Measuring Model Risk
A Toxicity Index for Exotic Derivatives

P. Hénaff
Euria
Université de Bretagne Occidentale

March 2011
What is Model Risk?
What is Model Risk?

Formal Problem Statement
What is Model Risk?

Formal Problem Statement

Model Risk within one model class

The Toxicity Index

Illustration
What is Model Risk?

Formal Problem Statement

Model Risk within one model class
   The Toxicity Index
   Illustration

A generic approach to model risk
   Computation of KL Divergence
      Kernel-based density estimation
      Calculation of the KL Divergence
   Numerical Experiments
What is Model Risk?

Formal Problem Statement

Model Risk within one model class

The Toxicity Index
Illustration

A generic approach to model risk

Computation of KL Divergence
Kernel-based density estimation
Calculation of the KL Divergence
Numerical Experiments

Conclusion
What is Model Risk?

Formal Problem Statement

Model Risk within one model class
   The Toxicity Index
   Illustration

A generic approach to model risk
   Computation of KL Divergence
     Kernel-based density estimation
     Calculation of the KL Divergence
   Numerical Experiments

Conclusion

Technical Details
Example 1: Bank of England Survey

Survey of investment banks that make markets in derivatives (1997) shows large price discrepancies:

- Up to 20% for simple OTC derivatives (swaptions)
- Up to 60% for complex derivatives (double barrier options)
What is Model Risk?

Example 2: Bank of Tokyo-Mitsubishi

- Black-Derman-Toy model is calibrated to at-the-money swaptions and used to price exotic interest rate derivatives, resulting in mispricing.
- 1999: Study (by Meridien Research) reveals that most of the loss was attributable to wrong choice of pricing model.
What is Model Risk?

Model Risk

- Risk related to the use of the wrong model (i.e. a model that does not accounts for important risk factors)
- Risk related to poor or unstable calibration of the model

Both issues translate into:

- Inaccurate initial pricing
- Inaccurate risk indicators, leading to ineffective hedging strategy

All of this while the model correctly prices benchmark instruments.
Historical Summary


*Overall, incorporating stochastic volatility and jumps is important for pricing and internal consistency. But for hedging, modeling stochastic volatility alone yields the best performance.*
Historical Summary : Model Uncertainty

From Observation to Measurement

Basel Committee directive of July 2009 on model risk:

- Quantify model risk
- Provision the risk with Tiers I capital

There is now a need to measure model risk in currency terms.

- A consistent risk measure
- Standardized across asset classes and models, to allow for meaningful comparisons.
Formal Problem Statement

Notation (from Cont, 2006)

- $I$ Set of benchmark instruments,
- $H_i$ Payoff of instrument $i$
- $E^Q(H_i)$ Price of benchmark instrument $i$ under model $Q$,
- $C^*_i$ the mid-market prices, $C^*_i \in [C^{bid}_i, C^{ask}_i]$.
- $Q$ Set of models, consistent with the market prices of benchmark instruments:

$$Q \in Q \Rightarrow E^Q[H_i] \in [C^{bid}_i, C^{ask}_i], \forall i \in I \quad (1)$$
A Model

Definition (Model)
A model is the specification of the stochastic processes for the risk factors that determine the price of a financial instrument.
Manifestation of Model Risk

Consider a new payoff $X$, with price $E^Q[X]$ under model $Q$. Upper and lower price bounds over the family of models, for a pay-off $X$:

$$
\overline{\pi}(X) = \sup_{j=1,\ldots,n} E^{Q_j}[X] \tag{2}
$$

$$
\underline{\pi}(X) = \inf_{j=1,\ldots,n} E^{Q_j}[X] \tag{3}
$$
Illustration

Consider two models for the dynamic of the forward price:

\[
\frac{dF(t, T)}{F(t, T)} = \sigma \, dW_t 
\tag{4}
\]

\[
\text{Var}(\ln(F(T, T))) = \sigma^2 T 
\tag{5}
\]

\[
\frac{dF(t, T)}{F(t, T)} = \sigma e^{-\beta(T-t)} \, dW_t 
\tag{6}
\]

\[
\text{Var}(\ln(F(T, T))) = \sigma^2 \left(1 - e^{-\beta T}\right) 
\tag{7}
\]
Illustration (cont’d)
Benchmark instruments : 2Y European option.

