Change-point models in finances and enlargements of filtration.

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Plan

1. Change-point models
2. The $f$-divergence minimal martingale measures
3. Optimal strategies for utility maximisation
4. Optimal strategies for utility maximisation in exponential Levy models
5. The $f$-divergence minimal martingale measures in change-point situation.
6. Optimal strategies for utility maximisation in change-point situation
Models with change points

- The parameters of financial models are generally highly dependent on time (information in the press, price of raw materials, stock price hits some psychological level)
- This time-dependency of the parameters can often be described using a piece-wise constant function
- the time of change (change-point) for the parameters is not explicitly known, but it is often possible to make reasonable assumptions about its nature and use statistical tests for its detection
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Exponential Levy model with change-point

- Let $L = (L_t)_{t \geq 0}$ be a Levy process with parameters $(b, c, \nu)$ where $b$ is the drift parameter, $c$ the diffusion parameter and $\nu$ the Levy measure.
- Let $\tilde{L} = (\tilde{L}_t)_{t \geq 0}$ be a Levy process which is independent from $L$ and with parameters $(\tilde{b}, \tilde{c}, \tilde{\nu})$.
- $X_t = L_t \mathbb{1}_{\{\tau > t\}} + (L_\tau - \tilde{L}_t) \mathbb{1}_{\{\tau \leq t\}}$
- $r(t) = r \mathbb{1}_{\{\tau > t\}} + \tilde{r} \mathbb{1}_{\{\tau \leq t\}}$
- $S_t = S_0 \exp(X_t), \quad B_t = B_0 \exp(\int_0^t r(s)ds)$
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Filtrations

- Natural filtration of $X$ denoted $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \leq t \leq T}$
- Progressively enlarged filtration denoted $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{0 \leq t \leq T}$

$$\hat{\mathcal{F}}_t = \bigcap_{s > t} (\mathcal{F}_s^X \vee \mathcal{H}_s)$$

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Minimal martingale measures

- risk-minimisation in an $L^2$-sense (Follmer, Schweizer; Schweizer)
- Hellinger integrals minimisation (Choulli, Stricker; Choulli, Stricker, Li)
- entropy minimisation (Miyahara; Fujiara, Miyahara)
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\( f \)-divergence

- \( P \) and \( Q \) two probability measures, \( Q \ll P \)
- \( f \) is a convex function on \( \mathbb{R}^+ \)
- \( f \)-divergence introduced by Czisar is
  \[
  f(Q|P) = \mathbb{E}_P[f\left(\frac{dQ}{dP}\right)]
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- In particular cases, when \( f(x) = -x^\alpha, 0 < \alpha < 1 \) we obtain
  - Hellinger integral, when \( f(x) = x \ln(x) \) we obtain entropy,
  - with \( f(x) = (1 - x)^2 \) we have squared variance distance,
  - with \( f(x) = |1 - x| \) we have variance distance.
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Definitions

We say that $Q^*$ is an $f$-divergence minimal martingale measure if $f(Q^*|P) < \infty$ and

$$f(Q^*|P) = \inf_{Q \in \mathcal{M}(P)} f(Q|P)$$

where $\mathcal{M}(P)$ is the set of equivalent martingale measures.

We say that an $f$-divergence minimal martingale measure $Q^*$ is invariant under scaling if for all $x \in \mathbb{R}^+$, *

$$f(xQ^*|P) = \inf_{Q \in \mathcal{M}(P)} f(xQ|P)$$

For a given exponential Levy model $S = e^L$, we say that an $f$-divergence minimal martingale measure $Q^*$ preserves the Levy property if $L$ remains a Levy process under $Q^*$.  

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Conservation of Levy property

**Theorem**

Let $f$ be a strictly convex two times differentiable function, $f''(x) = ax^\gamma$ with $a > 0$, $\gamma \in \mathbb{R}$, and such that an $f$-minimal martingale measure $Q^*$ exists. Then it preserves the Levy property of initial Levy process and it is invariant under scaling.
Conservation of Levy property

**Theorem**

Let $f$ be a strictly convex function, $f \in C^3(\mathbb{R}^+,\ast)$, such that an $f$-minimal martingale measure $Q^*$ exists and preserves the Levy property. Suppose that $L$ contains either a non-zero continuous martingale part either the Levy measure $\nu^P$ has a strictly positive density with respect to the Lebesgue measure. Then $f'''(x) = ax^\gamma$ with $a > 0$ and $\gamma \in \mathbb{R}$, and up to a multiplicative constant and a linear term, $f(x) = x \ln(x)$, or $f(x) = -\ln(x)$ or $f(x) = x^p$ with $p \neq 0,1$. Moreover, $Q^*$ is invariant under scaling.
Utility function

