Default times with given survival probability and their semimartingale decomposition formula in the progressively enlarged filtration

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Abstract. In credit risk modelling the basic ingredients are a probability space \((\Omega, \mathcal{A}, Q)\), a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\), a random time \(\tau\), and a \(\mathbb{F}\)-predictable increasing process \(\Lambda\). We require these ingredients to be displayed so that \(\mathbb{1}_{\{\tau \leq t\}} - \Lambda_t\) is a \(\mathcal{G}\)-local martingale, where \(\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}\) with \(\mathcal{G}_t = \sigma(\tau \wedge t) \vee \mathcal{F}_t\).

We consider here the existence problem. We know that, for \(\mathbb{1}_{\{\tau \leq t\}} - \Lambda_t \wedge \tau\) to be a \(\mathcal{G}\)-local martingale, a necessary and sufficient condition is that the multiplicative decomposition of \(Q[\tau > t | \mathcal{F}_t]\) is \(N_t e^{-\Lambda_t}\), where \(N\) is a positive \(\mathbb{F}\)-local martingale. We suppose given the filtration \(\mathbb{F}\), the processes \(N\) and \(\Lambda\), as well as a probability \(\mathbb{P}\) on \(\mathcal{F}_\infty\).

Can we then construct a probability \(Q\) and a random time \(\tau\) such that \(Q|_{\mathcal{F}_\infty} = \mathbb{P}\) and \(Q[\tau > t | \mathcal{F}_t] = N_t e^{-\Lambda_t}\)? In this talk, we present a method of such a construction. The method is based on a characterization of the problem by increasing families of positive martingales and the solutions can be obtained through a system of stochastic differential equations. We note that for a given \(N_t e^{-\Lambda_t}\), multiple solutions may be constructed by that method.

A second question we study is \(\mathbb{F}\)-martingales’ behaviour in the filtration \(\mathcal{G}\) (mainly : are the \(\mathbb{F}\)-martingales \(\mathcal{G}\)-semimartingales?). That question has always an answer before the default time \(\tau\). But the answer after the default time \(\tau\) depends on models. We show that, for the models we construct in the first part of the talk, the so-called \((\mathcal{H'})\) hypothesis holds and we give the semimartingale decomposition formula.
Outline of the talk

$$\mathcal{M}\text{-problem} \iff iM_Z \iff \begin{cases} \text{SDE(изм)} \\ \text{balayage} \end{cases} \Rightarrow \text{SDE with differentiable coefficients} \iff \text{semimartingale decomposition formula (SDF)}$$

↑ ↑ ↑ ↑
key notion effective common absolutely
for the construction property continuous
solution of the iM_Z of our iM_Z kernel method

Pour la mémoire, je dois écrire au tableau, après la présentation de chaque section, quelques mots qui représentent son contenu.
1 Introduction

1.1 The problem

Our problem can be described as follows

\textbf{M-problem} Let be given a local martingale \( N \) and an increasing process \( \Lambda \) such that \( 0 \leq Ne^{-\Lambda} \leq 1 \). We want to find all probability measures \( Q \) (eventually defined on some enlarged space) and random times \( \tau \) such that

1. (restriction condition) \( Q|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty} \) (we shall call \( Q \) an extension of \( \mathbb{P}|_{\mathcal{F}_\infty} \))
2. (projection condition) \( Q[\tau > t|\mathcal{F}_t] = Ne^{-\Lambda_t} \)

This formulation of the problem is natural. But it is not without ambiguity. In our writing paper we take in fact a different formulation of the problem. However, the above description of the problem is easier to understand and explains better the main ideas.

1.2 Precise formulation of the problem

\textbf{Initial data} The given data are the following:

- a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\),
- an increasing \( \mathbb{F} \)-predictable process \( \Lambda \),
- a \( \mathbb{P} \)-\( \mathbb{F} \) positive local martingale \( N \) such that \( 0 \leq N_t e^{-\Lambda_t} \leq 1, \forall 0 \leq t < \infty \).

\textbf{Convention on "positive" and "increasing".}

\textbf{Extension of a filtered probability space} Instead of the restriction condition, we introduce the notion of extension. A filtered probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{Q}, \pi)\) is an extension of \((\Omega, \mathcal{F}, \mathbb{P})\), if

- \( \pi : \hat{\Omega} \rightarrow \Omega \) a map exists
- \( \hat{\mathcal{F}}_t = \pi^{-1}(\mathcal{F}_t), \forall 0 \leq t \leq \infty \)
- \( \hat{\mathbb{P}} = Q|_{\hat{\mathcal{F}}_\infty} \circ \pi^{-1} \).

