Lévy driven financial models

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QQ plots for Deutsche Bank
zero-bond log-returns (1985-95), 5 years to maturity
empirical densities calculated from zero-yield data for Germany

-0.010 -0.005 0.0 0.005 0.010
0 100 200 ... (1985-95), 5 years to maturity
empirical normal

log return x of zero-bond

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log return x of zero-bond

\log(\text{density}(x))

-0.010 -0.005 0.0 0.005 0.010

-8 -6 -4 -2 0 2 4 ...

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empirical normal

log return x of zero-bond

\log(\text{density}(x))

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empirical normal
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Lévy processes

\( L = (L_t)_{t \geq 0} \) process with stationary and independent increments on a probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\)

càdlàg paths: right-continuous with left limits

canonical representation

\[
L_t = bt + \sqrt{c} W_t + Z_t + \sum_{s \leq t} \Delta L_s \mathbf{1}_{\{|\Delta L_s| > 1\}}
\]

\( b \) and \( c \geq 0 \) real numbers, \((W_t)_{t \geq 0}\) standard Brownian motion

\((Z_t)_{t \geq 0}\) purely discontinuous martingale independent of \((W_t)_{t \geq 0}\)

\( \Delta L_s = L_s - L_{s-} \) jump at time \( s > 0 \)
Semimartingale representation

$$(X_t)_{t \geq 0} \text{ semimartingale with } X_0 = 0$$

$$X_t - \sum_{s \leq t} \Delta X_s \mathbb{I}_{\{|\Delta X_s| > 1\}} \text{ process with bounded jumps}$$

→ special semimartingale: unique decomposition into a local martingale and a predictable process with finite variation

$$X_t = V_t + M_t + \sum_{s \leq t} \Delta X_s \mathbb{I}_{\{|\Delta X_s| > 1\}}$$

$$M_t = M^c_t + M^d_t \quad M^c \text{ continuous, } M^d \text{ purely discontinuous}$$

For Lévy processes: $V_t = bt$ and $M^c = \sqrt{c} W_t$

$$(Z_t)_{t \geq 0} = (M^d_t)_{t \geq 0} \quad \text{the purely discontinuous local martingale (of a Lévy process)}$$
The purely discontinuous martingale

\[ \sum_{s \leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s| \leq 1\}} \] does not converge in general \( \rightarrow \) compensating

\[
P \underset{\varepsilon \downarrow 0}{\lim} \left( \sum_{s \leq t} \Delta X_s \mathbb{1}_{\{\varepsilon \leq |\Delta X_s| \leq 1\}} - t \int \chi \mathbb{1}_{\{\varepsilon \leq |\chi| \leq 1\}} F(d\chi) \right)
\]

What is \( F \)?

\[
\mu^X(\omega; dt, d\chi) = \sum_{s > 0} \mathbb{1}_{\{\Delta X_s(\omega) \neq 0\}} \mathcal{E}(s, \Delta X_s(\omega)) (dt, d\chi)
\]

random measure of jumps of \( X = (X_t)_{t \geq 0} \)

\[
\mu^X(\omega; [0, t] \times A) \quad \text{counts how many jumps of size within } A \text{ occur for path } \omega \text{ from } 0 \text{ to } t
\]

\[
E[\mu^X(\cdot; [0, t] \times A)] = tF(A), \quad F \text{ intensity measure or Lévy measure}
\]

\[
M_t^d = \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| \leq 1\}} \left( \mu^X(ds, d\chi) - dsF(d\chi) \right)
\]

cannot be separated in general
Local characteristics of a Lévy process (1)

Triplet of local characteristics of \((L_t)_{t \geq 0}\) : \((b, c, F)\)

\[
L_t = bt + \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| \leq 1\}} \left( \mu^L(ds, dx) - dsF(dx) \right)
+ \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| > 1\}} \mu^L(ds, dx)
\]

\(\nu = \mathcal{L}(L_1)\) is infinitely divisible

\[
\mathcal{L}(L_1) = \mathcal{L}(L_{1/n}) \ast \cdots \ast \mathcal{L}(L_{1/n})
\]
Local characteristics of a Lévy process (2)

Fourier transform in Lévy–Khintchine form

\[
E[\exp(iuL_1)] = \exp \left[ iub - \frac{1}{2}u^2c + \int_{\mathbb{R}} \left( e^{iux} - 1 - iux1_{\{|x| \leq 1\}} \right) F(dx) \right] \\
= \exp(\psi(u))
\]

\[F\] Lévy measure: \( \int_{\mathbb{R}} \min(1, x^2) F(dx) < \infty \)

pricing of derivatives: \( E[f(L_T)] \)

uses \( E[\exp(iuL_T)] = \exp(\psi(u))^T \)
Examples of Lévy measures

The density of the Lévy measure of the normal inverse Gaussian (left) and the $\alpha$-stable process.
Integrability properties of the Lévy measure

Finiteness of the moments of the process depends on the frequency of the large jumps

**Proposition**

Let $L$ be a Lévy process with triplet $(b, c, F)$.

1. $E[|L_t|^p] < \infty$ for $p \in \mathbb{R}_+$ if and only if $\int_{\{|x|>1\}} |x|^p F(dx) < \infty$.

2. $E[\exp(pL_t)] < \infty$ for $p \in \mathbb{R}$ if and only if $\int_{\{|x|>1\}} \exp(px) F(dx) < \infty.$
Consequences for the representation

If \( \mathcal{L}(L_1) \) has a finite expectation then \( \int_{\{|x|>1\}} xF(dx) < \infty \)

→ add \( - \int iux \mathbb{1}_{\{|x|>1\}} F(dx) \) to the characteristic exponent

\[
E[\exp(iuL_1)] = \exp \left[ iub - \frac{1}{2} u^2 c + \int_{\mathbb{R}} (e^{iux} - 1 - iux) F(dx) \right]
\]

(different \( b \))

In the same way \( \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x|>1\}} (\mu^L(ds,dx) - dsF(dx)) \) can be added to \( (M^d_t)_{t \geq 0} \), consequently

\[
L_t = bt + \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} x(\mu^L(ds,dx) - dsF(dx))
\]

Martingale if \( b = E[X_1] = 0 \)

Submartingale if \( b > 0 \), supermartingale if \( b < 0 \)
Example (Poisson process)

Lévy measure \( F = \lambda \mathcal{E}_1 \), \( \lambda \) intensity parameter

No Gaussian component: \( c = 0 \)

→ jumps of size 1 occur with average rate \( \lambda \) per unit time

Fourier transform

\[
E[\exp(iuL_t)] = \exp[\lambda t(e^{iu} - 1)]
\]

Canonical representation

\[
L_t = \lambda t + (L_t - \lambda t)
\]

\[
= \lambda t + \left( \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}} - \lambda t \right)
\]
Generalized hyperbolic distributions
(O.E. Barndorff-Nielsen (1977))

Density: \( d_{GH}(x) = a(\lambda, \alpha, \beta, \delta) \left( \delta^2 + (x - \mu)^2 \right)^{(\lambda-1/2)/2} \times K_{\lambda-1/2} \left( a\sqrt{\delta^2 + (x - \mu)^2} \right) \exp(\beta(x - \mu)) \)

\[
a(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi \alpha^{\lambda-1/2} \delta^\lambda} K_{\lambda} \left( \delta \sqrt{\alpha^2 - \beta^2} \right)}
\]

\( K_{\lambda} \) modified Bessel function of the third kind with index \( \lambda \)

Parameters:
\[
\lambda \in \mathbb{R} \quad \text{Class parameter} \quad \mu \in \mathbb{R} \quad \text{Location}
\]
\[
\alpha > 0 \quad \text{Shape} \quad \delta > 0 \quad \text{Scale parameter}
\]
\[
\beta \text{ with } 0 \leq |\beta| < \alpha \quad \text{Skewness} \quad (\text{Volatility})
\]
Special cases

Hyperbolic
Normal inverse Gaussian (NIG)
Normal reciprocal inverse Gaussian (NRIG)
Variance gamma
Student $t$ (limiting case)
Cauchy (limiting case)
Skewed Laplace
Normal (limiting case)
Generalized inverse Gaussian (limiting case)
Hyperbolic distribution \( (\lambda = 1) \)

\[
d_H(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1\left(\delta \sqrt{\alpha^2 - \beta^2}\right)} \exp\left(-\alpha\sqrt{\delta^2 + (x - \mu)^2} + \beta(x - \mu)\right)
\]


Normal inverse Gaussian (NIG) \( (\lambda = -1/2) \)

\[
d_{NIG}(x) = \frac{\alpha\delta}{\pi} \exp\left(\delta \sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{K_1(\alpha g_\delta(x - \mu))}{g_\delta(x - \mu)}
\]

where \( g_\delta(x) = \sqrt{\delta^2 + x^2} \)

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Credit profit and loss distribution
Activity and variation

Proposition

Let $L$ be a Lévy process with triplet $(b, c, F)$.

1. If $F(\mathbb{R}) < \infty$ then almost all paths of $L$ have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.

2. If $F(\mathbb{R}) = \infty$ then almost all paths of $L$ have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.

Proposition

Let $L$ be a Lévy process with triplet $(b, c, F)$.

1. If $c = 0$ and $\int_{|x| \leq 1} |x| F(dx) < \infty$ then almost all paths of $L$ have finite variation.

2. If $c \neq 0$ or $\int_{|x| \leq 1} |x| F(dx) = \infty$ then almost all paths of $L$ have infinite variation.
Fine structure of the paths

1. $F(R) < \infty \iff \int_{\{|x| \leq 1\}} F(dx) < \infty \implies \text{finite activity}$

2. $F(R) = \infty \iff \int_{\{|x| \leq 1\}} F(dx) = \infty \implies \text{infinite activity}$

3. $\int_{\{|x| \leq 1\}} |x| F(dx) < \infty \text{ (and } c = 0) \implies \text{finite variation}$

→ the sum of the small jumps converges and

$$ \int_0^t \int_{\mathbb{R}} x(\mu^L(ds, dx) - dsF(dx)) = \int_0^t \int_{\mathbb{R}} x\mu^L(ds, dx) - t \int_{\mathbb{R}} xF(dx) $$
Financial modeling

Stock prices and indices: geometric Brownian motion (Samuelson 1965)

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \]

solution

\[ S_t = S_0 \exp \left( \sigma W_t + \left( \mu - \frac{\sigma^2}{2} \right) t \right) \]

Log returns:

\[ \log S_{t+1} - \log S_t \sim N \left( \mu - \frac{\sigma^2}{2}, \sigma^2 \right) \]

Correct return distributions: key ingredient

Consistency of the model along different time grids
zero-bond log-returns (1985-95), 5 years to maturity
empirical densities calculated from zero-yield data for Germany

log return \( x \) of zero-bond

density \( (x) \)

-0.010 -0.005 0.0 0.005 0.010

0 100 200 ... (1985-95), 5 years to maturity
empirical normal NIG
Exponential Lévy model

\[ S_t = S_0 \exp(L_t) \]

\[ L = (L_t)_{t \geq 0} \quad \text{Lévy process with } \mathcal{L}(L_1) = \nu \]

Along a time grid with span 1: exact log returns

Alternative description by a stochastic differential equation

\[ dS_t = S_t \left( dL_t + \frac{c}{2} \, dt + \int_{\mathbb{R}} (e^x - 1 - x)\mu^L(dt, dx) \right) \]

can be written as

\[ dS_t = S_t \, d\tilde{L}_t \]

where \((\tilde{L}_t)_{t \geq 0}\) is a Lévy process with jumps > \(-1\)
GH Levy process with marginal densities
Consistency along different time grids

Models are typically fitted (calibrated) on the basis of daily data (e.g. daily closing prices)

Does this model describe the price movements for an intraday or weekly horizon?

*Classical Gaussian model:* Log-returns are always normally distributed (selfsimilarity of Brownian motion)

*GH model:* Empirical investigation shows that the model provides rather good distributions along other time grids as well

- Daily returns
- One hour returns
- Fitted hyperbolic density

- **One hour returns**
- **Daily returns**

The graph compares the empirical densities of daily and one hour returns of Bayer data from January 1992 to August 1994.
Martingality of exponential Lévy models

\[ S_t = S_0 \exp(L_t) \]

Pricing of derivatives: martingale model

Necessary assumption: \( E[S_t] = S_0 E[\exp(L_t)] < \infty \)

This excludes a priori the class of stable processes in general

\[ E[\exp(L_t)] < \infty \implies E[L_t] < \infty \]

consequently

\[ L_t = bt + \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} x (\mu^L(ds, dx) - dsF(dx)) \]

\[ S_t = S_0 \exp(L_t) \] is a martingale if

\[ b = -\frac{c}{2} - \int_{\mathbb{R}} (e^x - 1 - x) F(dx) \]

Use either Itô’s formula or verify that

\[ M_t = \frac{\exp(L_t)}{E[\exp(L_t)]} \] is a martingale
Equivalent martingale measures (EMMs)

\[(e^{-rt}S_t)_{t \geq 0}\) has to be a martingale

In general large set of EMMs: market is incomplete

Characterization of the set of all EMMs (Eberlein and Jacod (1997))

Characterization of those EMMs under which \(L\) is again a Lévy process

The range of call option prices under all EMMs spans the whole no-arbitrage interval (Eberlein and Jacod (1997))

\[\left((S_0 - Ke^{-rT})^+, S_0\right)\]

Criteria to choose an EMM: Esscher transform, minimal distance MM, minimal entropy MM, utility functions, …

→ whole industry
Pricing of derivatives

\( f(S_T) \) payoff of the option at maturity \( T \)

\( f(x) = (x - K)^+ \) European call option

\( f(x) = (K - x)^+ \) European put option

Similarly: digitals, quantos, asset-or-nothing, power options, . . .

Given a specific martingale measure (calibration to market data)

\[
V = E[E^{-rT} f(S_T)]
\]

Explicit formula for European call

\[
V = S_0 \int_{\gamma}^{\infty} T^* d_{GH}(x; \theta + 1) \, dx - e^{-rT} K \int_{\gamma}^{\infty} T^* d_{GH}(x; \theta) \, dx
\]

where \( \gamma = \ln(K/S_0) \) and \( T^* \) GH-density under risk-neutral measure
Raible’s method

Numerical evaluation based on bilateral Laplace transforms (Raible (2000)).

We want to price a European call option;

\[ V = e^{-rT} \mathbb{E}[(S_T - K)^+] = e^{-rT} \int_{\Omega} (S_T - K)^+ \, d\mathbb{P} \]

\[ = e^{-rT} \int_{\mathbb{R}} (S_0 e^x - K)^+ \, d\mathbb{P}_L(x) = e^{-rT} \int_{\mathbb{R}} (S_0 e^x - K)^+ \rho(x) \, dx \]

if \( \mathbb{P}_L \) is absolutely continuous with respect to the Lebesgue measure \( \lambda \) with density \( \rho \). Define \( g(x) = (e^{-x} - K)^+ \) and \( \zeta = -\log S_0 \), then

\[ V = e^{-rT} \int_{\mathbb{R}} g(\zeta - x) \rho(x) \, dx = e^{-rT} (g * \rho)(\zeta) \]
Raible’s method (cont.)

is a convolution at point $\zeta$. Passing to bilateral Laplace transforms $\mathcal{L}_h(z)$, $z \in \mathbb{C}$

$$
\mathcal{L}_V(z) = e^{-rT} \int_{\mathbb{R}} e^{-zx} (g * \rho)(x) \, dx \\
= e^{-rT} \int_{\mathbb{R}} e^{-zx} g(x) \, dx \int_{\mathbb{R}} e^{-zx} \rho(x) \, dx = e^{-rT} \mathcal{L}_g(z) \mathcal{L}_\rho(z).
$$

$\mathcal{L}_g$ can be calculated explicitly; $\mathcal{L}_\rho$ can be expressed in terms of the characteristic function $\varphi_L$ (Lévy–Khintchine formula).

By numerically inverting the Laplace transform, we recover the option price.

The method applies to any European – hence path-independent – payoff, such as call, put, digital, self-quanto and power options.

The Lévy motions we are interested in, e.g. generalized hyperbolic, have a (known) Lebesgue density.
Supremum and infimum processes

Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastic process. Denote by

$$
\overline{X}_t = \sup_{0 \leq u \leq t} X_u \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq u \leq t} X_u
$$

the supremum and infimum process of $X$ respectively. Since the exponential function is monotone and increasing

$$
\overline{S}_T = \sup_{0 \leq t \leq T} S_t = \sup_{0 \leq t \leq T} \left( S_0 e^{L_t} \right) = S_0 e^{\sup_{0 \leq t \leq T} L_t} = S_0 e^{\overline{L}_T}.
$$

(1)

Similarly

$$
\underline{S}_T = S_0 e^{\underline{L}_T}.
$$

(2)
Valuation formulas – payoff functional

We want to price an option with payoff $\Phi(S_t, 0 \leq t \leq T)$, where $\Phi$ is a measurable, non-negative functional.

Separation of payoff function from the underlying process:

**Example**

Fixed strike lookback option

\[
(S_T - K)^+ = (S_0 e^{\tilde{L}} - K)^+ = (e^{\tilde{L} + \log S_0} - K)^+
\]

1. The *payoff function* is an arbitrary function $f : \mathbb{R} \to \mathbb{R}_+$; for example $f(x) = (e^x - K)^+$ or $f(x) = \mathbb{1}_{\{e^x > B\}}$, for $K, B \in \mathbb{R}_+$.

2. The *underlying process* denoted by $X$, can be the log-asset price process or the supremum/infimum or an average of the log-asset price process (e.g. $X = L$ or $X = \tilde{L}$).
Valuation formulas

Consider the option price as a function of $S_0$ or better of $s = -\log S_0$.

$X$ driving process ($X = L, \bar{L}, L$, etc.)

$$\Rightarrow \Phi(S_0 e^{Lt}, 0 \leq t \leq T) = f(X_T - s)$$

Time-0 price of the option (assuming $r \equiv 0$)

$$\nabla_f(X; s) = E[\Phi(S_t, 0 \leq t \leq T)] = E[f(X_T - s)]$$

Valuation formulas based on Fourier and Laplace transforms

Carr and Madan (1999) plain vanilla options

Raible (2000) general payoffs, Lebesgue densities

Borovkov and Novikov (2002) plain vanilla and lookback options

In these approaches: Some sort of continuity assumption (payoff or random variable)
Valuation formulas – assumptions

$M_{X_T}$ moment generating function of $X_T$

$g(x) = e^{-Rx}f(x)$ (for some $R \in \mathbb{R}$) dampened payoff function

$L_{bc}^1(\mathbb{R})$ bounded, continuous functions in $L^1(\mathbb{R})$

Assumptions

(C1) $g \in L_{bc}^1(\mathbb{R})$

(C2) $M_{X_T}(R)$ exists

(C3) $\hat{g} \in L^1(\mathbb{R})$
Valuation formulas

**Theorem**

Assume that (C1)–(C3) are in force. Then, the price $\nabla_f(X; s)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff $f(X_T)$ is given by

$$\nabla_f(X; s) = \frac{e^{-Rs}}{2\pi} \int_{\mathbb{R}} e^{ius} \varphi_{X_T}(-u - iR) \hat{f}(u + iR) du,$$

where $\varphi_{X_T}$ denotes the extended characteristic function of $X_T$ and $\hat{f}$ denotes the Fourier transform of $f$.