**Figure**: Integrated variance
Illustration: 3 models fitted to a smile curve

**Figure:** Calibration to benchmark volatility: (a) DEJD (b) Heston (c) Local Volatility
A definition of model risk (R. Cont)

Notation

\[ I \] Set of liquid instruments,

\[ H_{i \in I} \] pay-offs,

\[ C^*_{i \in I} \] the mid-market prices, \( C^*_{i} \in [C^\text{bid}_i, C^\text{ask}_i] \).

\[ Q \] Set of models, consistent with the market prices of benchmark instruments:

\[ Q \in Q \Rightarrow E^Q[H_i] \in [C^\text{bid}_i, C^\text{ask}_i], \forall i \in I \quad (8) \]

Upper and lower price bounds over the family of models, for a pay-off \( X \):

\[ \bar{\pi}(X) = \sup_{j=1,\ldots,n} E^{Q_j}[X] \quad \underline{\pi}(X) = \inf_{j=1,\ldots,n} E^{Q_j}[X] \]
A definition of model risk (R. Cont)

Risk measure:

\[ \mu_Q = \bar{\pi}(X) - \underline{\pi}(X) \]

Issues:

- How large should \( Q \) be?
- How can we compare risk across asset classes (i.e. sets \( Q \).)
Model Risk within one model class

The Toxicity Index

The Toxicity Index

- Create a large set $Q$ by perturbation of the original model
- Define a measure of similarity between the perturbed models and the original one (“size” of $Q$), in order to normalize the risk measure.
Toxicity Index at level $\alpha$

$$\mu_Q = \pi(X, \alpha) - \bar{\pi}(X, \alpha)$$

Over normalized set $Q : \{ Q \mid D(Q, P) < \alpha, E^Q[H_i] = C_i \}$. Where $P$ is the reference model.
Relative Entropy

Let $P$ and $Q$ be two random variables. The relative entropy of $P$ and $Q$ measures the closeness between probability distributions over the same set:

$$D(Q|P) = E_Q \{ \log Q - \log P \}$$

$$= \sum_x Q(x) \log \left( \frac{Q(x)}{P(x)} \right)$$
Let $0 < \alpha < 1$ such that:

$$D(Q|P) = \ln(N)(1 - \alpha)$$

Define $M = N^\alpha$:

$$K\text{-Index} = 100 \frac{M}{N}$$
Model Risk within one model class

The Toxicity Index

K-Index

![K-Index Diagram](image-url)

- K Index = 0
- K Index = 60%

Diagram showing distribution of K-Index values.
A Normalized Risk Measure

$$\pi(X, \alpha) = \min \sum q_k X_k$$

such that

$$\sum q_k A_{i,k} \leq b_i, \quad \forall i \in I$$

$$\sum q_k \log \left( \frac{q_k}{p_k} \right) \leq (1 - \alpha) \log(N)$$

$$\sum q_k = 1$$

Risk :

$$\overline{\pi}(X, \alpha) - \pi(X, \alpha)$$
Solution of Optimization Problem

\[
\min \sum q_k \ln \left( \frac{q_k}{p_k} \right)
\]

such that

\[
\sum q_k A_{i,k} \leq b_k, \quad i \in I
\]

\[
\sum q_k Z_k = p^*
\]

\[
\sum q_k = 1
\]
Feasible set for $p^*$

$p \in [p^-, p^+]$

With:

$p^- = \min \sum q_k Z_k$

such that

$\sum q_k A_{i,k} \leq b_k, \quad i \in I$

$\sum q_k = 1$
Solution of dual problem

\[ \min \sum q_k \ln \left( \frac{q_k}{p_k} \right) \]

such that

\[ \sum q_k A_{i,k} \leq b_k, \quad i \in I \]
\[ \sum q_k Z_k = p^* \]
\[ \sum q_k = 1 \]
Solution of dual problem

\[ \min_{\lambda \geq 0} \quad -b^T \lambda - \ln \left( \sum_{k} e^{-A_k^T \lambda} \right) \]

Then:

\[ q_k = e^{-\lambda^T A_k - \mu - 1} \]

\[ \mu = \ln \left( \sum_{k} e^{-A_k^T \lambda} \right) - 1 \]
Summary

Toxicity Index:

$$\bar{\pi}(X, \alpha) - \underline{\pi}(X, \alpha)$$

With $\bar{\pi}(X, \alpha), \underline{\pi}(X, \alpha)$ min and max value over set of models that are within distance $\alpha$ of the reference model.
Numerical Illustration

Benchmark instruments: 1Y European options.