\textbf{Convention} : Identification of elements on the extension \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{Q}, \pi)\) with their original counterparts on \((\Omega, \mathcal{F}, \mathbb{P})\).
such that $Q[\tau > t|\mathcal{F}_t] = N_t e^{-\Lambda t}$ for all $0 \leq t < \infty$. □

Note that the problem is a problem in law. So, if a solution to the $M$-problem existe, we can always transfer it on the product space $[0, \infty] \times \Omega$. On this space, we take always $\pi$ the project onto $\Omega$ and $\tau$ the projection onto $[0, \infty]$.

### 1.3 A short comment

The $M$-problem means Model Existence Problem. Le problème a été soulevé par Monique Jeanblanc et mis en avant avec force. Elle avait notamment trouvé la première solution à cette question délicate. Aujourd'hui on peut dire qu'elle avait raison de faire cela, car on constate maintenant que le $M$-problème engendre de nouveaux résultats sur les calculs des (sur)martingales ou sur le problème du grossissement des filtrations. Il apporte un nouveau regard sur les modèles des défauts, en particulier l’absence de l’unicité (qu’on va présenter ici) découverte à cause du $M$-problem.

### 1.4 What’s easy, what’s not easy

An almost good idea to solve the $M$-problem is the following remark. Any extension $Q$ of $P$ can be disintegrated into $P|\mathcal{F}_\infty$ and the conditional law $Q[\tau \in du|\mathcal{F}_\infty]$. Regarding $Q[\tau \in du|\mathcal{F}_\infty]$ as the terminal term of the martingale $M_t = Q[\tau \in du|\mathcal{F}_t]$, we hope to relate $M$ to the conditional survival probability $Z = Ne^{-\Lambda}$ through the projection condition $Q[\tau > t|\mathcal{F}_t] = Z_t$. The process $Z$ is the candidat to be the Azéma’s supermartingale of the random time $\tau$ once $\tau$ is constructed.

On the one hand,

$$Q[\tau > u|\mathcal{F}_t] = Q[Z_u|\mathcal{F}_t] = Q[Z_\infty|\mathcal{F}_t] + Q[\int_u^\infty Z_s d\Lambda_s|\mathcal{F}_t], \quad 0 \leq t \leq u$$

This yields

$$Q[\tau \in du, t \leq \tau \leq \infty|\mathcal{F}_t] = Q[Z_u d\Lambda_u 1_{\{t \leq u < \infty\}} + Z_\infty \delta_\infty(du)|\mathcal{F}_t]$$

On the other hand, however, one note quickly that no determining relation can be found between $Z$ and

$$Q[\tau \in du, 0 \leq \tau \leq t|\mathcal{F}_t]$$
We can have another natural idea to solve the problem. In fact, the projection condition recall straightforwardly the Follmer’s measure (see Meyer [9], it is the Doléans-Dade measure in our case of a class (D) supermartingale). We wonder if the solution of the $M$-problem is not simply the Follmer’s measure. To simplify the formula, we suppose $Z_{\infty} = 0$. The Follmer’s measure in this case is given by

$$
Q^F[F] = \mathbb{P}[\int_0^\infty F(s, \cdot)Z_s d\Lambda_s]
$$

This measure $Q^F$ satisfies the projection condition. In order to be a solution, $Q^F$ must satisfy also the restriction condition. The restriction of $Q^F$ onto $\mathcal{F}_\infty$ is calculated by

$$
Q^F[A] = \mathbb{P}[\mathbb{1}_A \int_0^\infty Z_s d\Lambda_s], \ A \in \mathcal{F}_\infty.
$$

Therefore, to be a solution, $\int_0^\infty Z_s d\Lambda_s$ must be equal to 1. And if it is the case the Follmer’s measure $Q^F$ is simply the Cox measure. The Follmer’s measure solves the problem only when $N = 1$. (Note that this situation is related to the pseudo-times. See Nikeghbali [10] for a complete discussion on the subject)

In fact the notion of Follmer’s measure is pertinent only on the $\mathbb{F}$-predictable $\sigma$-field. We notice that any solution of the $M$-problem coincide with the Follmer’s measure on that predictable $\sigma$-field. The difficulty of the problem is how to extend the Follmer’s measure onto the whole $\sigma$-field $\mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ so that the restriction condition will be satisfied.