**Proof**

$$\nabla_f(X; s) = \int_{\Omega} f(X_T - s) dP = e^{-Rs} \int_{\mathbb{R}} e^{Rx} g(x - s) P_{X_T}(dx).$$

(cont. next page)
Proof (cont.)

Under assumption (C1), $g \in L^1(\mathbb{R})$ and $\hat{g}$ is well-defined. With (C3) $\hat{g} \in L^1_{bc}(\mathbb{R})$.

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} \hat{g}(u) du.$$ (5)

Returning to the valuation problem (4) we get

$$\mathbb{V}_f(X; s) = e^{-Rs} \int_{\mathbb{R}} e^{Rx} \left( \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(x-s)u} \hat{g}(u) du \right) P_{X_T}(dx)$$

$$= \frac{e^{-Rs}}{2\pi} \int_{\mathbb{R}} e^{ius} \left( \int_{\mathbb{R}} e^{i(-u-iR)x} P_{X_T}(dx) \right) \hat{g}(u) du$$

$$= \frac{e^{-Rs}}{2\pi} \int_{\mathbb{R}} e^{ius} \varphi_{X_T}(-u - iR) \hat{f}(u + iR) du.$$ (6)
Discussion of assumptions

Alternative choice: \((C1')\) \(g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\)

\((C3')\) \(e^{R \cdot P_{X_T}} \in L^1(\mathbb{R})\)

\((C3')\) \(\Rightarrow e^{R \cdot P_{X_T}}\) has a cont. bounded Lebesgue density

Recall: \((C3)\) \(\hat{g} \in L^1(\mathbb{R})\)

Sobolov space

\[ H^1(\mathbb{R}) = \{ g \in L^2(\mathbb{R}) \mid \partial g \text{ exists and } \partial g \in L^2(\mathbb{R}) \} \]

**Lemma**

\(g \in H^1(\mathbb{R}) \Rightarrow \hat{g} \in L^1(\mathbb{R})\)

Similar for the Sobolev–Slobodeckij space \(H^s(\mathbb{R})\) \((s > \frac{1}{2})\)
Examples of payoff functions

Example (Call and put option)

Call payoff $f(x) = (e^x - K)^+, K \in \mathbb{R}_+$,

$$\hat{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \quad R \in I_1 = (1, \infty). \quad (7)$$

Similarly, if $f(x) = (K - e^x)^+, K \in \mathbb{R}_+$,

$$\hat{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \quad R \in I_1 = (-\infty, 0). \quad (8)$$
Example (Digital option)

**Call payoff** \( I_{\{e^{x} > B\}}, B \in \mathbb{R}^{+} \).

\[
\hat{f}(u + iR) = -B^{iu-R} \frac{1}{iu - R}, \quad R \in I_{1} = (0, \infty).
\] (9)

Similarly, for the payoff \( f(x) = I_{\{e^{x} < B\}}, B \in \mathbb{R}^{+} \),

\[
\hat{f}(u + iR) = B^{iu-R} \frac{1}{iu - R}, \quad R \in I_{1} = (-\infty, 0).
\] (10)

Example (Double digital option)

The payoff of a double digital call option is \( I_{\{B < e^{x} < \overline{B}\}}, B, \overline{B} \in \mathbb{R}^{+} \).

\[
\hat{f}(u + iR) = \frac{1}{iu - R} \left( \overline{B}^{iu-R} - B^{iu-R} \right), \quad R \in I_{1} = \mathbb{R} \setminus \{0\}.
\] (11)
Example (Asset-or-nothing digital)

**Call payoff**  
\[ f(x) = e^x 1_{\{e^x > B\}} \]

\[ \hat{f}(u + iR) = -\frac{B^{1+iu-R}}{1+iu-R}, \quad R \in I_1 = (1, \infty) \]

**Put payoff**  
\[ f(x) = e^x 1_{\{e^x < B\}} \]

\[ \hat{f}(u + iR) = \frac{B^{1+iu-R}}{1+iu-R}, \quad R \in I_1 = (-\infty, 1) \]

Example (Self-quanto option)

**Call payoff**  
\[ f(x) = e^x (e^x - K)^+ \]

\[ \hat{f}(u + iR) = \frac{K^{2+iu-R}}{(1+iu-R)(2+iu-R)}, \quad R \in I_1 = (2, \infty) \]
Non-path-dependent options

European option on an asset with price process $S_t = e^{L_t}$

Examples: call, put, digitals, asset-or-nothing, double digitals, self-quanto options

$\rightarrow X_T \equiv L_T$, i.e. we need $\varphi_{LT}$


$$\varphi_{L_1}(u) = e^{iu\mu} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_\lambda(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}{K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})}$$

$$I_2 = (-\alpha - \beta, \alpha - \beta)$$

$$\varphi_{LT}(u) = (\varphi_{L_1}(u))^T$$

similar: NIG, CGMY, Meixner
Non-path-dependent options II


Stochastic clock \( Y_t = \int_0^t y_s \, ds \quad (y_s > 0) \)
e.g. CIR process

\[
dy_t = K(\eta - y_t) \, dt + \lambda y_t^{1/2} \, dW_t
\]

Define for a pure jump Lévy process \( X = (X_t)_{t \geq 0} \)

\[
H_t = X_{Y_t} \quad (0 \leq t \leq T)
\]

Then

\[
\phi_{H_t}(u) = \frac{\phi_{Y_t}(-i\phi_{X_t}(u))}{(\phi_{Y_t}(-i\phi_{X_t}(-i)))^i u}
\]
### Classification of option types

<table>
<thead>
<tr>
<th>Lévy model</th>
<th>$S_t = S_0e^{L_t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>payoff</strong></td>
<td>$f(x) = (e^x - K)^+$</td>
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<tr>
<td><strong>payoff function</strong></td>
<td>$f(x) = 1_{{e^x &gt; B}}$</td>
</tr>
<tr>
<td><strong>distributional properties</strong></td>
<td>$P_{LT}$ usually has a density</td>
</tr>
<tr>
<td><strong>call</strong></td>
<td>$P_{LT}$ usually has a density</td>
</tr>
<tr>
<td><strong>digital</strong></td>
<td>$P_{LT}$ usually has a density</td>
</tr>
<tr>
<td><strong>digital barrier</strong></td>
<td>$P_{LT}$ usually has a density</td>
</tr>
<tr>
<td><strong>lookback</strong></td>
<td>$P_{LT}$ usually has a density</td>
</tr>
<tr>
<td><strong>= one touch</strong></td>
<td>$P_{LT}$ usually has a density</td>
</tr>
</tbody>
</table>
Valuation formula for the last case

Payoff function $f$ may be discontinuous $P_{X_T}$ does not necessarily possess a Lebesgue density

Assumption

(D1) $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$
(D2) $M_{X_T}(R)$ exists

Theorem

Assume (D1)–(D2) then

$$
\nabla_f(X; s) = \lim_{A \to \infty} \frac{e^{-Rs}}{2\pi} \int_{-A}^{A} e^{-ius} \varphi_{X_T}(u - iR) \hat{f}(iR - u) \, du
$$

(12)

if $\nabla_f(X; \cdot)$ is of bounded variation in a neighborhood of $s$ and $\nabla_f(X; \cdot)$ is continuous at $s$. 
Lookback options

Fixed strike lookback call: $\left( \bar{S}_T - K \right)^+$ (analogous for lookback put).

We get

$$ C_T(\bar{S}; K) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} \varphi_{\bar{L}_T}(-u - iR) \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)} \, du \quad (13) $$

where

$$ \varphi_{\bar{L}_T}(-u - iR) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} e^{T(Y+iv)} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, iu - R)} \, dv \quad (14) $$

for $R \in (1, M)$ and $Y > \alpha^*(M)$.

• The floating strike lookback option, $\left( \bar{S}_T - S_T \right)^+$, is treated by a duality formula (Eb., Papapantoleon (2005)).
One-touch options

One-touch call option: \( \mathbb{1}_{\{\bar{S}_T > B\}} \).

Driving Lévy process \( L \) is assumed to have infinite variation or has infinite activity and is regular upwards. \( L \) satisfies assumption (EM), then

\[
\mathbb{D}C_T(\bar{S}; B) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} S_0^{R+iu} \varphi_{\bar{L}_T}(u-iR) \frac{B^{-R-iu}}{R+iu} du \\
= P(\bar{L}_T > \log(B/S_0))
\]  

for \( R \in (0, M) \), \( \varphi = \alpha^*(M) \) and

\[
\varphi_{\bar{L}_T}(u-iR) = \lim_{N \to \infty} \frac{1}{2\pi} \int_{-N}^{N} e^{T(Y+iv)} \frac{\kappa(Y+iv, 0)}{\kappa(Y+iv, -R-iu)} dv.
\]
Equity default swap (EDS)

- Fixed premium exchanged for payment at “default”
- Default: drop of stock price by 30% or 50% of $S_0 \to$ first passage time
- Fixed leg pays premium $K$ at times $T_1, \ldots, T_N$, if $T_i \leq \tau_B$
- If $\tau_B \leq T$: protection payment $C$, paid at time $\tau_B$
- Premium of the EDS chosen such that initial value equals 0; hence

$$K = \frac{\mathbb{E} \left[ e^{-r \tau_B} \mathbb{1}_{\{\tau_B \leq T\}} \right]}{\sum_{i=1}^{N} \mathbb{E} \left[ e^{-r T_i} \mathbb{1}_{\{\tau_B > T_i\}} \right]}.$$  \hspace{1cm} (17)

- Calculations similar to touch options, since $\mathbb{1}_{\{\tau_B \leq T\}} = \mathbb{1}_{\{S_T \leq B\}}$. 

Options on multiple assets

Basket options
Options on the minimum: \((S_T^1 \land \cdots \land S_T^d - K)^+\)

Multiple functionals of one asset
Barrier options: \((S_T - K)^+\mathbb{1}_{\{S_T > B\}}\)

Slide-in or corridor options: \((S_T - K)^+ \sum_{i=1}^{N} \mathbb{1}_{\{L < S_{T_i} < H\}}\)

Modelling:
\[
S^i_t = S^i_0 \exp(L^i_t) \quad (1 \leq i \leq d)
\]
\[
f : \mathbb{R}^d \longrightarrow \mathbb{R}_+
\]
\[
g(x) = e^{-\langle R, x \rangle} f(x) \quad (x \in \mathbb{R}^d)
\]

Assumptions:
(A1) \(g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\)
(A2) \(M_{X_T}(R)\) exists
(A3) \(\hat{\varrho} \in L^1(\mathbb{R}^d)\) where \(\varrho(dx) = e^{\langle R, x \rangle} P_{X_T}(dx)\)
Sensitivities – Greeks

\[ W_f(X; S_0) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} M_{X_T}(R-iu) \hat{f}(u+iR) \, du \]

Delta of an option

\[ \Delta_f(X; S_0) = \frac{\partial W(X; S_0)}{\partial S_0} = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-1-iu} M_{X_T}(R-iu) \frac{\hat{f}(u+iR)}{(R-iu)^{-1}} \, du \]

Gamma of an option

\[ \Gamma_f(X; S_0) = \frac{\partial^2 W_f(X; S_0)}{\partial^2 S_0} = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-2-iu} M_{X_T}(R-iu) \frac{\hat{f}(u+iR)}{(R-1-iu)^{-1}(R-iu)^{-1}} \, du \]
Numerical examples

Option prices in the 2d Black-Scholes model with negative correlation.

Option prices in the 2d stochastic volatility model.

Option prices in the 2d GH model with positive (left) and negative (right) correlation.
Consequences for risk management

More precise quantification of market risk

The stochastic uncertainty of a book or portfolio corresponding to a specified time horizon is given by its

P&L-distribution

Risk measures (e.g. VaR, volatility, shortfall measure)

simple functions of the P&L-distribution

Chance measures (e.g. expected return)

also functions of the P&L-distribution

→ portfolio management
P&L-distribution

![Graph showing the probability distribution of Value at Risk (in $1000) vs. Loss and Profit. The graph includes lines indicating different confidence intervals.](image)

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Standard risk measure: Value at Risk

\[ P[X_t < u_\alpha] = \alpha \]

\( \alpha \)-quantile of return distribution

\[ \text{VaR}(\alpha) = S_0 - S_0 \exp(u_\alpha) \]

Functional value at risk

\[ \alpha \rightarrow \text{VaR}(\alpha) \]

Improvement: Shortfall measure

\[ \text{Shortfall}(\alpha, t) = E[S_0 - S_0 \exp(X_t) \mid X_t < u_\alpha] \]
Profit-and-Loss Distribution

Density

1%-Quantile

Value at Risk

0
Value at Risk

- empirical
- normal
- hyperbolic
- NIG
- GH

level of probability

Value at Risk

0.0 0.01 0.02 0.03 0.04 0.05
0.0 0.02 0.04 0.06 0.08

empirical
normal
hyperbolic
NIG
GH
Stochastic volatility

Basic model \( S_t = S_0 \exp(X_t) \) where \( X_t = \mu t + \sigma L_t \)

\( (L_t)_{t \geq 0} \) standardized Lévy process: \( E[L_1] = 0 \) and \( \text{Var}(L_1) = 1 \)

\( \sigma \longrightarrow (\sigma_t)_{t \geq 0} \) volatility process

Dynamic version: \( \text{d}X_t = \sigma_t \text{d}L_t \)

Discrete version: \( \Delta X_t = \sigma_t \Delta L_t \)

Various models for \( (\sigma_t)_{t \geq 0} \): historic volatility

Ornstein–Uhlenbeck process

\( \text{d}(\log \sigma_t^2) = -a(\log \sigma_t^2 - c) \, \text{d}t + b \, \text{d}B_t \)

GARCH model

implicit volatility
implied volatility

times of excessive losses

<table>
<thead>
<tr>
<th>implied volatility</th>
<th>times of excessive losses</th>
</tr>
</thead>
<tbody>
<tr>
<td>0100200300400500</td>
<td></td>
</tr>
<tr>
<td>600 800 1000 1200 1400 1600 1800</td>
<td></td>
</tr>
<tr>
<td>VaR and actual losses</td>
<td></td>
</tr>
</tbody>
</table>

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Interest rate models should be able to reproduce

- the observable term structures of interest rates,
- market prices of interest rate derivatives (caps, floors, swaptions)

but they should also be

- analytically tractable.

Idea: Use an HJM-type model driven by a (possibly non-homogeneous) Lévy process
Real and estimated interest rates of the USA

Svensson parameters: \( b_0 = 0.053 \), \( b_1 = -0.042 \), \( b_2 = -0.041 \), \( b_3 = -0.009 \), \( \tau_1 = 1.479 \), \( \tau_2 = 0.329 \)

Termstructure, February 17, 2004
Comparison of estimated interest rates (least squares Svensson)

Termstructure, February 17, 2004
Nelson–Siegel (1987) curves \( m = \text{maturity}, \quad \) parameters \( \beta_0, \beta_1, \beta_2, \tau_1 \)

\[
s(m) = \beta_0 + \beta_1 \left( 1 - \exp \left( -\frac{m}{\tau_1} \right) \right) \left( \frac{m}{\tau_1} \right)^{-1} \\
+ \beta_2 \left( \left( 1 - \exp \left( -\frac{m}{\tau_1} \right) \right) \left( \frac{m}{\tau_1} \right)^{-1} - \exp \left( -\frac{m}{\tau_1} \right) \right)
\]

Improved curves: Svensson (1994) \( \) parameters \( \beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2 \)

\[
s(m) = \beta_0 + \beta_1 \left( 1 - \exp \left( -\frac{m}{\tau_1} \right) \right) \left( \frac{m}{\tau_1} \right)^{-1} \\
+ \beta_2 \left( \left( 1 - \exp \left( -\frac{m}{\tau_1} \right) \right) \left( \frac{m}{\tau_1} \right)^{-1} - \exp \left( -\frac{m}{\tau_1} \right) \right) \\
+ \beta_3 \left( \left( 1 - \exp \left( -\frac{m}{\tau_2} \right) \right) \left( \frac{m}{\tau_2} \right)^{-1} - \exp \left( -\frac{m}{\tau_2} \right) \right)
\]
One month rate at the German market, March 01, 1967 – March 31, 1997
**Short rate dynamics**

Merton (1970) \[dr_t = \theta \, dt + \sigma \, dB_t\]

Vasiček (1977) \[dr_t = k(\theta - r_t) \, dt + \sigma \, dB_t\]

Dothan (1978) \[dr_t = ar_t \, dt + \sigma r_t \, dB_t\]

Brennan-Schwartz (1979) \[dr_t = (\theta(t) - ar_t) \, dt + \sigma r_t \, dB_t\]

Constantinides-Ingersoll (1984) \[dr_t = \sigma r_t^{3/2} \, dB_t\]

Cox-Ingersoll-Ross (1985) \[dr_t = k(\theta - r_t) \, dt + \sigma \sqrt{r_t} \, dB_t\]

Ho-Lee (1986) \[dr_t = \theta(t) \, dt + \sigma \, dB_t\]

Black-Derman-Toy (1990) \[dr_t = r_t \left( \theta(t) - a \ln r_t + \frac{1}{2} \sigma^2(t) \right) \, dt + \sigma(t)r_t \, dB_t\]

Hull-White (1990) \[dr_t = k(\theta(t) - r_t) \, dt + \sigma(t)\sqrt{r_t} \, dB_t\]

Sandmann-Sondermann (1993) \[dr_t = (1 - e^{-r_t}) \left[ (\theta(t) - \frac{1}{2}(1 - e^{-r_t})\sigma^2) \, dt + \sigma \, dB_t \right]\]
Classical modeling of the
dynamics of term structures

\[ B(t, T) \text{ price at time } t \in [0, T] \text{ of a default-free zero coupon bond with } \]
maturity \( T \in [0, T^*] \quad (B(T, T) = 1) \)

\[ f(t, T) \text{ instantaneous forward rate: } B(t, T) = \exp \left( - \int_t^T f(t, u) \, du \right) \]

Heath, Jarrow, Morton (HJM) framework

\[ df(t, T) = \alpha(t, T) \, dt + \sigma(t, T)^\top \, dW_t \]

\( (W_t)_{t \geq 0} \quad d\text{-dimensional Brownian motion} \)

\( \sigma(t, T) \quad \text{volatility structure (e.g. Vasiček)} \)

Under the risk-neutral measure

\[ B(t, T) = B(0, T) \exp \left[ \int_0^t r(s) \, ds - \frac{1}{2} \int_0^t |\sigma^*(s, T)|^2 \, ds + \int_0^t \sigma^*(s, T)^\top \, dW_s \right] \]

where \( r(t) = f(t, t) \quad \text{short rate} \)
zero-bond log-returns (1985-95), 10 years to maturity
empirical densities calculated from zero-yield data for Germany

-0.02 -0.01 0.0

0.01

0.02

0 20 40 60 80 ... (1985-95), 10 years to maturity
empirical densities calculated from zero-yield data for Germany

empirical

normal

log return \( x \) of zero-bond

density \( (x) \)
zero-bond log-returns (1985-95), 5 years to maturity

empirical densities calculated from zero-yield data for Germany

log return $x$ of zero-bond density $f(x)$

-0.010 -0.005 0.0 0.005 0.010

0 100 200 ...

