Information and Credit Risk

M. L. Bedini

Université de Bretagne Occidentale, Brest - Friedrich Schiller Universität, Jena

Jena, March 2011
Motivation (1/3)

European CDS Spreads
5-year, basis points

Hazard-process approach for Credit-risk. \( \left( \Omega, \mathcal{F}, \mathcal{G} = (\mathcal{G}_t)_{t \geq 0}, \mathbb{P} \right) \), the total information \( \mathcal{G} \) can be decomposed:

\[
\mathcal{G} = \mathcal{F} \lor \mathcal{H}
\]

- **Market risk**: given by the natural movements of interest rates, etc. It can be hedged with \( \mathcal{F} \)-adapted financial instruments.
- **Default risk**: risk that is directly associated with the default. It can be hedged with \( \mathcal{H} \)-adapted financial instruments.

What is \( \mathcal{H} = (\mathcal{H}_t)_{t \geq 0} \)?

\[
H = (H_t := \mathbb{I}_{\{\tau \geq t\}}, t \geq 0) \text{ is the default indicator process.}
\]
Objective

Our approach aims to give a qualitative description of the information on $\tau$ before the default, thus making $\tau$ “a little bit less inaccessible”.

In our approach the information will be carried by $\beta = (\beta_t, t \geq 0)$ a Brownian bridge between 0 and 0 on the stochastic interval $[0, \tau]$. Thus, in our market model we will deal with:

- $\mathcal{F}$: the reference filtration.
- $\mathcal{F}^\beta$: the information on the default event generated by the information process $\beta$. 
1. The Information Process $\beta$
   - Introduction and First Properties
   - Conditional Expectation
   - Application: Pricing a Credit Default Swap
   - Classification of $\tau$ in $\mathbb{F}^\beta$

2. The Enlarged Filtration $\mathbb{G} := \mathbb{F} \lor \mathbb{F}^\beta$
   - First Properties
   - Conditional Expectation in $\mathbb{G}$
   - Classification of $\tau$ in $\mathbb{G}$: two examples

3. Conclusion, Acknowledgments and References
(Ω, ℱ, ℙ) complete probability space, ℳP the collection of the ℙ-null sets. \( W = (W_t, t \geq 0) \) is a standard BM. \( \tau : \Omega \to (0, +\infty) \) random variable. 
\( F(t) := \mathbb{P}\{\tau \leq t\}. \)

**Assumption**

\( \tau \) is independent of \( W \).

**Definition**

The process \( \beta = (\beta_t, t \geq 0) \) is called *Information process*:

\[
\beta_t := W_t - \frac{t}{\tau \lor t} W_{\tau \lor t}
\]  
(1)
Let $\mathbb{F}^0 = (\mathcal{F}^0_t := \sigma \{\beta_s, 0 \leq s \leq t\})_{t \geq 0}$ be the natural filtration of the process $\beta$ and let:

$$\mathbb{F}^\beta = \left(\mathcal{F}^\beta_t := \mathcal{F}^0_t \vee \mathcal{N}_P\right)_{t \geq 0}$$

### Proposition

- $\tau$ is an $(\mathcal{F}^0_{t+})_{t \geq 0}$-stopping time, $\mathbb{P}$-a.s.
- For all $t > 0$, $\{\beta_t = 0\} = \{\tau \leq t\}$, $\mathbb{P}$-a.s. $\Rightarrow \tau$ is an $(\mathcal{F}^0_t \vee \mathcal{N}_P)_{t \geq 0}$-stopping time
- $\beta$ is an $(\mathcal{F}^0_{t+})_{t \geq 0}$-Markov process $\Rightarrow \mathcal{F}^\beta_t = \mathcal{F}^0_t \vee \mathcal{N}_P$
Lemma

The process $\beta$ is an $\mathbb{F}^\beta$-semi-martingale and the process $b = (b_t, t \geq 0)$ given by

$$b_t := \beta_t + \int_0^{t \wedge \tau} E \left[ \frac{\beta_s}{T - s} | \mathcal{F}^\beta_s \right] ds$$

is an $\mathbb{F}^\beta$-Brownian motion stopped at $\tau$.