### 1.5 What have been done since then

Two new ideas have been discovered since then and two solutions of the $M$-problem have been found:

1. Gapeev P.V., Jeanblanc M., Li L., Rutkowski M. [3]. The result is based on the idea:
   - Looking for solutions $Q$ among the probabilities absolutely continuous with respect to Cox model $\nu$ on $[0, \infty] \times \Omega$ associated with $\Lambda$.
   - Equation ($L$): If
     
     $$
     L_t = \left. \frac{dQ}{d\nu} \right|_{\mathcal{G}_t}, \ 0 \leq t \leq \infty
     $$

     $L$ satisfies the equation

     $$(L): \ N_t e^{-\Lambda t} + \int_0^t L_u(s, \cdot) e^{-\Lambda s} d\Lambda_s = 1, \ 0 \leq t \leq \infty.$$
II. Jeanblanc M., Song S. [5]. The result is based on the proportionality property: we look for the solution such that the map

\[ b \rightarrow \frac{Q[\tau \in du|\mathcal{F}_b]}{Q[\tau \leq b|\mathcal{F}_b]} \]

is a decreasing function on \( b \in [u, \infty) \), or equivalently such that the invariance property holds

\[ \frac{Q[\tau \in A|\mathcal{F}_a]}{Q[\tau \in B|\mathcal{F}_a]} = \frac{Q[\tau \in A|\mathcal{F}_b]}{Q[\tau \in B|\mathcal{F}_b]} \]
2 Family i$M_Z$

2.1 Why a new study

After the first success in solving the $M$-problem, we had nevertheless been annoyed by three questions:

i. What is the precise relation between the supermartingale $Z$ and conditional law $Q[\tau \in du, 0 \leq \tau \leq t|\mathcal{F}_t]$?

(Answer: a soft relation: $M_u^u = (1 - Z_u)$ and $M_t^u \leq 1 - Z_t$ are the only general relations)

ii. Why the proportionality has enabled us to find a solution?

(It corresponds to the simplest i$M_Z$)

iii. Is the solution unique? (No)

(We put aside the question on the equation $(L)$ which is a question of different nature.)

2.2 Because it's the problem, make it a notion

The main difficulty is the few relation between $Z$ and the family of conditional probabilities

\[ Q[\tau \leq u|\mathcal{F}_t], 0 \leq u \leq t < \infty \]

We exhibit easily the properties of this family, that we incorporate into the following notion

An increasing family of positive martingales precisely bounded by $1 - Z$ (in short i$M_Z$) is a family of processes $(M^u : 0 \leq u < \infty)$ satisfying the following conditions:

1. Each $M^u$ is a càdlàg $\mathbb{P}$-$\mathbb{F}$ martingale on $[u, \infty]$.
2. For any $u$, the martingale $M^u$ is positive and closed by $M^u_\infty = \lim_{t \to \infty} M^u_t$.
3. For each fixed $t$, $0 < t \leq \infty$, $u \in [0, t] \to M^u_t$ is a right continuous increasing map.
4. $M^u_u = 1 - Z_u$ and $M^u_t \leq 1 - Z_t$ for $u \leq t \leq \infty$.

2.3 Is i$M_Z$ an useful notion

What is pleasant is that these properties are exactly what we need to construct a solution of the $M$-problem. Given an i$M_Z$, let us denote by $d_u M^u_\infty$ the random measure on $(0, \infty)$ associated with the
increasing map $u \to M^u_\infty$. Define a probability measure $Q$ on $([0, \infty] \times \Omega, \mathcal{B} \otimes \mathcal{F}_\infty)$ by

$$Q[F] := \mathbb{P}\left[ \int_{[0, \infty]} F(u, \cdot) (M^0_\infty \delta_0(du) + d_u M^u_\infty + (1 - M^\infty_\infty)\delta_\infty(du)) \right]$$

Two properties for $Q$:

- (restriction condition) For $B \in \mathcal{F}_\infty$,

$$Q[B] = \mathbb{P}[\mathbb{I}_B \int_{[0, \infty]} (M^0_\infty \delta_0(dt) + d_t M^t_\infty + (1 - M^\infty_\infty)\delta_\infty(dt))] = \mathbb{P}[B]$$

- (projection condition) For $0 < t < \infty$, $A \in \mathcal{F}_t$,

$$Q[A \cap \{\tau \leq t\}] = \mathbb{P}[\mathbb{I}_A M^\tau_\infty] = \mathbb{P}[\mathbb{I}_A M^t_t] = Q[\mathbb{I}_A (1 - Z_t)]$$

$Q$ is effectively a solution of the $\mathcal{M}$-problem.

Notice that the advantage to consider an unknown $iM_Z$ instead of an unknown $Q$ is that $iM_Z$ is a process which can be worked on the initial space $(\Omega, \mathcal{F}, \mathbb{P})$, while $Q$ is probability on an unknown space.