(1985-95), 5 years to maturity

empirical normal NIG

log return $x$ of zero-bond density $f(x)$

-0.010 -0.005 0.0 0.005 0.010

0 100 200 ...

(1985-95), 5 years to maturity

empirical normal NIG
The driving process

\( L = (L^1, \ldots, L^d) \) is a \( d \)-dimensional time-inhomogeneous Lévy process, i.e. \( L \) has independent increments and the law of \( L_t \) is given by the characteristic function

\[
\mathbb{E}[\exp(i\langle u, L_t \rangle)] = \exp \int_0^t \theta_s(iu) \, ds \quad \text{with}
\]

\[
\theta_s(z) = \langle z, b_s \rangle + \frac{1}{2} \langle z, c_s z \rangle + \int_{\mathbb{R}^d} \left( e^{\langle z, x \rangle} - 1 - \langle z, x \rangle \right) F_s(dx)
\]

where \( b_t \in \mathbb{R}^d \), \( c_t \) is a symmetric nonnegative-definite \( d \times d \)-matrix and \( F_t \) is a Lévy measure.

Integrability:

\[
\int_0^{T^*} \left( |b_s| + |c_s| + \int_{\{|x| \leq 1\}} x^2 F_s(dx) \right) \, ds < \infty
\]

\[
\int_0^{T^*} \int_{\{|x| > 1\}} \exp(ux) F_s(dx) \, ds < \infty \quad \text{for} \ |u| \leq M
\]
Description in terms of modern stochastic analysis

$L = (L_t)$ is a special semimartingale with canonical representation

$$L_t = \int_0^t b_s \, ds + \int_0^t c_s^{1/2} \, dW_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(ds, dx)$$

and characteristics

$$A_t = \int_0^t b_s \, ds, \quad C_t = \int_0^t c_s \, ds, \quad \nu(ds, dx) = F_s(dx) \, ds$$

$W = (W_t)$ is a standard $d$-dimensional Brownian motion,

$\mu^L$ the random measure of jumps of $L$ and $\nu$ is the compensator of $\mu^L$

$L$ is also called a process with independent increments and absolutely continuous characteristics (PIIAC)
Simulation of a GH Lévy motion

NIG Levy process with marginal densities

values of NIG(100,0,1,0) Levy process

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Simulation of a Lévy process

$\text{NIG}(10,0,0.100,0)$ on $[0,1]$

$\text{NIG}(10,0,0.025,0)$ on $[1,3]$
Lévy forward rate approach

Eberlein, Raible (1999), Eberlein, Özkan (2003),

df(t, T) = \partial_2 A(t, T) \, dt - \partial_2 \Sigma(t, T) \, dL_t \quad (0 \leq t \leq T \leq T^*)

Σ and A are deterministic functions with values in \( \mathbb{R}^d \) and \( \mathbb{R} \) respectively whose paths are continuously differentiable in the second variable.

The volatility structure is bounded \( 0 \leq \Sigma^i(t, T) \leq M \) \((i \in \{1, \ldots, d\})\).

Furthermore, \( \Sigma(t, T) \neq 0 \) for \( t < T \) and \( \Sigma(T, T) = 0 \) for \( T \in [0, T^*] \).

The drift condition \( A(t, T) = \theta_s(\Sigma(t, T)) \) holds.
Implications

Savings account and default-free zero coupon bond prices are given by

\[ B_t = \frac{1}{B(0, t)} \exp \left( \int_0^t \theta_s(\Sigma(s, t)) \, ds - \int_0^t \Sigma(s, t) \, dL_s \right) \quad \text{and} \]

\[ B(t, T) = B(0, T)B_t \exp \left( -\int_0^t \theta_s(\Sigma(s, T)) \, ds + \int_0^t \Sigma(s, T) \, dL_s \right). \]

Bond prices, once discounted by the savings account, are martingales.

In case \( d = 1 \), the martingale measure is unique (see Eberlein, Jacod, and Raible (2004)).
Key tool

\[ L = (L^1, \ldots, L^d) \]  \hspace{1cm} \text{\textit{d}-dimensional time-inhomogeneous Lévy process}

\[ \mathbb{E}[\exp(i\langle u, L_t \rangle)] = \exp \int_0^t \theta_s(iu) \, ds \quad \text{where} \]

\[ \theta_s(z) = \langle z, b_s \rangle + \frac{1}{2} \langle z, c_s z \rangle + \int_{\mathbb{R}^d} \left( e^{\langle z, x \rangle} - 1 - \langle z, x \rangle \right) F_s(dx) \]

in case \( L \) is a (time-homogeneous) Lévy process, \( \theta_s = \theta \) is the cumulant (log-moment generating function) of \( L_1 \).

\textbf{Proposition} \hspace{1cm} \textbf{Eberlein, Raible (1999)}

Suppose \( f: \mathbb{R}_+ \to \mathbb{C}^d \) is a continuous function such that \( |\mathcal{R}(f^i(x))| \leq M \) for all \( i \in \{1, \ldots, d\} \) and \( x \in \mathbb{R}_+ \), then

\[ \mathbb{E} \left[ \exp \left( \int_0^t f(s) \, dL_s \right) \right] = \exp \left( \int_0^t \theta_s(f(s)) \, ds \right) \]

Take \( f(s) = \sum (s, T) \) for some \( T \in [0, T^*] \)
Pricing of European options

\[
B(t, T) = B(0, T) \exp \left[ \int_0^t (r(s) + \theta_s(\Sigma(s, T))) \, ds + \int_0^t \Sigma(s, T) \, dL_s \right]
\]

where \( r(t) = f(t, t) \) short rate

\[
V(0, t, T, w) = \mathbb{E}_{\mathbb{P}^*} [B_t^{-1} w(B(t, T), K)]
\]

Volatility structures

\[
\Sigma(t, T) = \frac{\hat{\sigma}}{a} (1 - \exp(-a(T - t))) \quad \text{(Vasiček)}
\]

\[
\Sigma(t, T) = \hat{\sigma} (T - t) \quad \text{(Ho–Lee)}
\]

Fast algorithms for Caps, Floors, Swaptions, Digitals, Range options
Forward measure associated with data $T \leq T^*$

Density $\frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \frac{1}{B_T B(0, T)}$ or $\mathbb{E}^* \left[ \frac{d\mathbb{P}_T}{d\mathbb{P}^*} \mid \mathcal{F}_t \right] = \frac{B(t, T)}{B_t B(0, T)}$

For the case of the Lévy term structure model this equals

$$\exp \left( \int_0^t \Sigma(s, T) \, dL_s - \int_0^t \theta_s(\Sigma(s, T)) \, ds \right)$$

Compensator of $\mu^L$ under $\mathbb{P}_T$: $\nu^T(dt, dx) = e^{\langle \Sigma(t, T), x \rangle} \nu(dt, dx)$

Standard Brownian motion under $\mathbb{P}_T$: $W_t^T = W_t - \int_0^t c_s^{1/2} \Sigma(s, T) \, ds$
Pricing formula for caps

(Eberlein, Kluge (2006))

\[ w(B(t, T), K) = (B(t, T) - K)^+ \]

Call with strike \( K \) and maturity \( t \) on a bond that matures at \( T \)

\[ C(0, t, T, K) = \mathbb{E}_t^{\mathbb{P}^*}[B_t^{-1}(B(t, T) - K)^+] \]

\[ = B(0, t)\mathbb{E}_t^{\mathbb{P}_t}[(B(t, T) - K)^+] \]

Assume \( X = \int_0^t (\Sigma(s, T) - \Sigma(s, t))dL_s \) has a Lebesgue density, then

\[ C(0, t, T, K) = \frac{1}{2\pi} KB(0, t) \exp(R\xi) \]

\[ \times \int_{-\infty}^{\infty} e^{iu\xi}(R + iu)^{-1}(R + 1 + iu)^{-1}M_t^X(-R - iu) \, du \]

where \( \xi \) is a constant and \( R < -1 \).

Analogous for the corresponding put and for swaptions
Calibration to market data
Eberlein–Kluge (2006)

Calibration performed for a driving homogeneous as well as for a time-inhomogeneous Lévy process

Time-inhomogeneous case: piecewise Lévy process (maturities up to 1 year, 1 to 5 years, greater than 5 years)

Minimize the sum of

\[
\left( \frac{\text{model price} - \text{market price}}{\text{ATM market price for the respective maturity}} \right)^2
\]
Caplet market data

Euro caplet implied volatility surface on February 19, 2002
Calibration results

Absolute differences between implied volatility of model and market price
Implied volatility curve for 2 years

- Market price
- Model price (homogeneous)
- Model price (non-homogeneous)
Implied volatility curve for 5 years

Market price
Model price (homogeneous)
Model price (non-homogeneous)
Implied volatility curve for 10 years

- Market price
- Model price (homogeneous)
- Model price (non-homogeneous)
**Basic interest rates**

\( B(t, T) \): price at time \( t \in [0, T] \) of a default-free zero coupon bond

\( f(t, T) \): instantaneous forward rate

\[
B(t, T) = \exp \left( - \int_t^T f(t, u) \, du \right)
\]

\( L(t, T) \): default-free forward Libor rate for the interval \( T \) to \( T + \delta \)

\[
L(t, T) := \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right)
\]

\( F_B(t, T, U) \): forward price process for the two maturities \( T \) and \( U \)

\[
F_B(t, T, U) := \frac{B(t, T)}{B(t, U)}
\]

\[
1 + \delta L(t, T) = \frac{B(t, T)}{B(t, T + \delta)} = F_B(t, T, T + \delta)
\]
LIBOR market model

\[ T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_{M+1} = T^* \]

with \( \delta = T_{n+1} - T_n \) (fixed accrual period)

\[ L(t, T) \] forward LIBOR rate for the interval \( T \) to \( T + \delta \) as of time \( t \leq T \)

\[ \delta \)-forward LIBOR rate \( L(t, T) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right) \]

For two maturities \( T, U \) define the forward process

\[ F_B(t, T, U) = \frac{B(t, T)}{B(t, U)} \]

\( \implies 1 + \delta L(t, T) = F_B(t, T, T + \delta) \)

Sandmann, Sondermann, Miltersen (1995); Miltersen, Sandmann, Sondermann (1997); Brace, Gatarek, Musiela (1997); Jamshidian (1997)
Forward measure associated with data $T \leq T^*$

Density $\frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \frac{1}{B_T B(0, T)}$ or $\mathbb{E}_{\mathbb{P}^*}\left[ \frac{d\mathbb{P}_T}{d\mathbb{P}^*} \mid \mathcal{F}_t \right] = \frac{B(t, T)}{B_t B(0, T)}$

For the case of the Lévy term structure model this equals

$$\exp\left( \int_0^t \Sigma(s, T) \, dL_s - \int_0^t \theta_s(\Sigma(s, T)) \, ds \right)$$

Compensator of $\mu^L$ under $\mathbb{P}_T$: $\nu^T(dt, dx) = e^{\langle \Sigma(t, T), x \rangle} \nu(dt, dx)$

Standard Brownian motion under $\mathbb{P}_T$: $W_t^T = W_t - \int_0^t c_s^{1/2} \Sigma(s, T) \, ds$
The Lévy Libor model

(Eberlein, Özkan (2005))

Tenor structure \( T_0 < T_1 < \cdots < T_M < T_{M+1} = T^* \)

with \( T_{i+1} - T_i = \delta \), set \( T_i^* = T^* - i\delta \) for \( i = 1, \ldots, M \)

\[
\begin{array}{cccccc}
T_M^* & T_{M-1}^* & T_2^* & T_1^* \\
T_M & T_{M-1} & T_2 & T_1 \\
T_0 & T_i & T_2 & T_3 & T_{M-1} & T_M & T^*
\end{array}
\]

Assumptions

(LR.1): For any maturity \( T_i \) there is a bounded deterministic function \( \lambda(\cdot, T_i) \), which represents the volatility of the forward Libor rate process \( L(\cdot, T_i) \).

(LR.2): We assume a strictly decreasing and strictly positive initial term structure \( B(0, T) \) \( (T \in ]0, T^*] \). Consequently the initial term structure of forward Libor rates is given by

\[
L(0, T) = \frac{1}{\delta} \left( \frac{B(0, T)}{B(0, T + \delta)} - 1 \right)
\]
Backward Induction (1)

Given a stochastic basis \((\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}, (\mathcal{F}_t)_{0 \leq t \leq T^*})\)

We postulate that under \(\mathbb{P}_{T^*}\)

\[
L(t, T_1^*) = L(0, T_1^*) \exp \left( \int_0^t \lambda(s, T_1^*) dL_{T^*}^s \right)
\]

where \(L_{T^*}^t = \int_0^t b_s ds + \int_0^t c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} x(\mu_L - \nu_{T^*,L})(ds, dx)\)

is a non-homogeneous Lévy process with random measure of jumps \(\mu_L\) and \(\mathbb{P}_{T^*}\)-compensator \(\nu_{T^*,L}(ds, dx) = F_s(dx) ds, F_s(\{0\}) = 0\), where \(F_s\) satisfies some integrability conditions
Backward Induction (2)

In order to make \( L(t, T^*_1) \) a \( \mathbb{P}_{T^*} \)-martingale, choose the drift characteristic \((b_s)\) s.t.

\[
\int_0^t \lambda(s, T^*_1) b_s \, ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T^*_1) \, ds \\
- \int_0^t \int_{\mathbb{R}} \left( e^{\lambda(s, T^*_1)x} - 1 - \lambda(s, T^*_1)x \right) \nu^{T^*,L}(ds, dx)
\]

Transform \( L(t, T^*_1) \) in a stochastic exponential

\[
L(t, T^*_1) = L(0, T^*_1) \mathcal{E}(H(t, T^*_1))
\]

where

\[
H(t, T^*_1) = \int_0^t \lambda(s, T^*_1) c_s^{1/2} \, dW^*_s + \int_0^t \int_{\mathbb{R}} \left( e^{\lambda(s, T^*_1)x} - 1 \right) (\mu^L - \nu^{T^*,L})(ds, dx)
\]
Backward Induction (3)

Equivalently

\[ dL(t, T^*_1) = L(t^-, T^*_1) \left( \lambda(t, T^*_1) c_t^{1/2} dW_t^{T^*} ight) + \int_{\mathbb{R}} \left( e^{\lambda(t, T^*_1)x} - 1 \right) \left( \mu^L - \nu^{T^*,L} \right) (dt, dx) \]

with initial condition

\[ L(0, T^*_1) = \frac{1}{\delta} \left( \frac{B(0, T^*_1)}{B(0, T^*)} - 1 \right) \]
Backward Induction (4)

Recall \( F_B(t, T_1^*, T^*) = 1 + \delta L(t, T_1^*) \), therefore,

\[
\begin{align*}
\frac{dF_B(t, T_1^*, T^*)}{dt} &= \delta dL(t, T_1^*) \\
&= F_B(t-, T_1^*, T^*) \left( \frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \lambda(t, T_1^*) c_t^{1/2} dW_t^{T^*} \right) \\
&\quad + \int_{\mathbb{R}} \frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \left( e^{\lambda(t, T_1^*) x - 1} \left( \mu^L - \nu^{T^*,L} \right) (dt, dx) \right)
\end{align*}
\]

Define the forward martingale measure associated with \( T_1^* \)

\[
\begin{align*}
\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}_{T^*}} &= \mathcal{E}_{T_1^*}(M^1) \quad \text{where} \\
M_t^1 &= \int_0^t \alpha(s, T_1^*, T^*) c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} \left( \beta(s, x, T_1^*, T^*) - 1 \right) (\mu^L - \nu^{T^*,L}) (ds, dx)
\end{align*}
\]
Backward Induction (5)

\[ F_B(t, T_1^*, T) = \frac{B(t, T_1^*)}{B(t, T^*)} \]

\[
\frac{dP_{T_1^*}}{dP_{T^*}} \bigg|_{\mathcal{F}_t} = \frac{dP_{T_1^*}}{dP^*} \bigg|_{\mathcal{F}_t} = \frac{B(t, T_1^*)}{B_t B(0, T_1^*)} \frac{B_t B(0, T^*)}{B(t, T^*)} = \frac{B(0, T^*)}{B(0, T_1^*)} F_B(t, T_1^*, T^*)
\]
Backward Induction (6)

Then \( W_{t}^{T_{1}^{*}} = W_{t}^{T^{*}} - \int_{0}^{t} \alpha(s, T_{1}^{*}, T^{*})c_{s}^{1/2} \, ds \)

is the forward Brownian motion for date \( T_{1}^{*} \) and

\( \nu^{T_{1}^{*}, L}(dt, dx) = \beta(t, x, T_{1}^{*}, T^{*})\nu^{T^{*}, L}(dt, dx) \)

is the \( \mathbb{P}_{T_{1}^{*}} \)-compensator for \( \mu^{L} \).