- two assets: a non-risky asset $B$, with interest rate $r$, and a risky asset $S$ on filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$
- $\tilde{S} = (B, S)$ is the price process and $\tilde{\Phi} = (\phi^0, \phi)$ is strategy
- A predictable $\tilde{S}$-integrable process $\tilde{\Phi}$ will be said to be a self-financing admissible strategy if for every $t \in [0, T]$ and $x$ initial capital
  \[ \tilde{\Phi}_t \cdot \tilde{S}_t = x + \int_0^t \tilde{\Phi}_u \cdot d\tilde{S}_u \]
  where the stochastic integral in the right-hand side is bounded from below.
- If the interest rate $r$ is 0, then terminal wealth at time $T$ is
  \[ V_T(\phi) = x + \int_0^T \phi_s dS_s \]
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Optimal strategies

Let $u$ denote a strictly increasing, strictly concave, continuously differentiable function on $\text{dom}(u) = \{x \in \mathbb{R} | u(x) > -\infty\}$ which satisfies

$$u'(+\infty) = \lim_{x \to +\infty} u'(x) = 0,$$

$$u'(<\bar{x}) = \lim_{x \to \bar{x}} u'(x) = +\infty$$

where $\bar{x} = \inf\{u \in \text{dom}(u)\}$.

Utility maximizing strategy $\phi^*$:

$$\sup_{\phi \in \mathcal{A}} E_P(u(V_T(\phi))) = E_P(u(V_T(\phi^*)))$$
As has been shown in Goll, Ruschendorf that there is a strong link between this optimisation problem and the previous problem of finding $f$-minimal martingale measures. Let $f$ be the convex conjugate function of $u$:

$$f(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\}$$

We recall that in particular

- if $u(x) = \ln(x)$ then $f(x) = -\ln(x) - 1$,
- if $u(x) = \frac{x^p}{p}$, $p < 1$ then $f(x) = -\frac{p - 1}{p}x^{\frac{p}{p-1}}$,
- if $u(x) = 1 - e^{-x}$ then $f(x) = 1 - x + x \ln(x)$. 
Theorem

Let $x \in \mathbb{R}^+$ be fixed. Let $Q^*$ be an equivalent $f$-minimal martingale measure which satisfies

$$\mathbb{E}_P[f(\lambda \frac{dQ^*_T}{dP_T})] < \infty, \quad \mathbb{E}_{Q^*}[|f'(\lambda \frac{dQ^*_T}{dP_T})|] < \infty$$

with $\lambda$ such that

$$\mathbb{E}_{Q^*}[f'(\lambda \frac{dQ^*_T}{dP_T})] = x.$$
Then
\[ -f'(\lambda \frac{dQ^*}{dP_T}) = x + \int_0^T \phi_u dS_u \]

where \( \phi \) is predictable function such that \((\int_0^* \phi_u dS_u)\) is \(Q^*\)-martingale. If the last relation holds, then \( \Phi = (\phi^0, \phi) \) with \( \phi^0_t = x + \int_0^t \phi_u dS_u - \phi_t S_t \) is an admissible optimal minimax portfolio strategy.
Let $P$ be the physical measure and $Q$ be an equivalent martingale measure obtained by minimisation of some $f$-divergence (or equivalently by maximisation of the corresponding utility function) and which preserves the Levy property of $L$.

Let $\zeta = (\zeta_t)_{t \geq 0}$ be the Radon-Nikodym density process of $Q$ with respect to $P$:

$$\zeta_t = \frac{dQ_t}{dP_t}.$$

Let $u$ be a utility function and $f$ its convex conjugate.

We set

$$q(t, x) = \mathbb{E}_P[\zeta_{T-t}^2 f''(x \lambda \zeta_{T-t})].$$
Optimal strategies in exponential Levy models

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Optimal strategies result

**Theorem**

We suppose that $f$ is a strictly convex function from $C^3(\mathbb{R}^+,*)$ such that the $f$-minimal martingale measure $Q$ exists and preserves the Levy property of $L$. We also suppose that uniformly on the compact sets of $\lambda > 0$ and uniformly in $0 \leq t \leq T$,

\[ \mathbb{E}_P|f(\lambda \zeta_T)| < \infty, \quad \mathbb{E}_P \left[ |\zeta_t f'(\lambda \zeta_t)| \right] < \infty, \quad \mathbb{E}_P \left[ (\zeta_t)^2 f''(\lambda \zeta_t) \right] < \infty \]
Then, for an initial capital $\rho(0, 1) = \mathbb{E}_Q f(\lambda \zeta_T)$ and a utility function $u$, the optimal strategy exists and is given by

$$\phi_t = -\frac{\lambda \beta \zeta_t}{S_t} q(t, \zeta_t)$$

If $f''(x) = ax^\gamma$ with $a > 0$ then

$$\phi_t = -\frac{a\beta \lambda^{\gamma+1} \zeta_t}{S_t} \mathbb{E}_P [\zeta_{T-t}^{\gamma+2}]$$
Hypothesis and notations

We introduce the following hypotheses:

- (H1): The $f$-divergence minimal equivalent martingale measures $Q^*$ and $\tilde{Q}^*$ relative to $L$ and $\tilde{L}$ exist.