**Theorem 2.1** A solution of the $\mathcal{M}$-problem exists if and only if an $iM_Z$ exists.
3 Constructions of $iM_Z$

The notion $iM_Z$ characterizes the solvability of the $M$-problem. However, the $M$-problem remains open as long as we have not constructed concretely an $iM_Z$.

3.1 A collection of hypotheses

We will need different hypotheses at different places.

Hypothesis $Hy(B)$: $Z_0 = 1$ and $\Lambda$ is continuous.

Hypothesis $Hy(N, \Lambda)$

1. For all $0 < t < \infty$, $0 \leq Z_t < 1$, $0 \leq Z_{t-} < 1$ (strictly smaller than 1).
2. Almost surely, the random measure $\frac{d\Lambda_t}{1-Z_t} = \frac{d\Lambda_t}{1-Z_{t-}}$ is finite on any finite interval contained in $(0, \infty)$.

Hypothesis $Hy(C)$: All $\mathbb{P}$-$\mathbb{F}$ martingales are continuous.

Hypothesis $Hy(I)$ $\int_0^t \frac{e^{-2\Lambda_s}}{(1-Z_{s-})^2} d\langle N \rangle_s < \infty$ for all $0 < t < \infty$.

We introduce $H = \{s : 1 - Z_s = 0\}$ and the random times

$$g_t := \sup\{0 \leq s \leq t : s \in H\}, 0 < t < \infty,$$

$$d_t := \inf\{s > t : s \in H\}, 0 < t < \infty,$$

$$g := \lim_{t \to \infty} g_t$$

Hypothesis $Hy(H)$ The set $H$ is not empty and is closed. The measure $d\Lambda$ has a decomposition $d\Lambda_s = dV_s + dA_s$ where $V, A$ are continuous increasing processes such that $dV$ charges only $H$ while $dA$ charges its complementary $H^c$. Moreover, we suppose

$$\mathbb{I}_{\{g_t \leq u < t\}} \int_u^t \frac{Z_s}{1-Z_s} dA_s \left( = \mathbb{I}_{\{u < t < d_u\}} \int_u^t \frac{Z_s}{1-Z_s} dA_s \right) < \infty$$

for any $0 < u < t < \infty$. ■
3.2 The simplest $iM_Z$

**Theorem 3.1** Assume $Hy(B)$ and $Hy(N, \Lambda)$. The family

$$B_t^u = (1 - Z_t) \exp \left( - \int_u^t \frac{Z_s d\Lambda_s}{1 - Z_s} \right) \quad 0 \leq u < \infty, u \leq t \leq \infty,$$

defines an $iM_Z$.

It consists simply to counterbalance the increasing factor in the multiplicative decomposition of $1 - Z$.

It can be easily seen if the martingale $N$ is continuous. We can then write

$$(1 - Z_t) = (1 - Z_u) e^{\int_u^t \frac{Z_s d\Lambda_s}{1 - Z_s}},$$

and

$$B_t^u = (1 - Z_u) e^{\int_u^t \frac{Z_s d\Lambda_s}{1 - Z_s}}, \quad 0 \leq u \leq t < \infty.$$

3.3 An idea for more $iM_Z$

- To illustrate the situation, Gapeev-Jeanblanc-Li-Rutkowski [3] gives a very concrete example:

$$dZ_t = -\lambda Z_t dt + \frac{b}{2} Z_t (1 - Z_t) dW_t$$

$$M_t^u = Q[\tau \leq u | \mathcal{F}_t] = \frac{X_t}{X_u} - X_t Y_u e^{-\lambda u} = (\frac{1}{X_u} - Y_u e^{-\lambda u}) X_t$$

$$dX_t = -\frac{b}{2} Z_t X_t dW_t$$

$$dY_t = e^{\lambda t} d\left( \frac{1}{X_t} \right)$$

We see clearly an $iM_Z$. We note that the $M^u$ satisfies a SDE on $[u, \infty)$.

- In Jeanblanc-Song [5] also, we have also an $iM_Z$ satisfying a SDE:

$$dM_t^u = -(1 - Z_u) M_t^u e^{-\lambda t} \frac{e^{-\lambda t}}{1 - Z_t} dN_t, \quad 0 \leq u \leq t < \infty.$$

(putting aside the integrability question)

- Let us recall that, to construct an $iM_Z$, we should respect four constraints:
i. $M_u = (1 - Z_u)$

ii. $0 \leq M_u$

iii. $M_u \leq 1 - Z$

iv. $M_u \leq M^v$ for $u < v$

What we remark is that these constraints are particularly "easy" to handle if $M^u$ are solutions of a SDE: The constraint i indicates the initial condition; the constraint ii means that we must take an exponential SDE; the constraint iv is a comparison theorem for one dimensional SDE, whiles the constraint iii can be handled by local time as described in the following result:

**Lemma 3.1** Let $M$ be a $\mathbb{P}$-$\mathbb{F}$ local martingale on $[u, \infty)$ such that $M_u \leq 1 - Z_u$. Then, $M_t \leq (1 - Z_t)$ on $t \in [u, \infty)$ if and only if the local time at zero of $M - (1 - Z)$ on $[u, \infty)$ is identically null. 

