
For $\delta_0 = E(\Lambda)c_1$, there is secured interbank lending. An unsecured interbank market develops iff

$$\beta_0 \frac{P_2}{P_0} < c_1 (\text{ess sup}(\Lambda) - E(\Lambda)).$$

In this case the equilibrium prices satisfy

$$r \in \left[\frac{1}{\hat{p}} - 1, \infty\right), \quad P_1 = \frac{P_2}{(1 + r)\hat{p}}.$$

If there is no unsecured lending, the bond price must satisfy

$$P_1 \in [P_0 \frac{c_1}{\beta_0} (\text{ess sup}(\Lambda) - E(\Lambda)), P_2].$$
In order to get a closed form for the optimal policies, we define

\[ S_s := \mathbb{E}[(\Lambda c_1 - \delta_0)1\{\frac{1}{c_1} \delta_0 \leq \Lambda \leq \frac{1}{c_1} (\beta_0 P_1 P_0 + \delta_0)\}] + \beta_0 \frac{P_1}{P_0} \mathbb{P}(\Lambda > \frac{1}{c_1} (\beta_0 P_1 P_0 + \delta_0)), \]

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$$D := E[(\delta_0 - \Lambda c_1)1\{\Lambda < \frac{1}{c_1}\}]$$

the excess capital of banks with liquidity surplus, which equals the aggregate demand for interbank loans plus demand for liquid asset.
The model
Optimal policies in $t_1$
Example

$$\beta_1^{\lambda^*} = \begin{cases} 0, & \lambda > \frac{1}{c_1}(\beta_0 \frac{P_1}{P_0} + \delta_0) \\ \beta_0 \frac{P_1}{P_0} + \delta_0 - \lambda c_1, & \frac{1}{c_1} \delta_0 \leq \lambda \leq \frac{1}{c_1}(\beta_0 \frac{P_1}{P_0} + \delta_0) \\ \beta_0 \frac{P_1}{P_0} + \frac{S_s}{D}(\delta_0 - \lambda c_1), & \lambda < \frac{1}{c_1} \delta_0 \end{cases}$$

$$\gamma_+^{\lambda^*} = \begin{cases} 0, & \lambda \geq \frac{1}{c_1} \delta_0 \\ \frac{S_u}{D}(\delta_0 - \lambda c_1), & \lambda < \frac{1}{c_1} \delta_0 \end{cases}$$

$$\gamma_-^{\lambda^*} = \begin{cases} \lambda c_1 - (\beta_0 \frac{P_1}{P_0} + \delta_0), & \lambda > \frac{1}{c_1}(\beta_0 \frac{P_1}{P_0} + \delta_0) \\ 0, & \lambda \leq \frac{1}{c_1}(\beta_0 \frac{P_1}{P_0} + \delta_0) \end{cases}$$

$$\delta_1^{\lambda^*} = \begin{cases} 0, & \lambda \geq \frac{1}{c_1} \delta_0 \\ (1 - \frac{S_u + S_s}{D})(\delta_0 - \lambda c_1), & \lambda < \frac{1}{c_1} \delta_0 \end{cases}$$
For $\delta_0 \geq E(\Lambda)c_1$ the strategy $(\beta_1^{\lambda*}, \gamma_+^{\lambda*}, \gamma_-^{\lambda*}, \delta_1^{\lambda*})$ is optimal and profit in $t = t_2$ is given by

$$V_2(\alpha, \beta_0, \delta_0) = \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} - (1 - \Lambda)c_2 + (\delta_0 - \Lambda c_1) \frac{P_2}{P_1} 1_{\{\Lambda \leq \frac{1}{c_1} \delta_0\}} +$$

$$+ (\delta_0 - \Lambda c_1) \frac{P_2}{P_1} 1_{\{\frac{1}{c_1} \delta_0 < \Lambda \leq \frac{1}{c_1} (\beta_0 \frac{P_1}{P_0} + \delta_0)\}} + (\delta_0 - \Lambda c_1) \frac{P_2}{P_1 \hat{p}} 1_{\{\Lambda > \frac{1}{c_1} (\beta_0 \frac{P_1}{P_0} + \delta_0)\}}$$

where $\frac{P_2}{P_1} = (1 + r)\hat{p}$.
Now we allow for banks which are not able to fulfill the deposit claims \((\delta_0 < \lambda c_1)\) to fail in \(t = t_1\). The cost of failure is the sum of deposit contracts not fulfilled in \(t_1\) and open contracts in \(t_2\).
Allowing for failure in $t = t_1$