**Sketch of the proof.** The process $B = \left( B_t := \beta_t + \int_0^{t \wedge \tau} \frac{\beta_s}{T - s} ds, t \geq 0 \right)$ is an $\mathbb{F}^\beta \vee \sigma \{ \tau \}$-Brownian motion stopped at $\tau$. Then apply standard results from filtering theory.

□
Conditional Expectation

**Theorem**

Let \( t > 0, \ g : \mathbb{R}^+ \to \mathbb{R} \) a Borel function such that \( \mathbb{E}[|g(\tau)|] < +\infty \). Then, \( \mathbb{P} \)-almost surely

\[
\mathbb{E}\left[g(\tau) \mid \mathcal{F}_t^\beta\right] = g(\tau) \mathbb{1}_{\{\tau \leq t\}} + \frac{\int_t^{+\infty} g(r) \varphi_t(r, \beta_t) F(dr)}{\int_t^{+\infty} \varphi_t(r, \beta_t) F(dr)} \mathbb{1}_{\{\tau > t\}}
\]

(2)

where \( \varphi_t(r, x) \), \( r > t > 0, x \in \mathbb{R} \) denotes the density of

\[
\beta_t^r \sim \mathcal{N}\left(0, \frac{t(r-t)}{r}\right)
\]

M. L. Bedini (Université de Bretagne Occidentale, Brest - Friedrich Schiller Universität, Jena)
A Credit Default Swap (CDS) is a financial contract between a buyer and a seller:

- The *buyer* wants to insure the risk of default. *Protection leg*:
  \[
  \mathbb{I}\{t < \tau \leq T\} \delta(\tau)
  \]

- The *seller* is paid by the buyer to provide such insurance. *Fee leg*:
  \[
  \mathbb{I}\{\tau > t\} k \left[ (\tau \wedge T) - t \right]
  \]

The price \( S_t(k, \delta, T) \) of the CDS is equal to:

\[
S_t(k, \delta, T) = \mathbb{E} \left[ \mathbb{I}\{t < \tau \leq T\} \delta(\tau) \mid \hat{F}_t \right] - \mathbb{E} \left[ \mathbb{I}\{\tau > t\} k \left[ (\tau \wedge T) - t \right] \mid \hat{F}_t \right]
\]

where \( \hat{F} = \left( \hat{F}_t \right)_{t \geq 0} \) is the (yet unspecified) market filtration.
Lemma

If the market filtration is $\mathbb{F}^\beta$, for $t \in [u, T]$ we have

\[
S_t (k, \delta, T) = \mathbb{I}_{\{\tau > t\}} \left[ - \int_t^T \delta(r) d\Psi_t(r) - k \int_t^T \Psi_t(r) dr \right]
\]

Where $\Psi_t(r) := \mathbb{P} \left\{ \tau > r \mid \mathcal{F}_t^\beta \right\}$.

Lemma

If the market filtration is $\mathbb{H}$, for $t \in [u, T]$ we have

\[
S_t (k, \delta, T) = \mathbb{I}_{\{\tau > t\}} \left[ - \int_t^T \delta(r) dG(r) - k \int_t^T G(r) dr \right]
\]

Where $G(r) := \mathbb{P} \left\{ \tau > r \right\}$.
Theorem

Suppose $F(t)$ admits a continuous density with respect to the Lebesgue measure: $dF(t) = f(t)dt$. Then $\tau$ is a totally inaccessible stopping time with respect to $F^\beta$ and the compensator $K = (K_t, t \geq 0)$ of $H = (H_t, t \geq 0)$ is given by

$$K_t = \int_0^{\tau \wedge t} \frac{f(r)}{\int_r^{+\infty} \varphi_r(v, 0) f(v) dv} dl_r$$

(3)

where $l = (l_t, t \geq 0)$ is the local time at 0 of the process $\beta$ at time $t$. 
First Properties (1/2)

Let $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual condition. $F_t(dr) := \mathbb{P} \{ \tau \in dr | \mathcal{F}_t \}$

**Assumption**

$\mathcal{F}_{+\infty} \lor \sigma \{ \tau \}$ is independent of $\mathcal{W}$.