Conclusion: our $iM_Z$ will be constructed through a SDE. This SDE should be chosen so that the above results are applicables.

### 3.4 Generating equation when $1 - Z > 0$

Here is a SDE to generate $iM_Z$. We assume $\text{Hy}(B)$, $\text{Hy}(N, \Lambda)$, $\text{Hy}(C)$ and $\text{Hy}(I)$. Let $Y$ be a $\mathbb{P}$-$\mathbb{F}$ local martingale and $f$ be a bounded Lipschitz function with $f(0) = 0$. For any $0 \leq u < \infty$, we consider the equation

$$
\begin{cases}
  dM_t = M_t \left( -\frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(M_t - (1 - Z_t))dY_t \right), & u \leq t < \infty \\
  M_u = x
\end{cases}
$$

**Theorem 3.2** Let $M^u$ be the solution on $[u, \infty)$ of the equation(iii) with initial condition $M^u_u = 1 - Z_u$. Then, $(M^u, 0 \leq u < \infty)$ defines an $iM_Z$. 

3.5 A remark

Our method remains valid if in SDE(\(\natural\)) \(dM_t = M_t \left( -\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + f(M_t - (1 - Z_t))dY_t \right)\), the term \(f(M_t - (1 - Z_t))\) is replaced by some more general function \(f(M_t - (1 - Z_t), M_t, t, \omega)\) such that

\[
|f(M_t - (1 - Z_t), M_t, t, \omega)| \leq K|M_t - (1 - Z_t)|
\]

or if some extra term \(g(M_t - (1 - Z_t), M_t, t, \omega)dU_t\) is added. We shall not work this generality in this talk, because otherwise we have to manage long computation expressions obscuring the essential factors.

3.6 Proof indication

We look at two factors. Firstly, let us look at the inequality \(M^u \leq 1 - Z\) on \([u, \infty)\). According to Lemma 3.1, we need only to prove that the local time of \(\Delta = M^u - (1 - Z)\) at zero is null. This will be the consequence of the following estimation:

\[
d\langle \Delta \rangle_t = \Delta_t^2 \left( \frac{e^{-\Lambda_t}}{1-Z_t} \right)^2 d\langle N \rangle_t + M_t^2 f^2(\Delta_t)d\langle Y \rangle_t - 2\Delta_t \frac{e^{-\Lambda_t}}{1-Z_t} M_t f(\Delta_t)d\langle N,Y \rangle_t
\]

\[
\leq 2\Delta_t^2 \left( \frac{e^{-\Lambda_t}}{1-Z_t} \right)^2 d\langle N \rangle_t + 2M_t^2 f^2(\Delta_t)d\langle Y \rangle_t
\]

\[
\leq 2\Delta_t^2 \left( \frac{e^{-\Lambda_t}}{1-Z_t} \right)^2 d\langle N \rangle_t + 2M_t^2 K^2 \Delta_t^2 d\langle Y \rangle_t
\]

From this, we can write

\[
\int_0^t \mathbb{I}_{\{0<\Delta_s<\epsilon\}} \frac{1}{\Delta_s^2} d\langle \Delta \rangle_s < \infty, \ 0 < \epsilon, 0 < t < \infty.
\]

and get the result according to Revuz-Yor[11]

Secondly we look at the inequality \(M^u \leq M^v\) on \([v, \infty)\) when \(u < v\). To see this we note that the comparison theorem holds for SDE(\(\natural\)). We note also that \(M^u\) and \(M^v\) satisfy the same SDE(\(\natural\)) on \([v, \infty)\). So, since \(M^u_t \leq (1 - Z_v) = M^v_v\), \(M^u_t \leq M^v_t\) for all \(t \in [v, \infty)\).

3.7 Balayage formula when \(1 - Z\) can take zero

In this case \(Hy(N, \Lambda)\) is invalid. We need new methods to construct \(iM_Z\). It is to use the balayage formula (see Revuz-Yor[11]).