- (H2): The $f$-divergence minimal equivalent martingale measures $Q^*$ and $\tilde{Q}^*$ preserve the Levy property and are invariant under scaling.

- (H3): For all $c > 0$ and $t \in [0, T]$, we have: $\mathbb{E}_Q |f'(c \zeta^*_t)| < \infty$, $\mathbb{E}_{\tilde{Q}} |f'(c \tilde{\zeta}^*_t)| < \infty$ where $\zeta^*$ and $\tilde{\zeta}^*$ are the densities of the $f$-minimal equivalent martingale measures $Q^*$ and $\tilde{Q}^*$ with respect to $P$ and $\tilde{P}$ respectively.
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  \[ E_Q | f'(c \, \zeta^*_t) | < \infty, \quad E_{\tilde{Q}} | f'(c \, \tilde{\zeta}^*_t) | < \infty \]
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The $f$-divergence minimal martingale measures

Optimal strategies for utility maximisation

Optimal strategies for utility maximisation in exponential Levy models

The $f$-divergence minimal martingale measures in change-point situation.

Optimal strategies for utility maximisation in change-point situation.

**Theorem**

Assume that $f$ is a strictly convex function, $f \in C^1(\mathbb{R}^+,\ast)$, and that (H1), (H2), (H3) hold. Then if the $f$-minimal martingale measure $Q^*$ for the change-point model exists, it has the following structure:

$$
\frac{dQ_T^*}{dP_T} = c(\tau) z_T^*(\tau)
$$

where $c(\cdot)$ is a measurable function $[0, T] \rightarrow \mathbb{R}^+$ such that $E c(\tau) = 1$. 

We set for $t \in [0, T]$

$$
z_T^*(t) = \zeta_T^* \frac{\tilde{\zeta}_T^*}{\tilde{\zeta}_T^*}
$$

Lioudmila Vostrikova

talk of 3 june 2010 on Workshop "Enlargement of Filtrations and Applications to Finance and Insurance, Jena, Germany"
$f$ divergence MME

For $c > 0$, let

$$\lambda_t(c) = \mathbb{E}[f'(c z_T^*(t)) z_T^*(t)]$$

where the expectation is taken with respect to $\mathbb{P}$ and $c_t(\lambda)$ is its right-continuous inverse.

Then, if there exists $\lambda^*$ such that

$$\int_0^T c_t(\lambda^*) d\alpha(t) = 1$$

the $f$- minimal equivalent measure for a change-point situation exists and is given by the previous expression with $c^*(t) = c_t(\lambda^*)$ for $t \in [0, T]$. 
In particular, if \( f'(x) = ax^\gamma \), for \( a > 0 \) and \( \gamma \in \mathbb{R} \), then

\[
c^*(t) = \frac{\left[ \mathbb{E}(z_T^*(t)^{\gamma+1}) \right]^{-\frac{1}{\gamma}}}{\int_0^T \left[ \mathbb{E}(z_T^*(t)^{\gamma+1}) \right]^{-\frac{1}{\gamma}} d\alpha(t)}
\]

and for \( f'(x) = \ln(x) \),

\[
c^*(t) = \frac{e^{-\mathbb{E}(z_T^*(t) \ln z_T^*(t))}}{\int_0^T e^{-\mathbb{E}(z_T^*(t) \ln z_T^*(t))} d\alpha(t)}.
\]
A change-point Black-Scholes model

- \( L \) and \( \tilde{L} \) are continuous Levy processes with characteristics \((b, c, 0)\) and \((\tilde{b}, c, 0)\) respectively.

- Initial models are complete, with a unique equivalent martingale measure which defines a unique price for options.