**Definition**

The filtration $\mathcal{G} = \left( G_t := \mathcal{F}_t \lor \mathcal{F}^{\beta}_t \right)_{t \geq 0}$ is called *enlarged filtration.*
Lemma

The process $\beta$ satisfies the following

$$
P \{ \beta_{t+h} \in \Gamma | \mathcal{G}_t \} = P \{ \beta_{t+h} \in \Gamma | \mathcal{F}_t \vee \sigma \{ \beta_t \} \}, \ P - a.s.
$$

for all $t, h \geq 0$ and $\Gamma \in \mathcal{B} (\mathbb{R})$.

Lemma

If $E[\sqrt{\tau}] < +\infty$ the process $\beta$ is a $\mathcal{G}$-semi-martingale and the process $b^G = (b^G_t, t \geq 0)$ given by

$$
b^G_t := \beta_t + \int_0^{t \wedge \tau} E\left[ \frac{\beta_s}{\tau - s} | \mathcal{G}_s \right] ds
$$

is a $\mathcal{G}$-Brownian motion stopped at $\tau$. 
Conditional Expectation in $G$

**Theorem**

Let $t > 0$, $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ a Borel function such that $E[g(\tau)] < +\infty$. Then, $P$-almost surely

$$E[g(\tau) | G_t] = g(\tau) \mathbb{I}_{\{\tau \leq t\}} + \frac{\int_t^{+\infty} g(r) \varphi_t(r, \beta_t) F_t(dr)}{\int_t^{+\infty} \varphi_t(r, \beta_t) F_t(dr)} \mathbb{I}_{\{\tau > t\}}$$

(4)

where $\varphi_t(r, x)$, $r > t > 0$, $x \in \mathbb{R}$ denotes the density of

$$\beta_t' \sim \mathcal{N}\left(0, \frac{t(r-t)}{r}\right)$$
Classification of $\tau$ in $\mathbb{G}$

Azéma super-martingale: $Z = (Z_t := \mathbb{P} \{\tau > t|\mathcal{F}_t\}, \ t \geq 0)$. If the random time $\tau$ avoids $\mathbb{F}$-stopping times (i.e. $\mathbb{P} \{\tau = \rho\} = 0$, $\rho$ $\mathbb{F}$-stopping time), the $\mathbb{F}$-dual predictable projection $A = (A_t, \ t \geq 0)$ of the single-jump process $H = (H_t, \ t \geq 0)$ is continuous and the canonical decomposition of the semi-martingale $Z$ is given by

$$Z_t = M_t - A_t$$

where $M = (M_t = \mathbb{E} [A_\infty |\mathcal{F}_t], \ t \geq 0)$. Moreover the process $N = (N_t, \ t \geq 0)$ given by

$$N_t := H_t - \int_0^{t \wedge \tau} \frac{1}{Z_s^-}dA_s$$

is a martingale with respect to $\mathbb{G}^* = (\mathcal{G}_t^* = \mathcal{F}_t \vee \mathcal{H}_t)_{t \geq 0}$. In particular $\tau$ is a $\mathbb{G}^*$-totally inaccessible stopping time.
Let $\tau$ and $\eta$ two independent exponentially distributed random variable with parameters $\lambda$ and $\mu$ respectively. Let

$$\theta := \tau + \eta$$

and define $\mathbb{F}^1 = (\mathcal{F}_t^1)_{t \geq 0}$ to be the minimal filtration making $\theta$ a stopping time ($\tau$ is not an $\mathbb{F}$-stopping time). $\mathbb{G}^1 := \mathbb{F}^1 \vee \mathbb{F}^\beta$.