Now we allow for banks which are not able to fulfill the deposit claims ($\delta_0 < \lambda c_1$) to fail in $t = t_1$. The cost of failure is the sum of deposit contracts not fulfilled in $t_1$ and open contracts in $t_2$.

Thus utility of a failed bank is

$$F^\lambda := u(\delta_0 - \lambda c_1 - (1 - \lambda)c_2).$$
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Banks which fail do not take part in the interbank market.

As in the situation without failure we have to consider three cases looking for possible equilibria.
For $\delta_0 < E(\Lambda)c_1$ we get the following result:

If there is $\tilde{r} \in \left[\frac{1}{\hat{p}} - 1, \infty\right)$ s.t.

$$E \left( \Lambda 1_{\{F^{\Lambda} > E(u(V_2^{\Lambda})), \Lambda c_1 > \delta_0\}} \right) = E(\Lambda) - \frac{1}{c_1} \delta_0,$$

then $(\tilde{r}, \frac{P_2}{(1+\tilde{r})\hat{p}})$ is an equilibrium price pair and banks with $F^{\lambda} > E(u(V_2^{\lambda}(\alpha, \beta_0, \delta_0)))$ decide to fail in $t = t_1$.

The remaining banks choose the strategy $(\beta^{\lambda*}_1, \gamma^{\lambda*}_+, \gamma^{\lambda*}_-, \delta^{\lambda*})$.

Else all banks with $\lambda c_1 > \delta_0$ fail, there is no interbank trading and the remaining banks hold all their bonds and invest spare liquidity in the liquid asset.
For $\delta_0 = E(\Lambda)c_1$ the interest rates can’t be arbitrarily large anymore:

If $(\alpha, \beta_0, \delta_0)$ satisfy

$$E \left( u(\alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} - (1 - \lambda)c_2 + (\delta_0 - \lambda c_1) \frac{1}{\hat{p}}) \right) \geq F^\lambda$$

for all $\lambda > \frac{1}{c_1}(\beta_0 \frac{P_2}{P_0} + \delta_0)$, then we get an upper bound for $r$

$$\bar{r} := \inf \{ r \in \left[ \frac{1}{\hat{p}} - 1, \infty \right) : P\left( F^\Lambda > E\left( V_2(\alpha, \beta_0, \delta_0) | \Lambda > \frac{1}{c_1} \delta_0 \right) \right) > 0 \},$$

else all banks with $\lambda c_1 > \delta_0$ fail and there is no interbank market.
The optimization problem in $t = 0$

The objective function ($\delta_0 \geq E(\Lambda)c_1$) is

$$\int_0^{\infty} \int_0^{\frac{1}{c_1}} (\beta_0 \frac{P_2}{P_1} + \delta_0) \, u(\alpha s + \beta_0 \frac{P_2}{P_0} + \lambda (c_2 - c_1 \frac{P_2}{P_1}) - c_2 + \delta_0 \frac{P_2}{P_1}) \, P^\Lambda(d\lambda) \, P^{\frac{s_{t_2}}{s_0}}(ds)$$

$$+ \int_0^{\infty} \int_{\frac{1}{c_1}}^{1} (\beta_0 \frac{P_2}{P_1} + \delta_0) \, u(\alpha s + \beta_0 \frac{P_2}{P_0} + \lambda (c_2 - c_1 \frac{P_2}{P_1}) - c_2 + \delta_0 \frac{P_2}{P_1}) \, P^\Lambda(d\lambda) \, P^{\frac{s_{t_2}}{s_0}}(ds).$$