We suppose \(Hy(B)\) and \(Hy(H)\). Recall that the measure \(d\Lambda\) has a decomposition \(d\Lambda_s = dV_s + dA_s\)
where \( V, A \) are continuous increasing processes such that \( dV \) charges only \( H = \{1 - Z = 0\} \) while \( dA \) charges its complementary \( H^c \). We introduce

\[
E_t^u = \exp\left(- \int_u^t \frac{Z_s}{1 - Z_s} dA_s\right)
\]

\[
M_t^u = \mathbb{I}_{\{g_t \leq u\}} E_t^u (1 - Z_t), \quad 0 < u < \infty, u \leq t \leq \infty.
\]

**Theorem 3.3** The family \((M^u : 0 \leq u < \infty)\) defines an \(iM_Z\). Moreover, we have the equation, for \(0 < u \leq t \leq \infty\)

\[
M_t^u = (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \leq u\}} E_s^u e^{-\Lambda_s} dN_s
\]

### 3.8 Proof indication

We need only to prove that each \(M^u\) satisfies the above equation, and therefore, that \(M^u\) is a local \(\mathbb{P}-\mathbb{F}\) martingale. We begin with, for \(0 < u \leq t < d_u\)

\[
d_t (E_t^u (1 - Z_t)) = E_t^u (-e^{-\Lambda_t} dN_t + Z_t dV_t)
\]

We apply the balayage formula (see [11]) and we obtain, for \(u \leq b < \infty\),

\[
M_t^u = \mathbb{I}_{\{g_t \leq u\}} E_t^u (1 - Z_t)
\]

\[= \mathbb{I}_{\{g_t \leq u\}} \left((1 - Z_u) - \int_u^t E_s^u e^{-\Lambda_s} dN_s + \int_u^t E_s^u Z_s dV_s\right)
\]

\[= \mathbb{I}_{\{g_u \leq u\}} (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \leq u\}} E_s^u e^{-\Lambda_s} dN_s + \int_u^t \mathbb{I}_{\{g_s \leq u\}} E_s^u Z_s dV_s
\]

\[= (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \leq u\}} E_s^u e^{-\Lambda_s} dN_s
\]

### 3.9 Solution of \(\mathcal{M}\)-problem if \(1 - Z\) can be zero

We recall that the goal is to construct solutions of the \(\mathcal{M}\)-problem. According to Theorem 2.1, each time we have an \(iM_Z\) we have a solution automatically. But in the case where \(1 - Z\) can take zero, more can be said.

Let us define a probability measure \(Q\) on \([0, \infty] \times \Omega\) by the following relations: for \(A \in \mathcal{F}_\infty, 0 \leq u < \infty\),

\[Q[A \cap \{\tau \leq u\}] = \mathbb{P} \left[ \mathbb{I}_A \exp\{- \int_u^\infty \frac{Z_s}{1 - Z_s} dA_s\} (1 - Z_\infty)\right],
\]

\[Q[A \cap \{\tau = \infty\}] = \mathbb{P} \left[ \mathbb{I}_A (1 - Z_\infty)\right].\]
(recall that \( \tau(s, \omega) = s \) is the projection form \([0, \infty] \times \Omega \) onto \([0, \infty] \). )

**Theorem 3.4** \(([0, \infty] \times \Omega, \mathbb{Q}, g \lor \tau) \) is a solution of the \( \mathcal{M} \)-problem. ■

We note that in this solution, the factor of \( 1 - Z = 0 \) is represented by \( g \) and the factor of \( 1 - Z > 0 \) is represented by \( \tau \), and the two factors are organized simply by a \( \lor \).
Recall that the $\mathcal{M}$-problem is considered as a credit risk modelling problem. After having constructed different solutions of the $\mathcal{M}$-problem, we need to study their properties. Here we study in particular the enlargement of filtration problem.

### 4.1 What we can say in general

- $\mathcal{G}$ is a progressive enlargement of $\mathcal{F}$.
- The $\mathcal{F}$-local martingales remain always $\mathcal{G}$-semimartingales on the interval $[0, \tau]$, whose semimartingale decomposition formula is given in Jeulin [6].
- The $\mathcal{F}$-local martingales' behaviours on the interval $[\tau, \infty)$ in the filtration $\mathcal{G}$ depend on the underground model.

The best known examples to deal with $\mathcal{F}$-local martingales on $[\tau, \infty)$ are that of honest time (see ex. [2, 8]) and that of initial time (see ex. [4]). These results are based on, for the first one, the particular structure of the $\mathcal{G}$-predictable $\sigma$-field and the computations of the dual projections, and for the second one, the density hypothesis.

We can not use directly these results.