**Lemma**

The compensator $K^{(1)} = \left( K_t^{(1)}, t \geq 0 \right)$ of the $\mathbb{G}^1$-sub-martingale $H^1 = \left( H_t := \mathbb{I}_{\{\theta \leq t\}}, t \geq 0 \right)$ is continuous. Thus $\tau$ is a totally inaccessible stopping time w.r.t. $\mathbb{G}^1$. 
Sketch of the proof. The compensator is obtained through the direct computation of the process \( K^{(1)} = \left( K_t^{(1)}, t \geq 0 \right) \).

\[ \square \]

Remark

The conditional density of \( \tau \) given \( \mathbb{F}^1 \), \( \mathbb{P} \{ \tau \in dr | \mathcal{F}_t \} = f_t (r) \, dr \), on the set \( \{ \theta > t \} \), is:

\[
    f_t (r) = \frac{ (\lambda - \mu) \left[ \lambda e^{-\lambda r} \mathbb{I}_{\{r \geq t\}} + \lambda e^{-\mu t} e^{r(\mu - \lambda)} \mathbb{I}_{\{0 < r < t\}} \right] }{ \lambda e^{-\mu t} - \mu e^{-\lambda t} } \mathbb{I}_{\{\theta > t\}}.
\]

Note that the condition \( f_t (r) = f_r (r), \forall r \leq t \) is not satisfied (no “immersion” in the standard progressive enlargement).
Let $B = (B_t, t \geq 0)$ be a standard Brownian motion independent of $W$. Let $\mathbb{F}^2 = (\mathcal{F}_t^2)_{t \geq 0}$ be the natural and completed filtration of $B$.

$$\tau := \sup_{0 \leq t \leq 1} \{B_t = 0\}$$

Define $\mathbb{G}^2 := \mathbb{F}^2 \vee \mathbb{F}^\beta$.

**Lemma**

$\tau$ is $\mathbb{P}$-a.s. the first hitting time of 0 of the continuous 2-dimensional process $X = (X_t := [B_t, \beta_t]', t \in [0, 1])$. Thus $\tau$ is a predictable stopping time.
Proof. Let $X = (X_t, \ t \in [0, 1])$ be the two-dimensional process given by:

$$X_t := \begin{bmatrix} \beta_t \\ B_t \end{bmatrix}, \ 0 \leq t \leq 1$$

and let $\sigma$ be the first hitting time of 0 of $X$:

$$\sigma := \inf_{t \in [0, 1]} \{ X_t = 0 \}$$

Since $\sigma$ is the first entry time in the closed set $\{0\}$ of the continuous process $X$ we have that $\sigma$ is a predictable stopping time. Our aim is to show that

$$\tau = \sigma, \quad \mathbb{P} \text{- a. s.}$$
By definition of $\tau$ we have that

$$\tau = \sup_{t \in [0,1]} \{X_t = 0\}$$

and so

$$\sigma \leq \tau, \quad P - a. s.$$  

Thus, it must be shown that

$$P \{\sigma < \tau\} = 0$$

and, since

$$P \{\sigma < \tau\} = E\left[P \{\sigma < \tau | \tau\}\right]$$

$$= \int_{[0,1]} P \{\sigma < \tau | \tau = r\} F(dr)$$

it suffices to show that, conditionally on $\tau = r$,

$$P \{\sigma < \tau | \tau = r\} = 0$$
Knowing $\tau = r$ we have that

1. $(B_t, 0 \leq t \leq r)$ is a Brownian bridge of length $r$ between 0 and 0 (see Mansuy and Yor [6]).

2. $(\beta_t, 0 \leq t \leq r)$ is a Brownian bridge of length $r$ between 0 and 0 (from the definition of $\beta$).

3. $\beta$ is independent of $\tau, B$ (because it depends only on $W$ which is, by hypothesis, independent of $\tau, B$).

This means that, conditionally on $\tau = r$, the process $(X_t, 0 \leq t \leq r)$ is a 2-dimensional Brownian bridge from 0 to 0.
Let now $X^r = (X^r_t, 0 \leq t \leq r)$ be a 2-dimensional Brownian bridge from 0 to 0 on the time interval $[0, r]$. Let $\sigma^r$ be the first hitting time of 0 of the process $X^r$:

$$\sigma^r := \inf_{t \in [0, r]} \{X^r_t = 0\}$$

Knowing $\tau = r$, the two processes $(X_t, 0 \leq t \leq r)$ and $(X^r_t, 0 \leq t \leq r)$ have the same finite dimensional distributions and