Second step

We postulate that under \( \mathbb{P}_{T_{1}^{*}} \)

\( L(t, T_{2}^{*}) = L(0, T_{2}^{*}) \exp \left( \int_{0}^{t} \lambda(s, T_{2}^{*}) \, dL_{s}^{T_{1}^{*}} \right) \)

where

\( L_{t}^{T_{1}^{*}} = \int_{0}^{t} b_{s}^{T_{1}^{*}} \, ds + \int_{0}^{t} c_{s}^{1/2} \, dW_{s}^{T_{1}^{*}} + \int_{0}^{t} \int_{\mathbb{R}} x(\mu^{L} - \nu^{T_{1}^{*}, L})(ds, dx) \)
Backward Induction (7)

$L(t, T^*_2)$ is a $\mathbb{P}_{T^*_1}$-martingale if $(b_{s1}^{T^*_1})$ is chosen s.t.

$$
\int_0^t \lambda(s, T^*_2) b_{s1}^{T^*_1} \, ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T^*_2) \, ds \\
- \int_0^t \int_{\mathbb{R}} \left( e^{\lambda(s, T^*_2)x} - 1 - \lambda(s, T^*_2)x \right) \nu_{T^*_1, L}^{T^*_1, L}(ds, dx)
$$
Backward Induction (8)

Second measure change

\[ \frac{\mathrm{d}\mathbb{P}_{T_2^*}}{\mathrm{d}\mathbb{P}_{T_1^*}} = \mathcal{E}_{T_2^*}(M^2) \]

where

\[ M_t^2 = \int_0^t \alpha(s, T_2^*, T_1^*) c_s^{1/2} \, dW_s^{T_1^*} \]

\[ + \int_0^t \int_{\mathbb{R}} (\beta(s, x, T_2^*, T_1^*) - 1) (\mu^L - \nu^{T_1^*, L})(ds, dx) \]

This way we get for each time point \( T_j^* \) in the tenor structure a Libor rate process which is under the forward martingale measure \( \mathbb{P}_{T_j^*-1} \) of the form

\[ L(t, T_j^*) = L(0, T_j^*) \exp \left( \int_0^t \lambda(s, T_j^*) \, dL_s^{T_j^*-1} \right) \]
Forward process model (1)

Postulate

\[ 1 + \delta L(t, T_1^*) = (1 + \delta L(0, T_1^*)) \exp \left( \int_0^t \lambda(s, T_1^*) \, dL_s^{T_1^*} \right) \]

equivalently

\[ F_B(t, T_1^*, T^*) = F_B(0, T_1^*, T^*) \exp \left( \int_0^t \lambda(s, T_1^*) \, dL_s^{T_1^*} \right) \]

In differential form

\[ dF_B(t, T_1^*, T^*) = F_B(t-, T_1^*, T^*) \left( \lambda(t, T_1^*) c_t^{1/2} \, dW_t^{T_1^*} \right. \]

\[ + \left. \int_{\mathbb{R}} \left( e^{\lambda(t, T_1^*) x} - 1 \right) (\mu_L - \nu_t^{T_1^* L})(dt, dx) \right) \]
Forward process model (2)

Define the forward martingale measure associated with $T_1^*$

\[
\frac{d\mathbb{P}_{T^*_{1}}}{d\mathbb{P}_{T^*}} = \mathcal{E}_{T^*_{1}}(\tilde{M}^1)
\]

where \[
\tilde{M}^1_t = \int_0^t \lambda(s, T^*_{1})c_s^{1/2} \, dW^s_{T^*} + \int_0^t \int_{\mathbb{R}} \left( e^{\lambda(s, T^*_{1})x} - 1 \right) (\mu^L - \nu^{T^*,L})(ds, dx).
\]
Forward process model (3)

Then \( W_{t}^{T_1^*} = W_{t}^{T^*} - \int_{0}^{t} \lambda(s, T_1^*)c_s^{1/2} \, ds \) is the forward Brownian motion for date \( T_1^* \) and

\( \nu^{T_1^*,L}(dt, dx) = \exp(\lambda(t, T_1^*)x) \nu^{T^*,L}(dt, dx) \) is the \( \mathbb{P}_{T_1^*} \)-compensator of \( \mu^L \).

Continuing this way we get for each time point \( T_j^* \) in the tenor structure a Libor rate process under \( \mathbb{P}_{T_{j-1}^*} \) in the form

\[
1 + \delta L(t, T_j^*) = (1 + \delta L(0, T_j^*)) \exp \left( \int_{0}^{t} \lambda(s, T_j^*) \, dL_{s}^{T_j^*} \right).
\]

with successive compensators

\[
\nu^{T_j^*,L}(dt, dx) = \exp \left( \sum_{i=1}^{j} \lambda(t, T_i^*)x \right) F_t(dx) \, dt.
\]

Consequence of this alternative approach:

negative Libor rates can occur
Pricing of caps and floors (1)

Time- $T_j$-payoff of a cap settled in arrears

$$N \delta(L(T_{j-1}, T_{j-1}) - K)^+$$

$N$ notional amount (set $N = 1$)

$K$ strike rate

Time-$t$ value

$$C_t = \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*} \left[ \frac{B_t}{B_{T_j}} \delta(L(T_{j-1}, T_{j-1}) - K)^+ \mid \mathcal{F}_t \right]$$

$$= \sum_{j=1}^{n} B(t, T_j) \mathbb{E}_{\mathbb{P}_{T_j}} \left[ \delta(L(T_{j-1}, T_{j-1}) - K)^+ \mid \mathcal{F}_t \right]$$

Analogous for floor

$$N \delta(K - L(T_{j-1}, T_{j-1}))^+$$
Pricing of caps and floors (2)

Dynamics of $L(t, T_{j-1})$ under $\mathbb{P}_{T_j}$ (purely discontinuous case)

$$dL(t, T_{j-1}) = L(t-, T_{j-1}) \int_{\mathbb{R}} \left( e^{\lambda(t, T_{j-1})x} - 1 \right) \left( \mu^L - \nu^{T_j, L} \right) (dt, dx)$$

Solution

$$L(t, T_{j-1}) = L(0, T_{j-1}) \exp \left( \int_0^t \lambda(s, T_{j-1}) dL_s^{T_j} \right)$$

$$= L(0, T_{j-1}) \exp \left( \int_0^t b_s^{T_j} \lambda(s, T_{j-1}) ds \right)$$

$$+ \int_0^t \int_{\mathbb{R}} (x\lambda(s, T_{j-1})) \left( \mu^L - \nu^{T_j, L} \right) (ds, dx)$$

Write $X_t = \int_0^t \lambda(s, T_{j-1}) dL_s^{T_j}$ then $L(t, T_{j-1}) = L(0, T_{j-1}) \exp(X_t)$ is a martingale with respect to $\mathbb{P}_{T_j}$
Denote $\zeta_j = -\ln(L(0, T_{j-1}))$ and $v_K(x) = (e^{-x} - K)^+$

Bilateral Laplace transform of $v_K$: $L[v_K](z) = \int_{-\infty}^{+\infty} e^{-zx} v_K(x) \, dx$

Characteristic function of $X_{T_{j-1}}$: $\chi(u) = \mathbb{E}_{\mathbb{P}_{T_j}}[\exp(iuX_{T_{j-1}})]$

Assume $\operatorname{mgf}(-R) < \infty$, then the time-0 price of the $j$-th caplet is given by

$$V_j(\zeta_j, K) = \delta B(0, T_j) \frac{e^{\zeta_j R}}{2\pi} \int_{-\infty}^{+\infty} e^{iu\zeta_j} L[v_K](R+iu) \chi(iR-u) \, du$$

whenever the right-hand side exists

$\chi(u)$ easy to compute for generalized hyperbolic Lévy motion
Representation as convolution

\[ V_j(\zeta_j, K) = \delta B(0, T_j) \frac{e^{\zeta_j R}}{2\pi} \int_{-\infty}^{+\infty} e^{iu\zeta_j} L[v_K](R + iu) \chi(iR - u) \, du \]

\[ V_j(\zeta_j, K) = \delta B(0, T_j) \mathbb{E}_{\mathbb{P}_{T_j}} [(L(T_{j-1}, T_{j-1}) - K)^+] \]

\[ = \delta B(0, T_j) \mathbb{E}_{\mathbb{P}_{T_j}} [v_K(\zeta_j - X_{T_{j-1}})] \]

\[ = \delta B(0, T_j) \int_{\mathbb{R}} v_K(\zeta_j - x) \mathbb{P}_{T_j}^{X_{T_{j-1}}} (dx) \]

\[ = \delta B(0, T_j) \int_{\mathbb{R}} v_K(\zeta_j - x) \rho(x) \, dx \]

\[ = \delta B(0, T_j) (v_K \ast \rho)(\zeta_j). \]

And \[ L[V_j](R + iu) = \delta B(0, T_j)L[v_K](R + iu)L[\rho](R + iu) \quad \text{for } u \in \mathbb{R}. \]
Extensions of the basic Lévy market model

Lévy market model (Eb–Özkan (2005))

- Multi-currency setting (Eb–Koval (2006))
- Credit risk model (Eb–Kluge–Schönbucher (2006))
- Swap rate model (Eb–Liinev (2006))
- Duality principle (Eb–Kluge–Papapantoleon (2006))
Cross-currency Lévy market model

Relationship between domestic and foreign fixed income markets in a discrete-tenor framework.
Libor rates in a cross currency setting

Discrete tenor structure \( T_0 < T_1 < \cdots < T_n < T_{n+1} = T^* \)
Accrual periods \( \delta = T_{j+1} - T_j \)

(m + 1) markets \( i = 0, \ldots, m \)
0 = domestic market

Want to model the dynamics of the Libor rate \( L^i(t, T_{j-1}) \) which applies to time period \([T_{j-1}, T_j]\) in market \( i \) \((i = 0, \ldots, m)\)

We target at the form

\[
L^i(t, T_{j-1}) = L^i(0, T_{j-1}) \exp \left( \int_0^t \lambda^i(s, T_{j-1}) \, dL^i_s(t, T_j) \right)
\]
The driving process

\( \mathbf{L}^{0,T^*} = (L_1^{0,T^*}, \ldots, L_d^{0,T^*}) \) is a \( d \)-dimensional time-inhomogeneous Lévy process. The law of \( \mathbf{L}^{0,T^*}_t \) is given by

\[
\mathbb{E}[\exp(iu^\top \mathbf{L}^{0,T^*}_t)] = \exp \int_0^t \theta_s^{0,T^*}(iu) \, ds \quad \text{with}
\]

\[
\theta_s^{0,T^*}(z) = z^\top b_s^{0,T^*} + \frac{1}{2} z^\top C_s z + \int_{\mathbb{R}^d} \left( e^{z^\top x} - 1 - z^\top x \right) \lambda_s^{0,T^*}(dx),
\]

where \( b_s^{0,T^*} \in \mathbb{R}^d \), \( C_s \) is a symmetric nonnegative-definite \( d \times d \)-matrix and \( \lambda_s^{0,T^*} \) is a Lévy measure.

Integrability assumptions
Description in terms of modern stochastic analysis

\[ L^{0,T^*} = (L_t^{0,T^*}) \] is a special semimartingale with canonical representation

\[ L_t^{0,T^*} = \int_0^t b_s^{0,T^*} \, ds + \int_0^t c_s \, dW_s^{0,T^*} + \int_0^t \int_{\mathbb{R}^d} \lambda_s (\mu - \nu_{0,T^*})(ds, dx) \]

\( (W_t^{0,T^*}) \) is a \( \mathbb{P}^{0,T^*} \)-standard Brownian motion with values in \( \mathbb{R}^d \)

\( c_t \) is a measurable version of the square root of \( C_t \)

\( \mu \) the random measure of jumps of \( (L_t^{0,T^*}) \)

\( \nu_{0,T^*}(ds, dx) = \lambda_s^{0,T^*}(dx) \, ds \) is the \( \mathbb{P}^{0,T^*} \)-compensator of \( \mu \)

\( (L_t^{0,T^*}) \) is also called a process with independent increments and absolutely continuous characteristics (PIIAC).
The foreign forward exchange rate for date $T^*$ \hspace{1cm} (1)

### Assumptions

\((FXR.1)\): For every market $i \in \{0, \ldots, m\}$ there are strictly decreasing and strictly positive zero-coupon bond prices $B^i(0, T_j)(j = 0, \ldots, N + 1)$ and for every foreign economy $i \in \{1, \ldots, m\}$ there are spot exchange rates $X^i(0)$ given.

Consequently the initial foreign forward exchange rate corresponding to $T^*$ is

$$F_{X^i}(0, T^*) = \frac{B^i(0, T^*)X^i(0)}{B^0(0, T^*)}$$
The foreign forward exchange rate for date $T^*$ \ (2)

**Assumptions**

**\(\text{(FXR.2)}\):** For every foreign market $i \in \{1, \ldots, m\}$ there is a continuous deterministic function $\xi^i(\cdot, T^*) : [0, T^*] \rightarrow \mathbb{R}^d$.

For every coordinate $1 \leq k \leq d$ we assume $$(\xi^i(s, T^*))_k \leq \bar{M} \quad (s \in [0, T^*], \ 1 \leq i \leq m)$$

where $\bar{M} < \frac{M}{N + 2}$. 
The foreign forward exchange rate for date $T^*$  (3)

Assumptions

\textbf{(FXR.3): For every $i \in \{1, \ldots, m\}$ the foreign forward exchange rate for date $T^*$ is given by}

$$F_{X^i}(t, T^*) = F_{X^i}(0, T^*) \exp \left( \int_0^t \gamma^i(s, T^*) \, ds + \int_0^t \xi^i(s, T^*)^\top \, dL_{s,T^*}^0 \right)$$

where

$$\gamma^i(s, T^*) = -\xi^i(s, T^*)^\top b_{s,T^*}^0 - \frac{1}{2} |\xi^i(s, T^*)^\top c_s|^2$$

$$- \int_{\mathbb{R}^d} \left( e^{\xi^i(s,T^*)^\top x} - 1 - \xi^i(s,T^*)^\top x \right) \lambda_{s,T^*}^0(dx)$$

Equivalently

$$F_{X^i}(t, T^*) = F_{X^i}(0, T^*) \mathcal{E}_t \left( \int_0^t \xi^i(s, T^*)^\top c_s \, dW_{s,T^*}^0 \right. \right.$$

$$+ \left. \int_0^t \int_{\mathbb{R}^d} \left( \exp \left( \xi^i(s,T^*)^\top x \right) - 1 \right) (\mu - \nu_{0,T^*})(ds, dx) \right)$$
The foreign forward exchange rate for date $T^*$  (4)

Consequences: $F_{X^i} (\cdot, T^*)$ is a $\mathbb{P}^{0,T^*}$-martingale

$$E_{\mathbb{P}^{0,T^*}} \left[ \frac{F_{X^i}(t, T^*)}{F_{X^i}(0, T^*)} \right] = 1$$

Define

$$\frac{d\mathbb{P}^{i,T^*}}{d\mathbb{P}^{0,T^*}} \bigg|_{\mathcal{F}_t} = \frac{F_{X^i}(t, T^*)}{F_{X^i}(0, T^*)}$$

By Girsanov's theorem we get a $\mathbb{P}^{i,T^*}$-standard Brownian motion

$$W_{t}^{i,T^*} = W_{t}^{0,T^*} - \int_0^t c_s \xi^i(s, T^*) \, ds$$

and a $\mathbb{P}^{i,T^*}$-compensator

$$\nu_{i,T^*}(dt, dx) = \exp(\xi^i(t, T^*)^\top x) \nu_{0,T^*}(dt, dx)$$
The Lévy Libor model
as in Eberlein–Özkan (2005)

Tenor structure  \( T_0 < T_1 < \cdots < T_N < T_{N+1} = T^* \)

with \( T_{j+1} - T_j = \delta \), set \( T_j^* = T^* - j\delta \) for \( j = 1, \ldots, N \)

\[
\begin{array}{cccc}
T_N^* & T_{N-1}^* & T_2^* & T_1^* \\
T_N & T_{N-1} & T_2 & T_1 \\
T_0 & T_1 & T_2 & T_3 & T_{N-1} & T_N & T^* \\
\end{array}
\]

Assumptions

(CLMM.1): For every market \( i \) and every maturity \( T_j \) there is a bounded deterministic function \( \lambda^i(\cdot, T_j) \), which represents the volatility of the forward Libor rate process \( L^i(\cdot, T_j) \) in market \( i \).