We maximize w.r.t. ($\alpha, \beta_0, \delta_0$) under constraints

$$\alpha, \beta_0 \geq 0, \delta_0 \geq E(\Lambda)c_1$$

$$\alpha + \beta_0 + \delta_0 = 1$$
CRR model for $S$, Binomial distribution for $\Lambda$

Let $\lambda_l < \lambda_h$, $d < \frac{P_2}{P_0} < u$ and $P(\Lambda = \lambda_l) = \pi_l = 1 - P(\Lambda = \lambda_h)$, $P\left(\frac{S_{t_2}}{S_0} = u\right) = p = 1 - P\left(\frac{S_{t_2}}{S_0} = d\right)$, with $p, \pi_l \in (0, 1)$. 
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Set $\bar{\lambda} := \pi_l \lambda_l + (1 - \pi_l) \lambda_h$. Assume exponential utility $u(x) := -\exp(-\kappa x)$ with $\kappa > 0$. 
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Set $\bar{\lambda} := \pi_l \lambda_l + (1 - \pi_l) \lambda_h$. Assume exponential utility $u(x) := -\exp(-\kappa x)$ with $\kappa > 0$.

In this case an unsecured interbank market develops and the optimal solution is interior iff

$$1 - p \frac{P_2}{P_0} - d \leq \frac{p}{u - \frac{P_2}{P_0}} \in \left(\exp(-(u-d)(1-\bar{\lambda})\kappa), \exp(-(u-d)(1-\bar{\lambda} - \pi_l(\lambda_h - \lambda_l)\frac{P_0}{P_2}))\kappa\right)$$
Then the optimal solution is

\[(\alpha^*, \beta_0^*, \delta^*) = \left( -\log \left( \frac{1-p}{p} \frac{p_2}{u-p_2} \frac{p_2}{p_0} - d \right) \frac{1}{\kappa(u-d)}, 1 - \bar{\lambda}c_1 + \log \left( \frac{1-p}{p} \frac{p_2}{u-p_2} \frac{p_2}{p_0} - d \right) \frac{1}{\kappa(u-d)}, \bar{\lambda}c_1 \right) \]
Then the optimal solution is

\[
(\alpha^*, \beta_0^*, \delta_0^*) = \left( \log \left( \frac{1-p}{p} \frac{P_2}{u-P_2} \right), 1 - \bar{\lambda} c_1 + \log \left( \frac{1-p}{p} \frac{P_2}{u-P_2} \right), \bar{\lambda} c_1 \right)
\]

For \( \kappa = 1, \lambda_l = \frac{1}{4}, \lambda_h = \frac{3}{4}, u = 1.2, d = 1, \frac{P_2}{P_0} = 1.1 \) and \( p = \frac{20}{39} \) we get

\[
(\alpha^*, \beta_0^*, \delta_0^*) = (-\log(0.95), \frac{1}{2} + \log(0.95), \frac{1}{2})
\]

\(
\approx (0.0222763947, 0.477723605, 0.5)
\)
Then the optimal solution is

\[
(\alpha^*, \beta^*_0, \delta^*_0) = \left( -\frac{\log \left( \frac{1-p}{p} \frac{P_2-P_0}{u-P_2} \right)}{\kappa(u-d)}, 1 - \bar{\lambda}c_1 + \frac{\log \left( \frac{1-p}{p} \frac{P_2-P_0}{u-P_2} \right)}{\kappa(u-d)}, \bar{\lambda}c_1 \right)
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For \( \kappa = 1, \lambda_l = \frac{1}{4}, \lambda_h = \frac{3}{4}, u = 1.2, d = 1, \frac{P_2}{P_0} = 1.1 \) and \( p = \frac{20}{39} \) we get

\[
(\alpha^*, \beta^*_0, \delta^*_0) = (-\log(0.95), \frac{1}{2} + \log(0.95), \frac{1}{2})
\]

\[
\approx (0.0222763947, 0.477723605, 0.5)
\]

An equilibrium price pair is \((1.145, 1)\) and the solvency probability for lenders in the unsecured interbank market is \(\frac{20}{39}\).
References