### 4.2 What have we done, what's the result

We have tried to computed directly the expectations

$$Q[1_A 1_{\{\tau \leq u\}}(X_{\tau \land t} - X_{\tau \land s})], A \in \mathcal{F}_s, 0 \leq u < \infty.$$

It happens that, for our models, the computation is possible. Thinks to this computation we prove that, for our models, the hypothesis ($H'$) holds between $\mathcal{F}$ and $\mathcal{G}$ and we obtain semimartingale decomposition formula.
4.3 Why it’s possible

But what makes this computational possibility? The reason is that the $iM_Z$ families in our models are all the solutions of SDE and these SDE have differentiable coefficients. See below for a concrete illustration.

4.4 Filtration $\mathcal{G}$ or filtration $\mathcal{G}_+$

Before going further, let us stop a moment at this question: why we consider the filtration $\mathcal{G}$, whilst normally we should consider the filtration $\mathcal{G}_+$. Here is the reason. In our study, the $\mathbb{F}$-local martingales $X$ have a $\mathcal{G}$-semimartingale decomposition $X = \overline{X} + \chi$, where $\overline{X}$ is a càdlàg $\mathcal{G}$-local martingale localizable with $\mathbb{F}$-stopping times, and $\chi$ is a càdlàg $\mathcal{G}$-predictable process with locally integrable bounded variations localizable with $\mathbb{F}$-stopping times. These elements yield that $X = \overline{X} + \chi$ is also the semimartingale decomposition in the filtration $\mathcal{G}_+$.

4.5 Semimartingale decomposition formula for the models constructed with SDE($\sharp$), the case of $1 - Z > 0$

In this subsection we suppose $\text{Hy}(B)$ and $\text{Hy}(N, \Lambda)$ and $\text{Hy}(C)$ and $\text{Hy}(I)$. We suppose $Z_\infty = 0$. We consider a generating equation ($\sharp_u$):

\[
\left\{ \begin{array}{l}
\displaystyle dM_t = M_t \left(-\frac{\Lambda_t}{Z_t} + f(M_t - (1 - Z_t))\right)dt + f(M_t - (1 - Z_t))dY_t, \\
M_u = x
\end{array} \right., \quad u \leq t < \infty
\]

with $f$ being a continuously differentiable Lipschitz function and $Y$ being a $\mathbb{P}$-$\mathcal{F}$ local martingale. Let $(M^u, 0 \leq u < \infty)$ be the $iM_Z$ family defined by the equation($\sharp$) (see Theorem 3.2) such that:

$\text{Hy}(Mc)$: For each $0 \leq t \leq \infty$, the map $u \to M^u_t$ is continuous on $[0, t]$.

Let $Q$ be the probability on the product space $[0, \infty] \otimes \Omega$ associated with the $iM_Z$ (see Theorem 2.1).
Theorem 4.1 Let $X$ be a $\mathbb{P}$-$\mathbb{F}$ local martingale. Then the process

$$X_t - X_0 - \int_0^t \mathbb{1}_{\{s \leq \tau\}} \frac{e^{-\Lambda_s}}{Z_s} d\langle N, X \rangle_s$$

$$- \int_0^t \mathbb{1}_{\{\tau < s\}} (-\frac{e^{-\Lambda_s}}{1-Z_s})d\langle N, X \rangle_s$$

$$- \int_0^t \mathbb{1}_{\{\tau < s\}} \left(f(M_s^\tau - (1 - Z_s)) + M_s^\tau f'(M_s^\tau - (1 - Z_s))\right) d\langle Y, X \rangle_s$$

is a $\mathbb{Q}$-$\mathbb{G}$ local martingale.

4.6 In what the SDE$(\mathcal{M})$ has been useful

We illustrate it through the following computation: Let $0 \leq a < b \leq s < t$ and $A \in \mathcal{F}_s$. Put aside the integrability question. We begin with

$$\mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{a < \tau \leq b\}} (X_t - X_s)]$$

$$= \mathbb{Q}[\mathbb{1}_A (M_b^\infty - M_a^\infty)(X_t - X_s)]$$

$$= \mathbb{Q}[\mathbb{1}_A \int_s^t \frac{(-1)e^{-\Lambda_r}}{1-Z_r} (M_r^b - M_r^a) d\langle N, X \rangle_r]$$

$$+ \mathbb{Q}[\mathbb{1}_A \int_s^t (M_r^b f(M_r^b - (1 - Z_r)) - M_r^a f(M_r^a - (1 - Z_r))) d\langle Y, X \rangle_r]$$

Compute separately the last two terms. Firstly

$$\mathbb{Q}[\mathbb{1}_A \int_s^t \frac{(-1)e^{-\Lambda_r}}{1-Z_r} (M_r^b - M_r^a) d\langle N, X \rangle_r]$$

$$= \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{a < \tau \leq b\}} \int_s^t (-\frac{e^{-\Lambda_r}}{1-Z_r})d\langle N, X \rangle_r]$$