$$P\{\sigma < \tau | \tau = r\} = P\{\sigma^r < r\}$$
If we prove that

$$\sigma^r = r, \ P - \text{a. s.}$$  \hspace{1cm} (5)

then we have that

$$\mathbb{P}\{\sigma < \tau\} = \mathbb{E}[\mathbb{P}\{\sigma < \tau | \tau\}] = \int_{[0,1]} \mathbb{P}\{\sigma < \tau | \tau = r\} F(dr)$$

$$= \int_{[0,1]} \mathbb{P}\{\sigma^r < r\} F(dr) = 0$$

which implies the thesis. Thus it remains to prove (5).
On \( (\Omega, \mathcal{F}, \mathbb{F}^r = (\mathcal{F}^r_t)_{t \geq 0}, \mathbb{P}^r) \) let \( X^r = (X^r_t, 0 \leq t \leq r) \) be a 2-dimensional Brownian bridge from 0 to 0 on the time interval \([0, r]\). Then, there exists a 2-dimensional \((\mathbb{F}^r, \mathbb{P}^r)\)-Brownian motion \( b = (b_t, t \geq 0) \) such that

\[
dX^r_t = db_t - \frac{X_t^r}{r - t} dt
\]

and, for every \( T \in (0, r) \) we can define the process \( Z^r = (Z^r_t, 0 \leq t \leq T) \) by

\[
Z^r_t := \exp \left( \xi^r_{t, T} - \frac{1}{2} \langle \xi^r_{t, T} \rangle_t \right)
\]

where

\[
\xi^r_{t, T} := \int_0^{t \land T} \frac{X^r_s}{r - s} db_s, \ 0 \leq t \leq T
\]
By Girsanov theorem, the process $Z^r$ is a $(\mathcal{F}^r, P^r)$-martingale, and the process $X^r = (X^r_t, 0 \leq t \leq T)$ is a $(\mathcal{F}^r, Q^r)$-Brownian motion, where

$$Q^r|_{\mathcal{F}_t} := Z^r_t \cdot P^r|_{\mathcal{F}_t}, \ 0 \leq t \leq T$$

Note that $Q^r$ is equivalent to $P^r$ only for $t \in [0, T]$. However, for a 2-dimensional Brownian motion, the one-point sets are polar, i.e.

$$Q^r \{ T_x \leq T \} = 0, \ \forall x \in \mathbb{R}^2$$

(see, for example Jeanblanc, Yor and Chesney [3]) where $T_x := \inf \{ t > 0 : X_t = x \}$. Thus

$$Q^r \{ \sigma^r \leq T \} = 0, \ \forall T \in (0, r)$$
Since $Q^r$ is equivalent to $P^r$, from the previous relation it follows that

$$P^r \{ \sigma^r \leq T \} = 0, \forall T \in (0, r) \Rightarrow P^r \{ \sigma^r > T \} = 1, \forall T \in (0, r)$$

On the other hand, by definition of Brownian bridge

$$P^r \{ X^r_r = 0 \} = 1$$

and so

$$P^r \{ \sigma^r = r \} = 1$$

Now the proof is concluded.
Conclusion

- Explicit formulas can be obtained and they appear to be an intuitive generalization of some simple models already present in literature. We can obtain pricing formulas (Credit Default Swap, Zero-Coupon Bond).

- The information process $\beta_t$ can substitute the default indicator process $H_t = \mathbb{I}_{\{\tau \leq t\}}$ in many models providing interesting insight on the role of information on financial markets.

- Under appropriate condition we can compute explicitly the compensator of the default indicator process in the enlarged filtration.

- We have the first example (at the best of our knowledge) of random time which is predictable in the enlarged filtration $G = F \vee F^\beta$ despite its “weak” properties in $F$ and $F^\beta$. 
Acknowledgments

Work supported in part by the European Community’s FP 7 Programme under contract PITN-GA-2008-213841, Marie Curie ITN ”Controlled Systems” and jointly supervised by prof. Rainer Buckdahn (UBO) and prof. Hans-Juergen Engelbert (FSU).