(CLMM.2): The initial term structure of forward Libor rates in market \( i \) is given by

\[
L^i(0, T_j) = \frac{1}{\delta} \left( \frac{B^i(0, T_j)}{B^i(0, T_j + \delta)} - 1 \right)
\]
Backward Induction

Given a stochastic basis \((\Omega, \mathcal{F}_{T^*}, \mathbb{P}^{0,T^*}, (\mathcal{F}_t)_{0 \leq t \leq T^*})\)

We postulate that under \(\mathbb{P}^{i,T^*}\)

\[
L^i(t, T_1^*) = L^i(0, T_1^*) \exp \left( \int_0^t \chi^i(s, T_1^*) \, dL^i_{s,T^*} \right)
\]

where

\[
L^i_{t,T^*} = \int_0^t b^i_{s,T^*} \, ds + \int_0^t c_s \, dW^i_{s,T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^i_{s,T^*}) (ds, dx)
\]

with \(W^i_{t,T^*}\) and \(\nu^i_{t,T^*}\) as given before.
Cross-currency Lévy market model

Relationship between domestic and foreign fixed income markets in a discrete-tenor framework.
Relationship between the domestic and the foreign market

The forward exchange rates in the $i$-th foreign market are related by

$$F_{X^i}(t, T_j) = F_{X^i}(t, T_{j+1}) \frac{F_{B^i}(t, T_j, T_{j+1})}{F_{B^0}(t, T_j, T_{j+1})}$$

From this one gets the dynamics of $F_{X^i}(t, T_j)$

$$\frac{dF_{X^i}(t, T_j)}{dF_{X^i}(t-, T_j)} = \zeta^i(t, T_j, T_{j+1}) \ dW^i_{t, T_j} + \int_\mathbb{R^d} (\bar{\zeta}^i(t, x, T_j, T_{j+1}) - 1)(\mu - \nu_0, T_j)(dt, dx)$$

where the coefficients are given recursively

$$\zeta^i(t, T_j, T_{j+1}) = \alpha^i(t, T_j, T_{j+1}) - \alpha^0(t, T_j, T_{j+1}) + \zeta^i(t, T_{j+1}, T_{j+2})$$

$$\bar{\zeta}^i(t, x, T_j, T_{j+1}) = \frac{\beta^i(t, x, T_j, T_{j+1})}{\beta^0(t, x, T_j, T_{j+1})} \zeta^i(t, x, T_{j+1}, T_{j+2})$$
Pricing cross-currency derivatives (1)

Foreign forward caps and floors

\[ \delta X[L^i(T_{j-1}, T_{j-1}) - K^i]^+ \]

Time-0-value of a foreign \( T_N \)-maturity cap

\[
FC^i(0, T_N) = \delta \sum_{j=1}^{N+1} B^i(0, T_j) \mathbb{E}_{p^i, T_j} \left[ \left( L^i(T_{j-1}, T_{j-1}) - K^i \right)^+ \right]
\]

Alternatively if we define \( \tilde{K}^i = 1 + \delta K^i \) (forward process approach)

\[
FC^i(0, T_N) = \sum_{j=1}^{N+1} B^i(0, T_j) \mathbb{E}_{p^i, T_j} \left[ \left( 1 + \delta L^i(T_{j-1}, T_{j-1}) - \tilde{K}^i \right)^+ \right],
\]

\[
= \sum_{j=1}^{N+1} C^i(0, T_j, \tilde{K}^i)
\]
Pricing cross-currency derivatives (2)

Numerical evaluation of the cap price

Define \( X_{T_{j-1}}^i(t) = \int_0^t \lambda^i(s, T_{j-1}) dL_s^{i,T_j} = \ln \frac{1 + \delta L^i(t, T_{j-1})}{1 + \delta L^i(0, T_{j-1})} \)

and let \( \chi^{i,T_{j-1}}(z) \) be its characteristic function, then via a convolution representation

\[
C^i(0, T_j, \tilde{K}^i) = B^i(0, T_j) \tilde{K}^i \frac{\exp(\tilde{\xi}_j R)}{2\pi} \int_{-\infty}^{\infty} \exp(iu\tilde{\xi}_j) \frac{\chi^{i,T_{j-1}}(iR - u)}{(R + iu)(1 + R + iu)} \, du
\]

where \( \tilde{\xi}_j = \ln(\tilde{K}^i) - \ln(1 + \delta L^i(0, T_{j-1})) \) and \( R \) is s.t. \( \chi^{i,T_{j-1}}(iR) < \infty \).
Pricing cross-currency derivatives (3)

Cross-currency swaps

Floating-for-floating cross-currency \((i; \ell; 0)\) swap

Libor rate \(L_i(T_{j-1}, T_{j-1})\) of currency \(i\) is received at each date \(T_j\)

Libor rate \(L_\ell(T_{j-1}, T_{j-1})\) of currency \(\ell\) is paid

Payments are made in units of the domestic currency

Thus the cashflow at time point \(T_j\) is (notional = 1)

\[
\delta (L_i(T_{j-1}, T_{j-1}) - L_\ell(T_{j-1}, T_{j-1}))
\]
Pricing cross-currency derivatives (4)

The time-0-value of a floating-for-floating \((i; \ell; 0)\) cross-currency forward swap in units of the domestic currency is

\[
CCFS_{[i;\ell;0]}(0) = B^0(0,T_j) \left( \sum_{j=1}^{N+1} \frac{B^i(0,T_{j-1})}{B^i(0,T_j)} \exp(\mathcal{D}^i(0,T_{j-1},T_j)) \right. \\
\left. - \sum_{j=1}^{N+1} \frac{B^\ell(0,T_{j-1})}{B^\ell(0,T_j)} \exp(\mathcal{D}^\ell(0,T_{j-1},T_j)) \right)
\]

where

\[
\mathcal{D}^i(0,T_{j-1},T_j) = - \int_0^{T_{j-1}} \lambda^i(s,T_{j-1})^\top c_s \zeta^i(s,T_j,T_{j+1}) \, ds \\
- \int_0^{T_{j-1}} \int_{\mathbb{R}^d} \left( \exp \left( \lambda^i(s,T_{j-1})^\top x \right) - 1 \right) (\zeta_i(s,x,T_j,T_{j+1}) - 1) \nu_{0,T_j}(ds, dx)
\]
Pricing cross-currency derivatives (5)

A quanto caplet with strike $K^i$, which expires at time $T_{j-1}$, pays at time $T_j$

$$QCpl^i(T_j, T_j, K^i) = \delta \overline{X}^i (L^i(T_{j-1}, T_{j-1}) - K^i)^+$$

where $\overline{X}^i$ is the preassigned foreign exchange rate

Time-0-value

$$QCpl^i(0, T_j, K^i) = B^0(0, T_j) \mathbb{E}_{\mathbb{P}^0, T_j} [\delta \overline{X}^i (L^i(T_{j-1}, T_{j-1}) - K^i)^+]$$

$$= B^0(0, T_j) \overline{X}^i \mathbb{E}_{\mathbb{P}^0, T_j} [(1 + \delta L^i(T_{j-1}, T_{j-1}) - (1 + \delta K^i))^+]$$

(forward process approach)
Pricing cross-currency derivatives (6)

Numerical evaluation of quanto caplets. Write

\[ 1 + \delta L^i(T_{j-1}, T_{j-1}) = (1 + \delta L^i(0, T_{j-1})) \exp \left( \int_0^{T_{j-1}} \lambda^i(s, T_{j-1}) \, dL^i_s \right) \]
\[ = (1 + \delta L^i(0, T_{j-1})) \exp \left( \underbrace{\mathcal{M}^i(0, T_{j-1}, T_j)}_{\text{assume density } \varrho} + \underbrace{\mathcal{D}^i(0, T_{j-1}, T_j)}_{\text{non-random}} \right) \]

then for \( \nu(x) = (e^{-x} - 1)^+ \)

\[ QCpl^i(0, T_j, K^i) = B^0(0, T_j) \overline{X}^i(1 + \delta K^i)(\nu \ast \varrho)(\xi_j) \]

Finally we get

\[ QCpl^i(0, T_j, K^i) = B^0(0, T_j) \overline{X}^i(1 + \delta K^i) \cdot \frac{\exp(\xi_j R)}{2\pi} \int_{-\infty}^{\infty} \exp(iu\xi_j) \frac{\chi^{\mathcal{M}^i,T_{j-1}}(iR - u)}{(R + iu)(R + 1 + iu)} \, du \]
Absolute errors of EUR caplet calibration
Absolute errors of USD caplet calibration
Basic interest rates

\( P(t, T) \): price at time \( t \in [0, T] \) of a default-free zero coupon bond with maturity \( T \in [0, T^*] \) \((P(T, T) = 1)\)

\( f(t, T) \): instantaneous forward rate

\[ P(t, T) = \exp \left( - \int_t^T f(t,u) \, du \right) \]

\( L(t, T) \): default-free forward Libor rate for the interval \( T \) to \( T + \delta \) as of time \( t \leq T \) \((\delta\)-forward Libor rate)

\[ L(t, T) := \frac{1}{\delta} \left( \frac{P(t, T)}{P(t, T + \delta)} - 1 \right) \]

\( F_P(t, T, U) \): forward price process for the two maturities \( T \) and \( U \)

\[ F_P(t, T, U) := \frac{P(t, T)}{P(t, U)} \]

\[ 1 + \delta L(t, T) = \frac{P(t, T)}{P(t, T + \delta)} = F_P(t, T, T + \delta) \]
Pricing of options on bonds

\( P(t, T) \): underlying
\( w(P(t, T), K) \): payoff of a European option with maturity \( t \) and strike \( K \)
\( V(0, t, T, w) \): time-0-price of the option

\[ V(0, t, T, w) = \mathbb{E}^*_{\mathbb{P}} [B_t^{-1} w(P(t, T), K)] \]

Caps, Floors, Swaptions, Digitals, Range options
Turnbull (1995): floating range notes in 1-factor Gaussian HJM
Navatte and Quittard-Pinon (1999): delayed digital options
Nunes (2004): multifactor Gaussian HJM
Forward measure and adjusted forward measure

Forward martingale measure for settlement day \( T \)

\[
\frac{dP_T}{dP} := \frac{1}{B_T P(0, T)} = \exp \left( - \int_0^T A(s, T) \, ds + \int_0^T \Sigma(s, T) \, dL_s \right)
\]

Adjusted forward measure \( P_{T', T} \) for \( T' < T \)

\[
\frac{dP_{T', T}}{dP_T} := \frac{F(T', T', T)}{F(0, T', T)} = \frac{P(0, T)}{P(0, T') P(T', T)}
\]

For \( 0 \leq t \leq T' \):

\[
\left. \frac{dP_{T', T}}{dP} \right|_{\mathcal{F}_t} = \left. \frac{dP_{T'}}{dP} \right|_{\mathcal{F}_t}
\]
Digital options (1)

Standard European interest rate digital call (put) with strike \( r_K \)

\[
SD(\Theta)_T[r_n(T, T + \delta); r_k; T] := 1_{\{\Theta r_n(T, T+\delta) > \Theta r_k\}},
\]

where \( r_n(T, T + \delta) \) is the reference rate (Libor)

\[
r_n(T, T + \delta) = \frac{1}{\delta} \left[ \frac{1}{P(T, T + \delta)} - 1 \right]
\]

and \( \Theta = 1 \) for a digital call, \( \Theta = -1 \) for a digital put

Delayed digital option for maturity \( T \) and payment date \( T_1 \)

\[
DD(\Theta)_{T_1}[r_n(T, T + \delta); r_k; T_1] := 1_{\{\Theta r_n(T, T+\delta) > \Theta r_k\}}
\]
Digital options (2)

Delayed range digital options ($T \leq T_1$)

$$DRD_{T_1}[r_n(T, T + \delta); r_\ell, r_u; T_1] := \mathbf{1}_{\{r_n(T, T + \delta) \in [r_\ell, r_u]\}}$$

Obvious relationship for time-$t$ prices

$$DRD_t[r_n(T, T + \delta); r_\ell, r_u; T_1] = P(t, T_1)$$
$$- DD(1)_t[r_n(T, T + \delta); r_u; T_1]$$
$$- DD(-1)_t[r_n(T, T + \delta); r_\ell; T_1].$$

Call-put parity only when $\mathcal{L}(P(T, T + \delta))$ without point masses

$$DD(1)_t[r_n(T, T + \delta); r_k; T_1] = P(t, T_1)$$
$$- DD(-1)_t[r_n(T, T + \delta); r_k; T_1].$$
Pricing formulae for delayed digital options (1)

\[
D_t := DD(1)_t[r_n(T, T + \delta); r_k; T_1] \\
= B_t \mathbb{E} \left[ \frac{1}{B_{T_1}} \mathbb{1}_{\{r_n(T, T + \delta) > r_k\}} \big| \mathcal{F}_t \right] \\
= P(t, T_1) \mathbb{E}_{T_1} \left[ \mathbb{1}_{\{r_n(T, T + \delta) > r_k\}} \big| \mathcal{F}_t \right] \\
= P(t, T_1) \mathbb{E}_{T_1} \left[ \mathbb{1}_{\{P(T, T + \delta) < \frac{1}{\delta r_{k+1}}\}} \big| \mathcal{F}_t \right] \\
= P(t, T_1) h \left( \frac{P(t, T + \delta)}{P(t, T)} \right)
\]

where \( h(y) = \mathbb{E}_{T_1} \left[ \mathbb{1}_{\{y \exp\left[-\int_t^T A(s, T, T + \delta) \, ds + \int_t^T \Sigma(s, T, T + \delta) \, dL_s\right]} < \frac{1}{\delta r_{k+1}}\} \right] \)

and \( A(s, T, T + \delta) = A(s, T + \delta) - A(s, T) \),
\( \Sigma(s, T, T + \delta) = \Sigma(s, T + \delta) - \Sigma(s, T) \)
Pricing formulae for delayed digital options (2)

Denote $X := \int_t^T \Sigma(s,T,T + \delta) \, dL_s$

$K := \frac{1}{\delta r_K + 1} \exp \left( \int_t^T A(s,T,T + \delta) \, ds \right)$

$\mathbb{P}^X_{T_1} = \text{distribution of } X \text{ under } \mathbb{P}_{T_1}$

then $h(y) = \int \mathbb{1}_{\{e^x < \frac{K}{y}\}} \, d\mathbb{P}^X_{T_1}(x)$

$= \int f_y(-x) \varphi(x) \, dx = (f_y \ast \varphi)(0) = V(0)$

for $f_y(x) = \mathbb{1}_{\{e^{-x} < \frac{K}{y}\}}$ and $V(\zeta) = (f_y \ast \varphi)(\zeta)$

Denote by $M^X_{T_1}$ the moment generating function of $X$ w.r.t. $\mathbb{P}_{T_1}$
Pricing formulae for delayed digital options (3)

**Theorem**

Suppose the distribution of $X$ possesses a Lebesgue density. Choose an $R > 0$ such that $M_{T_1}^X(-R) < \infty$. Then

$$D_t = \frac{1}{\pi} P(t, T_1) \int_0^\infty \Re \left( \left( \frac{P(t, T)}{P(t, T + \delta)} K \right)^{R + iu} \frac{1}{R + iu} M_{T_1}^X(-R - iu) \right) \, du$$

**Proof:**

$$L[V](z) = L[f_y](z)L[\varphi](z)$$

$$V(0) = \frac{1}{2\pi i} \lim_{Y \to \infty} \int_{R-iY}^{R+iY} L[V](z) \, dz$$

$$L[f_y](R + iu) = \frac{1}{R+iu} \left( \frac{K}{Y} \right)^{R+iu}$$
Pricing range notes (1)

\[ n_0^- \quad n_0^+ \quad n_1 \]

\[ T_0 \quad t \quad T_1 \quad T_2 \quad T_N \]

- \( T_j \) = coupon payment dates
- \( n_j \) = number of days between \( T_j \) and \( T_{j+1} \) based on some day count convention
- \( \delta_j \) = number of years between \( T_j \) and \( T_{j+1} \)
- \( T_{j,i} \) = \( T_j + i \)
- \( \delta_{j,i} \) = length (in years) of the compounding period starting at \( T_{j,i} \)
Pricing range notes (2)

For a floating range note, the coupon at time $T_{j+1}$ is

$$\nu_{j+1}(T_{j+1}) := \frac{r_n(T_j, T_j + \delta_j) + s_j}{D_j} H(T_j, T_{j+1})$$

where $s_j$ is the spread over the reference rate

$D_j$ number of days for the $(j+1)$-th compounding period

$$H(T_j, T_{j+1}) = \sum_{i=1}^{n_j} 1\{r_\ell(T_j, i) \leq r_n(T_j, i, T_j, i + \delta_j, i) \leq r_u(T_j, i)\}$$

Time-$t$ value of a floating range note

$$F_{\ell RN}(t) := P(t, T_N) + \sum_{j=0}^{N-1} \nu_{j+1}(t)$$
Valuation of $F\ell RN$ coupons (1)

\[ \nu_1(t) = B_t \mathbb{E} \left[ \frac{1}{B_{T_1}} \frac{r_n(T_0, T_0 + \delta_0) + s_0}{D_0} H(T_0, T_1) \mid \mathcal{F}_t \right] \]

\[ = \frac{r_n(T_0, T_0 + \delta_0) + s_0}{D_0} P(t, T_1) \mathbb{E}_{T_1} \left[ H(T_0, T_1) \mid \mathcal{F}_t \right] \]

\[ = \frac{r_n(T_0, T_0 + \delta_0) + s_0}{D_0} \left( P(t, T_1) H(T_0, t) \right) \]

\[ + \sum_{i=n_0^{-} + 1}^{n_0} P(t, T_1) \mathbb{E}_{T_1} \left[ 1_{\{r_\ell(T_0, i) \leq r_n(T_0, i, T_0, i + \delta_0, i) \leq r_u(T_0, i)\}} \mid \mathcal{F}_t \right] \]

\[ = DRD_t \left[ r_n(T_0, i, T_0, i + \delta_0, i); r_\ell(T_0, i); r_u(T_0, i); T_1 \right] \]
Valuation of $F\ell RN$ coupons (2)

\[\nu_{j+1}(t) = P(t,T_{j+1}) E_{T_{j+1}} \left[ \frac{r_n(T_j,T_{j+1}) + S_j}{D_j} H(T_j,T_{j+1}) \bigg| \mathcal{F}_t \right] \]

\[= \left( \frac{S_j}{D_j} - \frac{1}{\delta_j D_j} \right) P(t,T_{j+1}) \sum_{i=1}^{n_j} E_{T_{j+1}} \left[ \mathbb{1}\{r_\ell(T_j,i) \leq r_n(T_j,i,T_j,i+\delta_j,i) \leq r_u(T_j,i)\} \bigg| \mathcal{F}_t \right] \]

\[+ \frac{P(t,T_{j+1})}{\delta_j D_j} \sum_{i=1}^{n_j} E_{T_{j+1}} \left[ \frac{1}{P(T_j,T_{j+1})} \mathbb{1}\{r_\ell(T_j,i) \leq r_n(T_j,i,T_j,i+\delta_j,i) \leq r_u(T_j,i)\} \bigg| \mathcal{F}_t \right] \]

\[=: \nu_{j+1}^1(t) + \nu_{j+1}^2(t). \]

Note that

\[\nu_{j+1}^1(t) = \left( \frac{S_j}{D_j} - \frac{1}{\delta_j D_j} \right) \sum_{i=1}^{n_j} DRD_t[r_n(T_j,i,T_j,i + \delta_j,i); r_\ell(T_j,i); r_u(T_j,i); T_{j+1}] \]

and

\[\nu_{j+1}^2(t) = \sum_{i=1}^{n_j} \frac{P(t,T_j)}{\delta_j D_j} E_{T_j,T_{j+1}} \left[ \mathbb{1}\{r_\ell(T_j,i) \leq r_n(T_j,i,T_j,i+\delta_j,i) \leq r_u(T_j,i)\} \bigg| \mathcal{F}_t \right] \]

\[=: D_t^{j,i}. \]
Valuation of $\mathcal{F} \ell RN$ coupons (3)

Neglect now the indices $i,j$

**Theorem**

*Suppose the distribution of $X$ possesses a Lebesgue density. Choose an $R > 0$ such that $M^X(-R) < \infty$. Then*

$$D_t = \frac{1}{\pi} \int_0^\infty \Re \left( \left( \frac{P(t, T)}{P(t, T + \delta)} \overline{K} \right)^{R+iu} \frac{1}{R+iu} M^X(-R - iu) \right) du$$

$$- \frac{1}{\pi} \int_0^\infty \Re \left( \left( \frac{P(t, T)}{P(t, T + \delta)} K \right)^{R+iu} \frac{1}{R+iu} M^X(-R - iu) \right) du$$