**Figure:** Volatility smile - Benchmark Instruments
## Toxicity Index

<table>
<thead>
<tr>
<th>K-Index</th>
<th>Range Option</th>
<th>Lookback Call</th>
<th>Asian Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>17.12</td>
<td>16.39</td>
<td>15.90</td>
</tr>
<tr>
<td>92.5</td>
<td>14.25</td>
<td>13.76</td>
<td>12.70</td>
</tr>
<tr>
<td>95</td>
<td>11.17</td>
<td>10.71</td>
<td>9.37</td>
</tr>
<tr>
<td>97.5</td>
<td>7.61</td>
<td>7.30</td>
<td>5.65</td>
</tr>
<tr>
<td>99</td>
<td>4.47</td>
<td>4.39</td>
<td>3.00</td>
</tr>
</tbody>
</table>

*Table: Model risk ($\overline{\pi} - \overline{\p}$)*
Model Risk for Asian Call and Range Options
Model Comparison

- Given a set of benchmark instruments,
- Calibrate:
  - Double Exponential Jump Diffusion (Kou)
  - Heston
  - Local Volatility
- Compute model risk for range of K-Index
Asian Option

**Figure:** Model Risk - 1Y Asian Option
Lookback Put Option

**Figure:** Model Risk - 1Y Lookback Put
Summary

- We propose a variant of Cont’s model that allows to normalize the risk measure.
- By solving the dual optimization problem, we can effectively handle an unlimited number of scenarios.

But:

- Restricting the set $Q$ to scenario perturbations of the reference model is a limitation.
- Requiring a pricing by MC simulations limits the range of exotic derivatives that can be handled.
A generic measure of model risk

The model risk is still defined by:

$$\mu_Q = \pi(X) - \overline{\pi}(X)$$

with:

$$Q \in Q \Rightarrow E^Q[H_i] \in [C_i^{bid}, C_i^{ask}], \ \forall i \in I \text{ and } D(Q_j|P) \leq \epsilon_{KL} \ (10)$$
Model Features

Definition (Model Feature)
The model features is the set of observations on the risk factors that are needed to determine the payoff of a financial instrument.

Example:

- European option on a futures at expiry:
  \[ X = \{ F( T, T) \} \]

- Option on a spread between two futures contract:
  \[ X = \{ F( T_1, T_1), F( T_1, T_2) \} \]
Measure of similarity

Kullback-Leibler divergence between two equivalent probability distributions.

\[ D(Q|P) = E_Q \left( \ln \left( \frac{dQ}{dP} \right) \right) \]
\[ = E_P \left( \frac{dQ}{dP} \ln \left( \frac{dQ}{dP} \right) \right) \]
Kernel-based density estimation

Let \( X_i, i = 1, \ldots, n \) be a sample of random vectors in \( U \subseteq R^d \), drawn from a distribution with density function \( f \). The kernel density estimate of \( f \), noted \( \hat{f} \) is defined by:

\[
\hat{f}(X) = \frac{1}{n} \sum_{i=1}^{n} K_H(X - X_i)
\]

where:

- \( X \) is a vector in \( R^d \)
- \( H \) is a symmetric, positive definite matrix of rank \( d \)
- \( K_H \) is the kernel function: a symmetric multivariate density:

\[
K_H(X) = \|H\|^{-\frac{1}{2}} K(H^{-\frac{1}{2}}X)
\]
Kernel-based density estimation