- In our change-point situation the martingale measure is not unique, but we have an infinite set of martingale measures of the form

\[
\frac{dQ_t}{dP_t}(X) = c(\tau) \exp \left( \int_0^t \beta_s dX_s^c - \frac{1}{2} \int_0^t \beta_s^2 c ds \right)
\]

with \( E[c(\tau)] = 1 \) and

\[
\beta_s = -\frac{1}{c} \left[ (b + c/2) I_{[0,\tau]}(s) + (\tilde{b} + c/2) I_{[\tau,\infty]}(s) \right]
\]
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A change-point Black-Scholes model

1. If for example \( f'(x) = ax^\gamma \), applying Theorem, we get

\[
c^*(t) = \frac{e^{-\gamma+1 \over 2c}((b\gamma/c)^2t+(\bar{b}+c/2)^2(T-t))}{\int_0^T e^{-\gamma+1 \over 2c}((b\gamma/c)^2t+(\bar{b}+c/2)^2(T-t)) \, d\alpha(t)}
\]

2. If \( f'(x) = \ln(x) \),

\[
c^*(t) = \frac{e^{-1 \over 2c}((b\gamma/c)^2t+(\bar{b}+c/2)^2(T-t))}{\int_0^T e^{-1 \over 2c}((b\gamma/c)^2t+(\bar{b}+c/2)^2(T-t)) \, d\alpha(t)}
\]
A change-point Black-Scholes model

- If for example $f'(x) = ax^\gamma$, applying Theorem, we get

$$c^*(t) = \frac{e^{-\frac{\gamma+1}{2c}((b+c/2)^2t+(\tilde{b}+c/2)^2(T-t))}}{\int_0^Te^{-\frac{\gamma+1}{2c}((b+c/2)^2t+(\tilde{b}+c/2)^2(T-t))}d\alpha(t)}$$

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Optimal strategies in a change-point situation

- We denote by \((\beta, Y)\) and \((\tilde{\beta}, \tilde{Y})\) the Girsanov parameters corresponding to the changes of measure from \(P\) and \(\tilde{P}\) to the \(f\)-divergence minimal measures \(Q^*\) and \(\tilde{Q}^*\) respectively.

- Then the first Girsanov parameter for the change of the measure \(P\) to \(Q^*\) will be:

  \[
  \beta_t = \beta I_{[0,\tau]}(t) + \tilde{\beta} I_{\tau, +\infty}(t)
  \]

- For \(0 \leq t \leq T\) we denote

  \[
  z^*_t = \zeta^*_t I_{[0,\tau]}(t) + \zeta^*_\tau \tilde{\zeta}^*_\tau I_{\tau, +\infty}(t)
  \]

where \(\zeta^*, \tilde{\zeta}^*\) are the densities of \(Q^*\) and \(\tilde{Q}^*\) with respect to \(P\) and \(\tilde{P}\).
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\]

- For \(0 \leq t \leq T\) we denote

\[
z_t^* = \zeta_t^* I_{[0,\tau]}(t) + \tilde{\zeta}_t^* \frac{\tilde{\zeta}_T}{\zeta_T} I_{\tau, +\infty}(t)
\]

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Optimal strategies in a change-point situation

- The Radon-Nikodym derivative of $\mathbb{Q}^*$ with respect to $\mathbb{P}$ is

$$Z_T^*(\tau) = c^*(\tau)z_T^*$$

where $c^*(\tau)$ is defined in Theorem 1 and

- We denote for $0 \leq v \leq T$

$$q^{(v)}(t, x) = \mathbb{E}[(z_{T-t}^*((q - t)^+))^2f''(x\lambda z_{T-t}^*((q - t)^+))]$$
Optimal strategies in a change-point situation

- The Radon-Nikodym derivative of $Q^*$ with respect to $P$ is

$$Z^*_T(\tau) = c^*(\tau)z^*_T$$

where $c^*(\tau)$ is defined in Theorem 1 and

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$$q^{(\nu)}(t, x) = \mathbb{E}[(z^*_T-t((q-t)^+))^2f''(x\lambda z^*_T-t((q-t)^+))]$$
Optimal strategies in a change-point situation

Theorem

Let $f$ be a three times continuously differentiable strictly convex function satisfying (H1), (H2), (H3) and mentioned integrability conditions for $Q$ and $\tilde{Q}$. Then there exists an $F$-optimal strategy $\phi^*$ for our change-point model. In addition, it is $\hat{F}$-adapted, and

$$
\phi_t^* = -\lambda \beta_t Z_{t-}^*(\tau) S_{t-} q^{(\tau)}(t, Z_{t-}^*(\tau))
$$

In particular, when $f''(x) = ax^\gamma$ with $a > 0$, we have:

$$
\phi_t^* = \frac{a \lambda^{\gamma+1} \beta_t Z_{t-}^*(\tau)}{S_{t-}} \mathbb{E}([Z_{T-t}^* ((q - t)^+) \gamma + 2)|q = \tau]
$$