*with*

$$\overline{K} := \frac{1}{\delta r_\ell(T) + 1} \exp \left( \int_t^T A(s, T, T + \delta) ds \right)$$

$$K := \frac{1}{\delta r_u(T) + 1} \exp \left( \int_t^T A(s, T, T + \delta) ds \right)$$
Lévy credit risk model

![Graph of Lévy credit risk model with maturity in years on the x-axis and interest rate on the y-axis for different credit ratings such as Aaa, Caa, B3, Baa3, etc.](image-url)
Basic interest rates

\( B(t, T) \): price at time \( t \in [0, T] \) of a default-free zero coupon bond with maturity \( T \in [0, T^*] \) \( (B(T, T) = 1) \)

\( f(t, T) \): instantaneous forward rate

\[
B(t, T) = \exp \left( - \int_t^T f(t, u) \, du \right)
\]

\( L(t, T) \): default-free forward Libor rate for the interval \( T \) to \( T + \delta \) as of time \( t \leq T \) \( (\delta\text{-forward Libor rate}) \)

\[
L(t, T) := \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right)
\]

\( F_B(t, T, U) \): forward price process for the two maturities \( T < U \)

\[
F_B(t, T, U) := \frac{B(t, T)}{B(t, U)}
\]

\[ 1 + \delta L(t, T) = \frac{B(t, T)}{B(t, T + \delta)} = F_B(t, T, T + \delta) \]
The Lévy Libor model with default risk
(Eberlein, Kluge, Schönbucher 2006)

\(B^0(t, T_k):\) time-\(t\) price of a defaultable zero coupon bond with zero recovery and maturity \(T_k\)

\(\tau:\) time of default

\(\overline{B}(t, T_k):\) pre-default value of the defaultable bond

\[
\implies B^0(t, T_k) = 1_{\{\tau > t\}} \overline{B}(t, T_k), \quad \overline{B}(T_k, T_k) = 1 \quad (k = 1, \ldots, n)
\]

Terminal value of the defaultable bond

\[
B^0(T_k, T_k) = 1_{\{\tau > T_k\}} \overline{B}(T_k, T_k) = 1_{\{\tau > T_k\}}
\]
The Lévy Libor model with default risk (2)

- The *defaultable forward Libor rates* for the interval $[T_k, T_{k+1}]$ are given by
  \[ \bar{L}(t, T_k) := \frac{1}{\delta_k} \left( \frac{\bar{B}(t, T_k)}{\bar{B}(t, T_{k+1})} - 1 \right). \]

- The *forward Libor spreads* are given by
  \[ S(t, T_k) := \bar{L}(t, T_k) - L(t, T_k). \]

- The *default risk factors* or *forward survival processes* are given by
  \[ D(t, T_k) := \frac{\bar{B}(t, T_k)}{B(t, T_k)}. \]

- The discrete-tenor *forward default intensities* are given by
  \[ H(t, T_k) := \frac{1}{\delta_k} \left( \frac{D(t, T_k)}{D(t, T_{k+1})} - 1 \right) = \frac{S(t, T_k)}{1 + \delta L(t, T_k)}. \]
Canonical construction of the time of default

Let $\Gamma = (\Gamma_t)_{t \geq 0}$ be an $(\mathcal{F}_t)$-adapted, right-continuous, increasing process on $(\Omega, \mathcal{F}, \mathbb{P}_{T*})$, $\Gamma_0 = 0$, $\lim_{t \to \infty} \Gamma_t = \infty$.

Let $\eta$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ uniformly distributed on $[0, 1]$.

Define $\Omega := \tilde{\Omega} \times \tilde{\Omega}$, $\mathcal{G} := \mathcal{\tilde{F}} \otimes \mathcal{\tilde{F}}$, $\mathbb{Q}_{T^*} := \mathbb{P}_{T^*} \otimes \mathbb{P}$

$(\mathcal{F}_t)$ trivial extension of $(\mathcal{\tilde{F}}_t)$ to $(\Omega, \mathcal{G}, \mathbb{Q}_{T^*})$

\[ \tau := \inf\{t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \eta\} \]

Denote $\mathcal{H}_t := \sigma(\mathbb{1}_{\{\tau \leq u\}} | 0 \leq u \leq t)$, $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$

\[ \implies \tau \text{ is a (}\mathcal{G}_t\text{-stopping time)} \]

\[ \mathbb{Q}_{T^*}\{\tau > s|\mathcal{F}_{T^*}\} = \mathbb{Q}_{T^*}\{\tau > s|\mathcal{F}_s\} = e^{-\Gamma_s} \quad (0 \leq s \leq T^*) \]

\[ \implies (\Gamma_t) \text{ is the (}\mathcal{F}_t\text{-hazard process of } \tau \text{ under } \mathbb{Q}_{T^*} \quad (\text{and also under all } \mathbb{Q}_{T_k}) \]
Consequences for the price of a defaultable bond

Payoff at maturity: \( B^0(t, T_k) = \mathbb{1}_{\{\tau > T_k\}} \)

\[ \implies B^0(t, T_k) = B(t, T_k) \mathbb{E}_{Q_T} \left[ \mathbb{1}_{\{\tau > T_k\}} | G_t \right] \]

\[ = B(t, T_k) \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{Q_T} \left[ \mathbb{1}_{\{\tau > T_k\}} | F_t \right]}{e^{-\Gamma t}} \]

Therefore, define

\[ \overline{B}(t, T_k) := B(t, T_k) \frac{\mathbb{E}_{Q_{T_k}} \left[ \mathbb{1}_{\{\tau > T_k\}} | F_t \right]}{e^{-\Gamma t}} \]

\[ \implies H(t, T_k) = \frac{1}{\delta_k} \left( \frac{\mathbb{E}_{Q_T} \left[ e^{-\Gamma T_k} \right] | F_t \right]}{\mathbb{E}_{Q_{T_{k+1}}} \left[ e^{-\Gamma T_{k+1}} | F_t \right]} - 1 \]

\((\Gamma_{T_k})_{k=1,\ldots,n}\) can be chosen such that \(H(t, T_k)\) has the form

\[ H(t, T_k) = H(0, T_k) \exp \left( \int_0^t b^H(s, T_k, T_{k+1}) \, ds + \int_0^t c_s^{1/2} \gamma(s, T_k) \, dW_s^{T_{k+1}} \right. \]

\[ + \left. \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s, T_k), x \rangle \left( \mu - \nu(T_{k+1}) \right) (ds, dx) \right) \].

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Defaultable forward measures

The defaultable forward measure (or survival measure) $\overline{Q}_{T_i}$ for the settlement day $T_i$ is defined on $(\Omega, \mathcal{G}_{T_i})$ by

$$
\frac{d\overline{Q}_{T_i}}{dQ_{T_i}} := \frac{B(0, T_i)}{B^0(0, T_i)} B^0(T_i, T_i) = \frac{B(0, T_i)}{B(0, T_i)} 1\{\tau > T_i\}.
$$

$$
\implies \overline{Q}_{T_i}(A) = Q_{T_i}(A | \{\tau > T_i\}) \quad (A \in \mathcal{G}_{T_i}),
$$

forward measure conditioned on survival until $T_i \longrightarrow$ survival measure

Denote $\mathbb{P}_{T_i} := Q_{T_i} | \mathcal{F}_{T_i}$

The restricted defaultable forward measure $\overline{P}_{T_i}$ for the settlement day $T_i$ is defined on $(\Omega, \mathcal{F}_{T_i})$ by

$$
\frac{d\overline{P}_{T_i}}{d\mathbb{P}_{T_i}} = \frac{B(0, T_i)}{B(0, T_i)} Q_{T_i}(\{\tau > T_i\} | \mathcal{F}_{T_i}) = \frac{B(0, T_i)}{B(0, T_i)} \prod_{k=0}^{i-1} \frac{1}{1 + \delta_k H(T_k, T_k)}.
$$
Successive restricted defaultable forward measures

The defaultable Libor rate \((\overline{L}(t, T_i))_{0 \leq t \leq T_i}\) turns out to be a \(\overline{P}_{T_{i+1}}\)-martingale and

\[
\left. \frac{d\overline{P}_{T_i}}{d\overline{P}_{T_{i+1}}} \right|_{\mathcal{F}_t} = \frac{\overline{B}(0, T_{i+1})}{\overline{B}(0, T_i)} (1 + \delta_i \overline{L}(t, T_i)) = \frac{1 + \delta_i \overline{L}(t, T_i)}{1 + \delta_i \overline{L}(0, T_i)}
\]
Pricing contingent claims with defaultable forward measures

\( X \) promised payoff at day \( T_i \) with zero recovery upon default
\( \pi_t^X \) its price at time \( t \in [0, T_i] \)
\[
\pi_t^X = \mathbf{1}_{\{\tau > t\}} B(t, T_i) \mathbb{E}_{\mathbb{Q}_{T_i}} [X \mathbf{1}_{\{\tau > t\}} | \mathcal{G}_t] \quad (t \in [0, T_i])
\]

The defaultable forward measures \( \mathbb{Q}_{T_i} \) and \( \mathbb{P}_{T_i} \) are the appropriate tools.

If \( X \) is \( \mathcal{G}_{T_i} \)-measurable
\[
\pi_t^X = \mathbf{1}_{\{\tau > t\}} \mathbb{B}(t, T_i) \mathbb{E}_{\mathbb{Q}_{T_i}} [X | \mathcal{G}_t] = B^0(t, T_i) \mathbb{E}_{\mathbb{Q}_{T_i}} [X | \mathcal{G}_t].
\]

If \( X \) is \( \mathcal{F}_{T_i} \)-measurable
\[
\pi_t^X = \mathbf{1}_{\{\tau > t\}} \mathbb{B}(t, T_i) \mathbb{E}_{\mathbb{P}_{T_i}} [X | \mathcal{F}_t] = B^0(t, T_i) \mathbb{E}_{\mathbb{P}_{T_i}} [X | \mathcal{F}_t].
\]
Recovery rules and bond prices

Defaultable zero coupon bonds

\[ B^\pi(T, T) = \mathbb{1}_{\{\tau>T\}} + \pi \mathbb{1}_{\{\tau\leq T\}} = \pi + (1 - \pi) \mathbb{1}_{\{\tau>T\}} \]

At maturity of the bond

Time-\(t\) value (\(t \in [0, T]\))

\[ B^\pi(t, T) = \pi B(t, T) + (1 - \pi) \mathbb{1}_{\{\tau>t\}} \bar{B}(t, T) \]

Defaultable coupon bonds \(\longrightarrow\) recovery of par scheme

Recovery of par: If default occurs in the time interval \((T_k, T_{k+1}]\), recovery is given by the recovery rate \(\pi\) times the sum of the notional and the accrued interest over \((T_k, T_{k+1}]\). It is paid at \(T_{k+1}\).

Corresponding cashflow pattern

- at \(T_{k+1}\) (\(k = 0, \ldots, m - 1\)):
  \[ c \mathbb{1}_{\{\tau>T_{k+1}\}} + \pi (1 + c) \mathbb{1}_{\{T_k<\tau\leq T_{k+1}\}} \]
- at \(T_m\):
  \[ \mathbb{1}_{\{\tau>T_m\}} \]
Default payments

Denote by $e_k^X(t)$ the time-$t$ value of receiving an amount of $X$ at $T_{k+1}$ if default occurred in period $(T_k, T_{k+1})$.

\begin{lemma}
If $X$ is $\mathcal{F}_{T_k}$-measurable, then for $t \leq T_k$

\[ e_k^X(t) = \mathbb{1}_{\{\tau > t\}} \bar{B}(t, T_{k+1}) \delta_k \mathbb{E}_{\mathbb{P}_{T_{k+1}}} [XH(T_k, T_k) | \mathcal{F}_t] \]
\end{lemma}
Pricing of defaultable coupon bonds

Fixed coupon of $c$ to be paid at dates $T_1, \ldots, T_m$

$$B^\pi_{\text{fixed}}(0) = \overline{B}(0, T_m) + \sum_{k=0}^{m-1} \overline{B}(0, T_{k+1}) \left( c + \pi(1 + c)\delta_k \mathbb{E}_{\mathbb{P}^T_{k+1}} [H(T_k, T_k)] \right).$$

Floating coupon bond that pays Libor plus a constant spread $x$

Promised payoff at the date $T_{k+1}$: $\delta_k (L(T_k, T_k) + x)$

$$B^\pi_{\text{floating}}(0) = \overline{B}(0, T_m) + \sum_{k=0}^{m-1} \delta_k \overline{B}(0, T_{k+1}) \left( x + \mathbb{E}_{\mathbb{P}^T_{k+1}} [L(T_k, T_k)] + \pi(1 + \delta_k x) \mathbb{E}_{\mathbb{P}^T_{k+1}} [H(T_k, T_k)] + \pi \delta_k \mathbb{E}_{\mathbb{P}^T_{k+1}} [H(T_k, T_k)L(T_k, T_k)] \right).$$
Numerical aspects

\[ H(t, T_k) = H(0, T_k) \exp \left( \int_0^t \bar{b}^H(s, T_k, T_{k+1}) \, ds + \int_0^t c_s^{1/2} \gamma(s, T_k) \, d\bar{W}_s^{T_{k+1}} \right. \]

\[ + \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s, T_k), x \rangle (\mu - \bar{\nu}^{T_{k+1}}(ds, dx)) \, dx \]

Drift coefficient \( \bar{b}^H(s, T_k, T_{k+1}) \) to be approximated

\[
\mathbb{E}_{\mathbb{P}_{T_{k+1}}} [H(T_k, T_k)L(T_k, T_k)] = \frac{1}{\delta_k} \left( L(0, T_k) - \mathbb{E}_{\mathbb{P}_{T_{k+1}}} [L(T_k, T_k)] - \mathbb{E}_{\mathbb{P}_{T_{k+1}}} [H(T_k, T_k)] \right)
\]
Credit default swaps (CDS)

Standard default swap: Default of a coupon bond

A receives: \(1 - \pi(1 + c)\) (fixed coupon)
\(1 - \pi(1 + \delta_k(L(T_k, T_k) + x))\) (floating coupon)

Time-0 value of the fee payments: \(s \sum_{k=1}^{m} \bar{B}(0, T_k-1)\)

\(s\) default swap rate

\[
S_{\text{fixed}} = \frac{1 - \pi(1 + c)}{\sum_{k=1}^{m} \bar{B}(0, T_k-1)} \sum_{k=1}^{m} \left( \bar{B}(0, T_k)\delta_{k-1} \mathbb{E}_{\mathbb{P}_{T_k}} [H(T_{k-1}, T_{k-1})] \right)
\]

\[
S_{\text{floating}} = \frac{1}{\sum_{k=1}^{m} \bar{B}(0, T_k-1)} \sum_{k=1}^{m} \left( \bar{B}(0, T_k)\delta_{k-1} \left( (1 - \pi(1 + \delta_{k-1} x)) \right) \right)
\times \mathbb{E}_{\mathbb{P}_{T_k}} [H(T_{k-1}, T_{k-1})] - \pi \delta_{k-1} \mathbb{E}_{\mathbb{P}_{T_k}} [H(T_{k-1}, T_{k-1})L(T_{k-1}, T_{k-1})]
\]
Credit default swaptions (1)

**Assumption:** The volatility structures factorize in the following way:

\[ \lambda(s, T_i) = \lambda_i \sigma(s) \quad \text{and} \quad \gamma(s, T_i) = \gamma_i \sigma(s) \quad (0 \leq s \leq T_i). \]

Payoff of a credit default swaption that is knocked out at default with strike \( S \) and maturity \( T_i \) on a CDS that terminates at \( T_m \):

\[
\mathbb{1}_{\{\tau > T_i\}} \left( (s(T_i; T_i, T_m) - S)^+ \sum_{k=i}^{m-1} B(T_i, T_k) \right)
\]

where \( s(T_i; T_i, T_m) \) denotes the default swap rate at \( T_i \).

Price at time 0:

\[
\pi_0^{\text{CDS}} = E_{\mathbb{P}_{T_i}} \left[ \left( \frac{(1 - \pi(1 + c))\delta_{m-1} C^{i,m-1} H(T_i, T_{m-1})}{\prod_{l=i}^{m-1} (1 + \delta_l L(T_i, T_l))(1 + \delta_l H(T_i, T_l))} + \sum_{k=i}^{m-2} \frac{(1 - \pi(1 + c))\delta_k C^{i,k} H(T_i, T_k) - S}{\prod_{l=i}^{k} (1 + \delta_l L(T_i, T_l))(1 + \delta_l H(T_i, T_l))} - S \right)^+ \right].
\]
Credit default swaptions (2)

Forward Libor rates and default intensities can be written as

\[
L(T_i, T_l) = L(0, T_l) \exp \left( \frac{\lambda_l}{\sigma_{\text{sum}}} X_{T_l} + B^L_l \right),
\]

\[
H(T_i, T_l) = H(0, T_l) \exp \left( \frac{\gamma_l}{\sigma_{\text{sum}}} X_{T_l} + B^H_l \right)
\]

with \( \sigma_{\text{sum}} := \sum_{i=1}^{m-1} (\lambda_i + \gamma_i) \), \( X_{T_l} := \sigma_{\text{sum}} \int_0^{T_l} \sigma(s) \, dL^*_s \) and constants \( B^L_l \), \( B^H_l \).

Assume the distribution of \( X_{T_l} \) w.r.t. \( \mathbb{P}_{T_l} \) has a Lebesgue-density \( \varphi \), then

\[
\pi^\text{CDS}_0 = \mathbb{B}(0, T_l) \int_{\mathbb{R}} g(-x) \varphi(x) \, dx = \mathbb{B}(0, T_l)(g \ast \varphi)(0)
\]

for some (explicitly given) function \( g \).