Let $\rho_k(U, X)$ be the distance to the $k$-th nearest neighbor of $X$ in $U$.
Using $\rho_k(U, X)$ as bandwidth and the uniform kernel, we get:

$$\hat{f}(X) = \frac{k}{n \rho_k^d(X) v_d}$$
The KL divergence can be written:

\[
D(Q|P) = \int_{R^d} f_Q(x) \ln \left( \frac{f_Q(x)}{f_P(x)} \right) dx \tag{11}
\]

\[
= H^\times(f_Q, f_P) - H(f_Q) \tag{12}
\]

where \( H \) is the differential entropy

\[
H(f_Q) = -\int_{R^d} f_Q(x) \ln(f_Q(x)) dx \tag{13}
\]

and \( H^\times \) is the cross entropy

\[
H^\times(f_Q, f_P) = -\int_{R^d} f_Q(x) \ln(f_P(x)) dx
\]
KL Divergence

Use the expression for density in terms of k-nn statistics to get:

\[
D(f_Q, f_P) = \ln \left( \frac{|U|}{|V|} \right) + \frac{d}{|V|} \sum_{X \in V} \ln \left( \frac{\rho_k(U, X)}{\rho_k(V, X)} \right)
\]  

(14)
Algorithm

Calculation of model risk associated with the use of model $P$ for pricing derivative $X$.

1. Calibrate model $P$ on a set of benchmark instruments.
2. Simulate a set of paths for the feature vector $X$ under $P$. Let $V$ be this set.
3. Compute the minimum and maximum value of the exotic derivative according to each model in set $Q$.
4. Model risk is finally computed by

\[ \mu_Q = \bar{\pi}(X) - \underline{\pi}(X) \]
Construction of $\mathcal{Q}$

1. Build a list $\{Q_j\}$ of candidate models. This list may be built by perturbation of the parameters of $P$, or by postulating alternate stochastic processes. There are no constraints on the method used.

2. Price the benchmark data with the perturbed models, and only retain models $Q_j$ such that

$$E^{Q_j}(H_i) \in [C^{\text{bid}_i}, C^{\text{ask}_i}] \quad \forall i \in I$$

3. Simulate a set of paths of the feature vector $X$ for each model $Q_j$. Let $U_j$ be the corresponding set.

4. Compute the KL divergence by (14).

5. Only retain the models $Q_j$ such that:

$$D(Q_j|P) \leq \epsilon_{\text{KL}}$$

6. The elements of $\{Q_j\}$ that satisfy (2) and (5) form the set $\mathcal{Q}$. 
Generation of Candidate Models

**Figure:** First 100 Sobol sequences in 2D
KL Divergence by Features

- KL−D for min/max
- KL−D for even sample

Theta levels:
- (0.273,0.31]
- (0.241,0.273]

Kappa levels:
- (0.0953,0.109]
- (0.109,0.123]
- (0.123,0.14]
- (0.14,0.159]
KL divergence and pricing error

**Figure:** KL divergence vs. benchmark pricing error
Model Risk

**Figure:** Model risk as a function of KL divergence
Conclusion

- We have described a procedure, inspired from the field of image processing, for computing the degree of similarity between models, independently of the definition of these processes.
- We then use this measure to compute a normalized measure of model risk.
- The definition of the set of alternate models $Q$ is crucial to the measure.
- We have presented two implementations:
  - a computationally efficient method that restricts the set $Q$ to perturbations of the reference model
  - a more computationally intensive method that puts no restrictions on the composition of $Q$. 
Model 1 (perturbation of scenario weights)

Problem dimensions:
- 50,000 scenarios
- Feasible set calculation (LP): 50,000 variables
- Min/Max value (Dual): 50 variables

Technology:
- Asset pricing: R/Quantlib and Rmetrics
- Scenario generation: R
- Calculation of feasible set: R/GLPK
- Solution of dual problem: geometric programming with CVXOPT
Model 2 (KL divergence by k-nn algorithm)

- model calibration: QuantLib
- size of $Q$: 10000
- calculation of $D(Q||P)$:
  - 10,000 scenarios per $Q_j$
  - R/FNN package
- exotic pricing: R
Bibliography & Resources

- P. Hénaff & C. Martini: *Model Validation: theory, practice and perspectives*
- P. Hénaff: *A Normalized Measure of Model Risk*
- www.QuantLib.org
- FNN package in R
- cvxopt