Performing Laplace and inverse Laplace transformations and denoting by \( \mathcal{M}^{X_{T_i}}_{T_i} \) the \( \mathbb{P}_{T_l} \)-moment generating function of \( X_{T_l} \) yields

\[
\pi^\text{CDS}_0 = \mathbb{B}(0, T_l) \frac{1}{\pi} \int_0^{\infty} \Re \left( L[g](R + iu) \mathcal{M}^{X_{T_l}}_{T_i}(-R - iu) \right) \, du.
\]
Options on defaultable bonds

Payoff of a call with maturity $T_i$ and strike $K \in (0, 1)$ on a defaultable zero coupon bond with maturity $T_m \ (i < m)$ which is knocked out at default

$$\pi_{T_i}^{CO}(K, T_i, T_m) = 1_{\{\tau > T_i\}}(B^\pi(T_i, T_m) - K)^+$$

Price at time 0:

$$\pi_0^{CO} = \bar{B}(0, T_m)E_{\bar{\mathbb{P}}_{T_m}}\left[\left(\pi \prod_{\ell=i}^{m-1} (1 + \delta_{\ell}H(T_i, T_\ell)) + (1 - \pi) \right.ight.$$  
$$\left. - K \prod_{\ell=i}^{m-1} \left(1 + \delta_{\ell}L(T_i, T_\ell)\right)\left(1 + \delta_{\ell}H(T_i, T_\ell)\right)\right)^+\right]$$
Options on defaultable bonds

(2)

Using again a convolution representation

\[ \pi_0^{CO} = \overline{B}(0, T_m) \int_{\mathbb{R}} g(-x)\varphi(x) \, dx = \overline{B}(0, T_m)(g * \varphi)(0) \]

one gets for an \( R > 0 \) such that \( \overline{M}_{T_m}^{X_{T_i}}(-R) < \infty \) the following (approximate) formula

\[ \pi_0^{CO}(K, T_i, T_m) = \overline{B}(0, T_m) \frac{1}{\pi} \int_0^{\infty} \Re(L[g](R + iu) \cdot \overline{M}_{T_m}^{X_{T_i}}(-R - iu)) \, du \]
Further credit derivatives

- Total rate of return swaps
- Asset package swaps
- Credit spread options
The Theme

Call option in FX market: Euro/Dollar

Gives you the right to buy Euros paying in Dollars.
At the same time a right to sell Dollars getting Euros.

Payoff \((S_T - K)^+\) \((S_t)\) exchange rate, \(K\) strike

\[
(S_T - K)^+ = KS_T\left(\frac{1}{K} - \frac{1}{S_T}\right)^+
\]

\[
= KS_T(K' - S'_T)^+
\]

\[
\uparrow
\]

Dollar/Euro rate

Call price = \(K\cdot\) put price (in the dual market)

\[\rightarrow\text{ duality principle}\]
Brief literature survey

- Carr (1994) put-call duality in BS-setting and for diffusions
- Chesney and Gibson (1995) two-factor diffusion model
- Bates (1997) diffusions and jump-diffusions
- Schroder (1999) various payoffs in diffusions and jump-diffusions
- Carr, Ellis, and Gupta (1998) static hedging strategies for exotic derivatives
- Carr and Chesney (1996) put-call for American options
- Detemple (2001) American options with general payoffs
- Eberlein and Papapantoleon (2005a,b) Exotic options for Lévy and time-inhomogeneous Lévy models
- Fajardo and Mordecki (2006a,b) Lévy models
- Eberlein, Kluge, and Papapantoleon (2006) Interest rate options
- Eberlein, Papapantoleon, Shiryaev (2006) Semimartingales
Exponential semimartingale models

Let \( \mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, P) \) be a stochastic basis, where \( \mathcal{F} = \mathcal{F}_T \) and \( \mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \). We model the price process of a financial asset as an exponential semimartingale

\[
S_t = e^{H_t}, \quad 0 \leq t \leq T.
\]

\( H = (H_t)_{0 \leq t \leq T} \) is a semimartingale with canonical representation

\[
H = H_0 + B + H^c + h(x) \ast (\mu^H - \nu) + (x - h(x)) \ast \mu^H
\]

or, in detail

\[
H_t = H_0 + B_t + H^c_t + \int_0^t \int_{\mathbb{R}} h(x)d(\mu^H - \nu) + \int_0^t \int_{\mathbb{R}} (x - h(x))d\mu^H,
\]

where
• $h = h(x)$ is a truncation function; canonical choice $h(x) = x 1_{\{|x| \leq 1\}}$;

• $B = (B_t)_{0 \leq t \leq T}$ is a predictable process of bounded variation;

• $H^c = (H^c_t)_{0 \leq t \leq T}$ is the continuous martingale part of $H$;

• $\nu = \nu(\omega; dt, dx)$ is the predictable compensator of the random measure of jumps $\mu^H = \mu^H(\omega; dt, dx)$ of $H$.

For the processes $B$, $C = \langle H^c \rangle$, and the measure $\nu$ we use the notation

$$\mathbb{T}(H|P) = (B, C, \nu)$$

which will be called the *triplet of predictable characteristics* of the semimartingale $H$ with respect to the measure $P$.

**Assumption:** The truncation function satisfies the *antisymmetry* property

$$h(-x) = -h(x).$$
Alternative model description

$$\mathcal{E}(X) = (\mathcal{E}(X)_t)_{0 \leq t \leq T}$$  stochastic exponential

$$S_t = \mathcal{E}(\tilde{H})_t, \quad 0 \leq t \leq T$$

$$dS_t = S_t d\tilde{H}_t$$

where

$$\tilde{H}_t = H_t + \frac{1}{2} \langle H^c \rangle_t + \int_0^t \int_\mathbb{R} (e^x - 1 - x) \mu^H(ds, dx)$$

Note

$$\mathcal{E}(\tilde{H})_t = \exp(\tilde{H}_t - \frac{1}{2} \langle \tilde{H}^c \rangle_t) \prod_{0 < s \leq t} (1 + \Delta \tilde{H}_s) \exp(-\Delta \tilde{H}_s)$$

Asset price positive only if $\Delta \tilde{H} > -1$. 
Martingale and dual martingale measures

Assumption (ES)
The process $1_{\{x>1\}} e^x \ast \nu$ has bounded variation.

Then, $H$ is exponentially special and

$$S = e^H \in \mathcal{M}_{\text{loc}}(P) \iff B + \frac{C}{2} + (e^x - 1 - h(x)) \ast \nu^H = 0.$$

Moreover, we assume that $S \in \mathcal{M}(P)$, therefore $ES_T = 1$. Define on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T})$ a new probability measure $P'$ with

$$\frac{dP'}{dP} = S_T.$$

Since $S > 0$ ($P$-a.s.), we have $P \ll P'$ and

$$\frac{dP}{dP'} = \frac{1}{S_T}.$$
Introduce the process

\[ S' = \frac{1}{S}. \]

Then, denoting by \( H' \) the dual of the semimartingale \( H \), i.e. \( H' = -H \), we have

\[ S' = e^{H'}. \]

**Proposition**

Suppose \( S = e^H \in \mathcal{M}(P) \) i.e. \( S \) is a \( P \)-martingale. Then the process \( S' \in \mathcal{M}(P') \) i.e. \( S' \) is a \( P' \)-martingale.

**Lemma**

Let \( f \) be a predictable, bounded process. The triplet of predictable characteristics of the stochastic integral process \( J = \int_0^\cdot f dH \), denoted by \( \mathbb{T}(J|P) = (B_J, C_J, \nu_J) \), is

\[
B_J = f \cdot B + [h(fx) - fh(x)] \ast \nu \\
C_J = f^2 \cdot C \\
1_A(x) \ast \nu_J = 1_A(fx) \ast \nu, \quad A \in \mathcal{B}(\mathbb{R}).
\]
Theorem

The triplet $\mathbb{T}(H'|P') = (B', C', \nu')$ can be expressed via the triplet $\mathbb{T}(H|P) = (B, C, \nu)$ by the following formulae:

$$
B' = -B - C - h(x)(e^x - 1) \ast \nu
$$

$$
C' = C
$$

$$
1_A(x) \ast \nu' = 1_A(-x)e^x \ast \nu, \quad A \in \mathcal{B}(\mathbb{R}).
$$

Structure of the proof:

\[ \begin{array}{ccc}
\mathbb{T}(H|P) & \xrightarrow{(G)} & \mathbb{T}(H|P') \\
\xrightarrow{(a)} & & \xrightarrow{(-)} \\
\mathbb{T}(H'|P) & \xrightarrow{(G)} & \mathbb{T}(H'|P') \\
\xrightarrow{(d)} & & \xrightarrow{(-)} \\
\mathbb{T}(H'|P) & \xrightarrow{(-)} & \mathbb{T}(H|P)
\end{array} \]

\[ \begin{array}{cc}
\xrightarrow{(G)} & \text{: Girsanov's theorem,} \\
\xrightarrow{(-)} & \text{: dual of a semimartingale.}
\end{array} \]
Symmetry of markets

If the original market \((S, P)\) and the dual market \((S', P')\) satisfy

\[
\text{Law}(S|P) = \text{Law}(S'|P')
\]

then we say these markets are symmetric.

In cases where the triplets \(\mathbb{T}(H|P)\) and \(\mathbb{T}(H'|P')\) determine these laws completely (e.g. for Lévy processes \(H\) and \(H'\))

symmetry holds iff \(\nu' = \nu\)

The equation in the Theorem is then

\[
1_A(x) \ast \nu = 1_A(-x)e^x \ast \nu, \quad A \in B(\mathbb{R})
\]
Example 1: Diffusion models

\[ dS_t = S_t \sigma(t, S_t) \, dW_t, \quad S_0 = 1 \]

*local volatility models* (Dupire (1994), Skiadopoulos (2001))

\[
H_t = \int_0^t \sigma(u, e^{Hu}) \, dW_u - \frac{1}{2} \int_0^t \sigma^2(u, e^{Hu}) \, du \\
\Rightarrow B = -\frac{1}{2} \int_0^t \sigma^2(u, e^{Hu}) \, du, \quad C = \int_0^t \sigma^2(u, e^{Hu}) \, du, \quad \nu \equiv 0
\]

Theorem \( \Rightarrow B' = -B - C = -\frac{1}{2} \int_0^t \sigma^2(u, e^{Hu}) \, du \), \( C' = C \), \( \nu' \equiv 0 \)
Example 2: Purely discontinuous Lévy models

\[ S_t = e^{H_t}, \quad \mathbb{T}(H, P) = (B, 0, \nu) \]

Local characteristics:
\[ B_t(\omega) = bt, \quad \nu(\omega; dt, dx) = dtF(dx), \]
\[ F \text{ Lévy measure} \]

\[ S \in \mathcal{M}_{\text{loc}}(P) \iff b = -\int_{\mathbb{R}} (e^x - 1 - h(x))F(dx) \]

Actually: \( S \in \mathcal{M}(P) \)

Parametric models: \( F(dx) = e^{\vartheta x}f(x)dx \quad f \text{ even} \)

Generalized hyperbolic (includes hyperbolic, NIG, VG, \ldots)

CGMY, Meixner

Dual process \( H' \):
\[ \int 1_A(x)F'(dx) = \int 1_A(-x)e^{(1+\vartheta)x}f(x)dx \]
\[ b' = -\int_{\mathbb{R}} (e^x - 1 - h(x))F'(dx) \]
European options (1)

**Theorem**

The prices of standard call and put options satisfy the following duality relations:

\[
C_T(S; K) = K \mathbb{P}_T'(K'; S')
\]

and

\[
\mathbb{P}_T(K; S) = K C'_T(S'; K').
\]

**Proof:** Using the dual measure

\[
C_T(S; K) = E\left[ S_T \left( \frac{S_T - K}{S_T} \right)^+ \right] = E'\left[ \left( \frac{S_T - K}{S_T} \right)^+ \right] = E'[\left(1 - KS_T\right)^+] = KE'\left[\left( \frac{1}{K} - S'_T \right)^+ \right] = KE'\left[\left( K' - S'_T \right)^+ \right],
\]

where \( K' = \frac{1}{K} \).

\[\square\]
Corollary

Call and put prices in a dual pair of markets \((S, P)\) and \((S', P')\) satisfy a call-call parity

\[
C_T(S; K) = K C'_T(S'; K') + 1 - K
\]

and a put-put parity

\[
P_T(K; S) = K P'_T(K'; S') + K - 1
\]

**Proof:** Combine with classical call-put parity

\[
C_T(S; K) = P_T(K; S) + 1 - K
\]
Floating strike lookback options (1)

Payoff of a call: \( (S_T - \alpha \inf_{0 \leq t \leq T} S_t)^+ \) for an \( \alpha \geq 1 \)

Assume \( H' = (H'_t)_{0 \leq t \leq T} \) satisfies the reflection principle

\[
\text{Law} \left( \sup_{t \leq T} H'_t - H'_T \right) = \text{Law} \left( - \inf_{t \leq T} H'_t \right)
\]

(holds for Lévy processes), then

\[
C_T(S; \alpha \inf S) = \alpha \mathbb{P}'_T \left( \frac{1}{\alpha}; \inf S' \right)
\]

Value of a floating strike lookback call
→ value of a fixed strike lookback put
Floating strike lookback options (2)

Payoff of a put: \( \left( \beta \sup_{0 \leq t \leq T} S_t - S_T \right)^+ \) for a \( 0 < \beta \leq 1 \)

Assume \( H' = (H'_t)_{0 \leq t \leq T} \) satisfies

\[
\text{Law} \left( H'_T - \inf_{t \leq T} H'_t \mid P' \right) = \text{Law} \left( \sup_{t \leq T} H'_t \mid P' \right)
\]

(holds for Lévy processes), then

\[
P_T(\beta \sup S; S) = \beta C'_T \left( \sup S'; \frac{1}{\beta} \right)
\]

Value of a floating strike lookback put
→ value of a fixed strike lookback call
Floating strike Asian options

Payoff of a call: \( \left( S_T - \frac{1}{T} \int_0^T S_t \, dt \right)^+ \)

Assume \( H' = (H'_t)_{0 \leq t \leq T} \) satisfies

\[
\text{Law}(H'_T - H'_{(T-t)}; 0 \leq t < T | P') = \text{Law}(H'_t; 0 \leq t < T | P')
\]

(holds for Lévy processes), then

\[
\mathbb{C}_T(S; \frac{1}{T} \int S) = \mathbb{P}'_T \left( 1; \frac{1}{T} \int S' \right)
\]

Value of a floating strike Asian call \\
\( \rightarrow \) value of a fixed strike Asian put

Similarly \( \mathbb{P}_T \left( \frac{1}{T} \int S; S \right) = \mathbb{C}'_T \left( \frac{1}{T} \int S', 1 \right) \)
Forward-start options

Payoff of a call: \((S_T - S_t)^+\)
Payoff of a put: \((S_t - S_T)^+\)

Assume \(H' = (H'_t)_{0 \leq t \leq T}\) satisfies
\[
\text{Law}(H'_T - H'_{(T-t)} - ; 0 \leq t < T | P') = \text{Law}(H'_t ; 0 \leq t < T | P')
\]
then
\[
C_{t,T}(S;S) = P'_{T-t}(1; S')
\]
and
\[
P_{t,T}(S;S) = C'_{T-t}(S'; 1)
\]

Value of a forward-start call
→ value of a plain vanilla put
Multiasset price model

Each component $S^i$ of the vector of asset price processes $S = (S^1, \ldots, S^d)^\top$ is an exponential time-inhomogeneous Lévy process

$$S^i_t = S^i_0 \exp L^i_t, \quad 0 \leq t \leq T.$$ 

The driving process $L = (L_t)_{0 \leq t \leq T}$ is an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process that satisfies Assumption (EM), with canonical decomposition

$$L_t = \int_0^t b_s \, ds + \int_0^t c^{1/2}_s \, dW_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(ds, dx).$$
Multiasset price model (2)

**Theorem**

Let \( L = (L_t)_{0 \leq t \leq T} \) be an \( \mathbb{R}^d \)-valued PI1AC that satisfies Assumption (EM), with characteristics \( \mathbb{T}(L|P) = (B, C, \nu) \). Let \( u, v \) be vectors in \( \mathbb{R}^d \) such that \( v \in (-M, M)^d \) and \( u + v \in [-M, M]^d \). Define the measure \( P' \)

\[
\frac{dP'}{dP} = \frac{e^{\langle v, L_T \rangle}}{E[e^{\langle v, L_T \rangle}]}.
\]

Then, the process \( L^u = (L^u_t)_{0 \leq t \leq T} \), where \( L^u_t := \langle u, L_t \rangle \), is a 1-dimensional PI1AC with characteristics \( \mathbb{T}(L^u|P') = (B^u, C^u, \nu^u) \) with

\[
\begin{align*}
b^u_s &= \langle u, b_s \rangle + \langle u, C_s v \rangle + \int_{\mathbb{R}^d} \langle u, x \rangle (e^{\langle v, x \rangle} - 1) \lambda_s(dx) \\
c^u_s &= \langle u, C_s u \rangle \\
\lambda^u_s(E) &= \lambda'_s(\{x \in \mathbb{R}^d : \langle u, x \rangle \in E\}), \quad E \in \mathcal{B}(\mathbb{R})
\end{align*}
\]

where \( \lambda'_s \) is a measure defined by

\[
\lambda'_s(A) = \int_A e^{\langle v, x \rangle} \lambda_s(dx), \quad A \in \mathcal{B}(\mathbb{R}^d).
\]

Application: Multiasset options
Example: Swap option (Margrabe)

**Theorem**

We can relate the value of a swap, with payoff \((S_T^1 - S_T^2)^+\), and a plain vanilla option via the following duality:

\[
\mathbb{M}(S_0^1, S_0^2; C, \nu) = S_0^1 \mathbb{P}(1, S_0^2 / S_0^1; C', \nu')
\]

where the characteristics \((C', \nu')\) are given in the previous Theorem for \(\nu = (1, 0)^T\) and \(u = (-1, 1)^T\).

**Proof:** Using asset \(S^1\) to form the Radon–Nikodym derivative

\[
\mathbb{M} = E \left[ \left( S_T^1 - S_T^2 \right)^+ \right] = S_0^1 E \left[ \frac{S_T^1}{S_0^1} \left( 1 - \frac{S_T^2}{S_T^1} \right)^+ \right]
\]

\[
= S_0^1 E \left[ e^{L_T} \left( 1 - \frac{S_T^2}{S_T^1} \right)^+ \right] = S_0^1 E' \left[ \left( 1 - \frac{S_T^2}{S_T^1} \right)^+ \right],
\]
where \( \nu = (1, 0)^\top \). Now, note that

\[
\frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1} \frac{e^{L_t^2}}{e^{L_t^1}} = \frac{S_0^2}{S_0^1} e^{\langle u, L_t \rangle}, \quad 0 \leq t \leq T
\]

where \( u = (-1, 1)^\top \) and

\[
e^{\langle u, L \rangle} \in \mathcal{M}(P') \quad \text{since} \quad e^{\langle u, L \rangle} e^{\langle \nu, L \rangle} = e^{L^2} \in \mathcal{M}(P).
\]

Then, we have that

\[
\mathbb{M} = S_0^1 E' \left[ (1 - S_T')^+ \right]
\]

where \( S' \) is an exponential PIIAC with characteristics \( C' \) and \( \nu' \). \( \square \)
Model with interest rates

Asset price processes

\[ S^i_t = S^i_0 \exp[(r - \delta^i) t + L^i_t] \]

where \( L = (L^1, \ldots, L^d) \) is a PI\( \lambda \)AC with triplet \((B, C, \nu)\)

payoff of a Margrabe option: \((S^1_T - S^2_T)^+\)

value

\[ \mathbb{M}(S^1_0, S^2_0; r, \delta, C, \nu) = e^{-rT} E[(S^1_T - S^2_T)^+] \]

then

\[ \mathbb{M}(S^1_0, S^2_0; r, \delta, C, \nu) = E[S^1_T] e^{C_T} P(K, S^2_0/S^1_0, \delta^1, r, C', \nu') \]

where \( K = e^{-C_T} \) and \( C_T \) is a constant.
Duality in the Lévy forward rate model

Denote the value of a call option on a zero coupon bond by

\[ V_c \left( B(0, T); \frac{B(0, U)}{B(0, T)}, K; C, \nu \right) = \mathbb{E} \left[ \frac{1}{B_T} (B(T, U) - K)^+ \right], \]

and similarly for a put option

\[ V_p \left( B(0, T); \frac{B(0, U)}{B(0, T)}, K; C, \nu \right) = \mathbb{E} \left[ \frac{1}{B_T} (K - B(T, U))^+ \right]. \]

Theorem

Assume that bond prices are modeled according to the Lévy forward rate model. Then, the value of a call and a put option on a bond are related via:

\[ V_c \left( B(0, T); \frac{B(0, U)}{B(0, T)}, K; C, \nu \right) = V_p \left( B(0, T); K, \frac{B(0, U)}{B(0, T)}; C, -f \nu \right), \]

where \( f(s, x) = \exp \left( (\Sigma(s, U) + \Sigma(s, T))x \right). \)
Idea of proof: Define the constant $D := \mathbb{E} \left[ \frac{B(T, U)}{(B_T)^2} \right]$ and the measure $\tilde{\mathbb{P}}$ via

$$
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \frac{B(T, U)}{D(B_T)^2} = \eta_T.
$$

$\mathbb{P} \sim \tilde{\mathbb{P}}$ and the density process $(\eta_t)_{t \in [0, T]}$ is $\eta_t = \mathbb{E} \left[ \frac{B(T, U)}{D(B_T)^2} \bigg| \mathcal{F}_t \right]$. Using Girsanov’s theorem for semimartingales we deduce the $\tilde{\mathbb{P}}$-characteristics of the driving process $L$. Now,

$$
V_c = \mathbb{E} \left[ \frac{1}{B_T} (B(T, U) - K)^+ \right] = \mathbb{E} \left[ \frac{B(T, U)}{D(B_T)^2} KDB_T(K^{-1} - B(T, U)^{-1})^+ \right]
$$
and changing measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$, we get that

$$V_c = \tilde{\mathbb{E}} \left[ KDB_T(K^{-1} - B(T, U)^{-1})^+ \right].$$

This can be re-written as

$$V_c = \tilde{\mathbb{E}} \left[ \frac{1}{B_T} \left( \hat{K} - \hat{B}(T, U) \right)^+ \right],$$

for $(\hat{B}_T)^{-1} := \frac{B(0,T)}{B(0,U)}DB_T$, $\hat{K} := \frac{B(0,U)}{B(0,T)}$ and $\hat{B}(T, U) := K \frac{B(0,U)}{B(0,T)} B(T, U)^{-1}$.

Showing that $\hat{B}_T$ and $\hat{B}(T, U)$ have dynamics analogous to that of $B_T$ and $B(T, U)$ concludes the proof. $\square$
Equivalent formulation (1)

time- \( T_{i+1} \) payoff of a caplet: \( N\delta(L(T_i, T_i) - K)^+ \)

Recall \[ 1 + \delta L(T_i, T_i) = \frac{B(T_i, T_i)}{B(T_i, T_{i+1})} \]

\[
\delta(L(T_i, T_i) - K)^+ = (1 + \delta L(T_i, T_i) - (1 + \delta K)) = \left( \frac{1}{B(T_i, T_{i+1})} - K \right)
\]

time- \( T_i \) value of this payoff

\[
B(T_i, T_{i+1})\left( \frac{1}{B(T_i, T_{i+1})} - K \right)^+ = K \left( \frac{1}{K} - B(T_i, T_{i+1}) \right)^+
\]

→ payoff of a put option on a bond with strike \( \frac{1}{1 + \delta K} \)

Analogously for a floorlet.
Equivalent formulation (2)

Value of a floorlet with strike $K$ maturing at time $T_i$ that settles in arrears at time $T_{i+1}$

\[
\mathbb{F}_{L}(L(0, T_i), K; C, \nu) = \mathbb{E}\left[\frac{1}{B_{T_{i+1}}}\delta(K - L(T_i, T_i))^+\right]
\]

\[
= (1 + \delta K)\mathbb{E}\left[\frac{1}{B_{T_i}}(B(T_i, T_{i+1}) - K)^+\right]
\]

where $L(0, T_i) = \frac{1}{\delta} \left( \frac{B(0, T_i)}{B(0, T_{i+1})} - 1 \right)$ initial forward Libor rate

Therefore

\[
\mathbb{F}_{L}(L(0, T_i), K; C, \nu) = C \mathbb{C}_{\mathbb{L}}(K, L(0, T_i); C, -f\nu)
\]

where $C = \frac{1 + \delta K}{1 + \delta L(0, T_i)}$
Duality in the Lévy Libor model

Value of a caplet with strike $K$ maturing at time $T_i$ that settles in arrears at time $T_{i+1}$

$$\mathcal{C}L(L(0, T_i), K; C, \nu_{T_{i+1}}) = B(0, T_{i+1}) \mathbb{E}_{\mathbb{P}_{T_{i+1}}} [\delta(L(T_i, T_i) - K)^+]$$

Duality result

$$\mathcal{C}L(L(0, T_i), K; C, \nu_{T_{i+1}}) = \mathcal{F}L(K, L(0, T_i); C, -f \nu_{T_{i+1}})$$

where $f(s, x) = \exp(\lambda(s, T_i)x)$
Duality in the Lévy forward process model

Value of a call option on the forward process with strike $K$ which is settled in arrears at time $T_{i+1}$

$$C(F(0, T_i, T_{i+1}), K; C, \nu^{T_{i+1}}) = B(0, T_{i+1}) \mathbb{E}_{P_{T_{i+1}}} [(F(T_i, T_i, T_{i+1}) - K)^+]$$

Duality for call and put options on the forward process

$$C(F(0, T_i, T_{i+1}), K; C, \nu^{T_{i+1}}) = \mathbb{P}(K, F(0, T_i, T_{i+1}); C, -f\nu^{T_{i+1}})$$
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References

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## Financial statement Deutsche Bank

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## Financial liabilities held at fair value:

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<th>Valuation technique unobservable parameters</th>
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<td>Deposits</td>
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<td>[28], [30]</td>
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<td>1,358</td>
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<td>[30]</td>
<td>14,961</td>
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<td>(2,819)</td>
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<td>(3)</td>
<td>(3,552)</td>
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<td>(882)</td>
<td>3,635</td>
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</table>
Asset prices
Decomposition of the balance sheet

Cash + Risky Assets = Equity + Risky Debt + Risky Liabilities

\[ M(t) + A(t) = J(t) + D(t) + L(t) \]

\( M(t): \) Cash + short term investments (cash equivalent reserve)
   relatively nonrandom: \( M(t) = Me^{rt} \)

\( L(t): \) Short positions in stocks
   Negative side of a swap contract
   Payouts on writing credit protections
   Payouts on selling options
   Short positions in variance swaps

\( \rightarrow \) potentially unbounded
Equity as a Spread Option

Company set up with limited liability

At debt maturity (face value $F$)

$$J(T) = (Me^{rT} + A(T) - L(T) - F)^+$$

Debt holders receive

$$D(T) = (Me^{rT} + A(T) - L(T))^+ \wedge F$$

Consequently: Initial equity and debt value

$$J = E_0^Q \left[ e^{-rT} (A(T) - L(T) - (F - Me^{rT}))^+ \right]$$

$$D = E_0^Q \left[ e^{-rT} ((Me^{rT} + A(T) - L(T))^+ \wedge F) \right]$$
Equity as a Spread Option (2)

Value of the limited liability firm at debt maturity

$$ (Me^{rT} + A(T) - L(T))^+ $$

→ call option struck at $-Me^{rT}$

Value of the firm at time 0

$$ V = J + D = E_0^Q \left[ e^{-rT} (Me^{rT} + A(T) - L(T))^+ \right] $$

Negative part of this variable:

put option on $A(T) - L(T)$ struck at $-Me^{rT}$

Value

$$ P = E_0^Q \left[ e^{-rT} ( - Me^{rT} - (A(T) - L(T)))^+ \right] $$

Capital requirements set by external regulators: $M =$?
Equity as a Spread Option (3)

Architecture of this approach:

Model $A(T) - L(T)$ as the difference of two exponential Lévy processes

Compute equity prices

$$J(t) = E_t^Q \left[ e^{-r(T-t)} (A(T) - L(T) - (F - Me^{rT}))^+ \right]$$

Derive prices of equity options for strike $K$ and maturity $t$

$$W(K, t) = e^{-rt} E_0^Q \left[ (J(t) - K)^+ \right] \quad \text{(compound option)}$$

Calibration to the observed option price surface
Volatility smile and surface

Volatility surfaces

- Volatilities vary in strike (smile)
- Volatilities vary in time to maturity (term structure)
Net Asset Value Process

Model for the risky asset

\[ A(t) = A(0) \exp(X(t) + (r + \omega_X)t) \]

Model for the risky liability

\[ L(t) = L(0) \exp(Y(t) + (r + \omega_Y)t) \]

In order to create the right level of dependence between \( X(t) \) and \( Y(t) \)

\[ \rightarrow \text{linear mixture of 4 independent VG Lévy processes} \]

\[
\begin{bmatrix}
X(t) \\
Y(t)
\end{bmatrix}
= \begin{bmatrix}
\cos(\eta_1) & \cos(\eta_2) & \cos(\eta_3) & \cos(\eta_4) \\
\sin(\eta_1) & \sin(\eta_2) & \sin(\eta_3) & \sin(\eta_4)
\end{bmatrix}
\begin{bmatrix}
U_1(t) \\
U_2(t) \\
U_3(t) \\
U_4(t)
\end{bmatrix}
\]

Characteristic function of \( \text{VG}(\sigma, \theta, \nu) \)

\[
\chi_{\text{VG}(\sigma, \theta, \nu)}(u) = \left( \frac{1}{1 - i\theta \nu u + (\sigma^2 \nu/2)u^2} \right)^{1/\nu}
\]
Net Asset Value Process (2)

Joint characteristic function \( E [\exp (iuX(t) + ivY(t))] = \phi(u, v) \)

\[
= \prod_{j=1}^{4} \left( \frac{1}{1 - i(u \cos(\eta_j) + v \sin(\eta_j))\theta_j \nu_j + \frac{\sigma_j^2 \nu_j}{2} (u \cos(\eta_j) + v \sin(\eta_j))^2} \right)^{\frac{t}{\nu_j}}
\]

The values for the exponential compensators are

\[
\omega_X = \sum_{j=1}^{4} \frac{1}{\nu_j} \ln \left( 1 - \cos(\eta_j)\theta_j \nu_j - \frac{\sigma_j^2 \nu_j \cos^2(\eta_j)}{2} \right)
\]

\[
\omega_Y = \sum_{j=1}^{4} \frac{1}{\nu_j} \ln \left( 1 - \sin(\eta_j)\theta_j \nu_j - \frac{\sigma_j^2 \nu_j \sin^2(\eta_j)}{2} \right)
\]

Consequently \( E \left[ e^{iu \ln(A(t)) + iv \ln(L(t))} \right] \)

\[
= \phi(u, v) \exp(iu \ln(A(0)) + iv \ln(L(0)) + iu(r + \omega_X)t + iv(r + \omega_Y)t)
\]
Balance Sheet data

Balance sheet data for six major US banks from Wharton Research Data Service (end of 2008 in millions of dollars)

<table>
<thead>
<tr>
<th></th>
<th>M in millions of dollars</th>
<th>A in millions of dollars</th>
<th>L in millions of dollars</th>
<th>D in millions of dollars</th>
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</table>
Introduction
Lévy processes
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Valuation
Risk Managem.
Interest rate
Calibration
Lévy LIBOR
Cross-currency
Range options
Credit risk
Duality theory
Capital Requirements
Spread Option
Net Asset Value
Reserve Capital
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Required Reserve Capital

$X$ random variable: outcome (cashflow) of a risky position

For setting capital requirements: non-dynamic

In complete markets: unique pricing kernel given by a probability measure $Q$

value of the position: $E^Q[X]$  

position is acceptable if: $E^Q[X] \geq 0$  

company’s objective is: maximizing $E^Q[X]$  

Real markets: incomplete

Instead of a unique probability measure $Q$ we have to consider a set of probability measures $Q \in \mathcal{M}$

$E^Q[X] \geq 0$ for all $Q \in \mathcal{M}$ or $\inf_{Q \in \mathcal{M}} E^Q[X] \geq 0$
Required Reserve Capital (2)

Specification of $\mathcal{M}$ (test measures, generalized scenarios)

Axiomatic theory of risk measures: desirable properties

Monotonicity: $X \geq Y \implies \varrho(X) \leq \varrho(Y)$

Cash invariance: $\varrho(X + c) = \varrho(X) - c$

Scale invariance: $\varrho(\lambda X) = \lambda \varrho(X), \lambda \geq 0$

Subadditivity: $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$

Examples: Value at Risk (VaR)

Tail-VaR (expected shortfall)

General risk measure: $\varrho_m(X) = - \int_0^1 q_u(X)m(du)$

Any coherent risk measure has a representation

$$\varrho(X) = - \inf_{Q \in \mathcal{M}} E^Q[X]$$
Required Reserve Capital (3)

Acceptability of a cash flow?

Maybe it exposes the general economy to too much risk of loss

Business set up with limited liability and insufficient capital

\[ \rightarrow \text{Add capital } C \text{ such that cash flow } C + X \text{ is acceptable} \]

\[ \inf_{Q \in \mathcal{M}} E^Q[C + X] \geq 0 \]

Smallest such capital

\[ C = - \inf_{Q \in \mathcal{M}} E^Q[X] \]
Required Reserve Capital (4)

Computation of this required reserve capital

Link between acceptability and concave distortions
(Cherny and Madan (2009))

→ Concave distortions

Assume acceptability is completely defined by the distribution function of the risk

\( \Psi(u) \): concave distribution function on \([0, 1]\)

\( \Rightarrow \mathcal{M} \) the set of supporting measures is given by all measures \( Q \) with density \( Z = \frac{dQ}{dP} \) s.t.

\[
E^P[(Z - a)^+] \leq \sup_{u \in [0,1]} (\Psi(u) - ua) \quad \text{for all} \ a \geq 0
\]

Acceptability of \( X \) with distribution function \( F(x) \)

\[
\int_{-\infty}^{+\infty} xd\Psi(F(x)) \geq 0
\]
Distortion

\[ \Psi(x) \]

- \( \gamma = 2 \)
- \( \gamma = 10 \)
- \( \gamma = 20 \)
- \( \gamma = 100 \)
Required Reserve Capital (5)

Consider families of distortions \((\psi_\gamma)_{\gamma \geq 0}\)

\(\gamma\) stress level

Example: MIN VaR

\[\psi_\gamma(x) = 1 - (1 - x)^{1+\gamma} \quad (0 \leq x \leq 1, \gamma \geq 0)\]

Statistical interpretation:

Let \(\gamma\) be an integer, then \(\varrho_\gamma(X) = -E(Y)\) where

\[Y \overset{\text{law}}{=} \min\{X_1, \ldots, X_{\gamma+1}\}\]

and \(X_1, \ldots, X_{\gamma+1}\) are independent draws of \(X\)
Required Reserve Capital (6)

Further examples: MAX VaR

\[ \Psi^\gamma(x) = x^{\frac{1}{1+\gamma}} \quad (0 \leq x \leq 1, \gamma \geq 0) \]

Statistical interpretation: \( \rho_\gamma(X) = -E[Y] \)
where \( Y \) is a random variable s.t.

\[ \max\{Y_1, \ldots, Y_{\gamma+1}\} \overset{\text{law}}{=} X \]

and \( Y_1, \ldots, Y_{\gamma+1} \) are independent draws of \( Y \).

Combining MIN VaR and MAX VaR: MAX MIN VaR

\[ \Psi^\gamma(x) = (1 - (1 - x)^{1+\gamma})^{\frac{1}{1+\gamma}} \quad (0 \leq x \leq 1, \gamma \geq 0) \]

Interpretation: \( \rho_\gamma(X) = -E[Y] \) with \( Y \) s.t.

\[ \max\{Y_1, \ldots, Y_{\gamma+1}\} \overset{\text{law}}{=} \min\{X_1, \ldots, X_{\gamma+1}\} \]
Required Reserve Capital (7)

Distortion used: MIN MAX VaR

\[ \psi^\gamma(x) = 1 - \left(1 - x^{\frac{1}{1+\gamma}}\right)^{1+\gamma} \quad (0 \leq x \leq 1, \gamma \geq 0) \]

\[ \varrho^\gamma(X) = -E[Y] \text{ with } Y \text{ s.t. } Y \overset{\text{law}}{=} \min\{Z_1, \ldots, Z_{\gamma+1}\}, \]
\[ \max\{Z_1, \ldots, Z_{\gamma+1}\} \overset{\text{law}}{=} X \]

Capital required at (stress) level \( \gamma \)

\[ C = -\int_{-\infty}^{\infty} xd\psi^\gamma(F_X(x)) \]

Computationally: Let \( x_1 \leq x_2 \leq \cdots \leq x_N \) be historic or Monte Carlo realizations of the cashflow \( X \)

\[ C \approx \sum_{j=1}^{N} x_j \left( \psi^\gamma \left( \frac{j}{N} \right) - \psi^\gamma \left( \frac{j-1}{N} \right) \right) \]
## Computation of Required Reserve Capital and the value of the taxpayer put

In Billions of US Dollars

<table>
<thead>
<tr>
<th></th>
<th>Reserve Capital Required</th>
<th>Reserve Capital Held</th>
<th>Limited Liability Put Value</th>
<th>Required to Actual Ratio</th>
<th>Adjustment Factor</th>
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<tbody>
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</table>
References

References (cont.)

References (cont.)

References (cont.)

References (cont.)


References (cont.)