Nextly

$$\mathbb{Q}[\mathbb{1}_A \int_s^t (M_r^b f(M_r^b - (1 - Z_r)) - M_r^a f(M_r^a - (1 - Z_r))) d\langle Y, X \rangle_r]$$

$$= \mathbb{Q}[\mathbb{1}_A \int_s^t \int_a^b \left(f(M_r^a - (1 - Z_r)) + M_r^a f'(M_r^a - (1 - Z_r))\right) d_v M_v^a d\langle Y, X \rangle_r]$$

$$= \mathbb{Q}[\mathbb{1}_A \int_s^t \mathbb{1}_{\{a < \tau \leq b\}} \left(f(M_r^a - (1 - Z_r)) + M_r^a f'(M_r^a - (1 - Z_r))\right) d\langle Y, X \rangle_r]$$

It is now easy to deduce the semimartingale decomposition formula from this computation. •
4.7 Semimartingale decomposition formula for the model constructed with 
the balayage formula, the case of eventual $1 - Z = 0$

We suppose $\text{Hy}(B)$ and $\text{Hy}(H)$. We consider the \( iM_Z \) constructed in the theorem 3.3 and its associated 
probability measure $Q$ on $[0, \infty] \times \Omega$ (see Theorem 2.1).

**Theorem 4.2** Let $X$ be a $\mathbb{P}$-$\mathbb{F}$ local martingale. Then

\[
X_t - X_0 - \int_0^t \mathbf{1}_{\{s < g \vee \tau\}} \frac{e^{-\Lambda_s}}{Z_s} d\langle N, X \rangle_s + \int_0^t \mathbf{1}_{\{g < s < \tau\}} \frac{e^{-\Lambda_s}}{1 - Z_{s-}} d\langle N, X \rangle_s, \quad 0 \leq t < \infty,
\]

is a $Q$-$\mathbb{G}$ local martingale.\*

It is noted that the above formula has the same form as the formula for honest time, whilst $g \vee \tau$ is not 
a honest time in the filtration $\mathbb{F}$.

4.8 Proof with the SDE

The theorem can be proved in quite the same way as in the preceding theorem, except some precaution 
on the zeros of $1 - Z$. Recall that the elements in $iM_Z$ satisfy the equation :

\[
M_t^u = (1 - Z_u) - \int_u^t \mathbf{1}_{\{g < r \leq \tau\}} E_r^u e^{-\Lambda_r} dN_r, \quad u \leq t < \infty.
\]

Let $0 \leq a < b \leq s < t$ and $A \in \mathcal{F}_s$. Put aside the integrability question. We have

\[
\begin{align*}
= & \quad Q[\mathbf{1}_A \mathbf{1}_{\{a < g \vee \tau \leq b\}} (X_t - X_s)] \\
= & \quad Q[\mathbf{1}_A (M^b_t - M^a_t)(X_t - X_s)] \\
= & \quad Q[\mathbf{1}_A \int_s^t \mathbf{1}_{\{g \leq \tau \leq b\}} E^b_r e^{-\Lambda_r} d\langle N, X \rangle_r] \\
& \quad - Q[\mathbf{1}_A \int_s^t \mathbf{1}_{\{a < g \leq \tau \}} E^a_r e^{-\Lambda_r} d\langle N, X \rangle_r] \\
= & \quad Q[\mathbf{1}_A \int_s^t \mathbf{1}_{\{\tau < g \leq b\}} \frac{(-e^{-\Lambda_r})}{1 - Z_r} d\langle N, X \rangle_r] \\
& \quad - Q[\mathbf{1}_A \int_s^t \mathbf{1}_{\{a < \tau \}} \frac{(-e^{-\Lambda_r})}{1 - Z_r} d\langle N, X \rangle_r]
\end{align*}
\]
5 Reciprocity

We have seen that the SDE of our $iM_Z$ implies SDF. Now we consider the inverse problem: Can we determine the SDE for $iM_Z$ from SDF. We shall prove here that the answer is "Yes".

We suppose so to be given a solution $(Q, \tau)$ of the $M$-problem. Let $iM_Z$ be the family associated with this solution $(Q, \tau)$. We have the following theorem:

**Theorem 5.1** We assume the hypothesis $Hy(B)$, $Hy(N, \Lambda)$, $Hy(C)$, $Hy(I)$, $Z_\infty = 0$ and $Hy(Mc)$. If the SDF in the Theorem 4.1 holds under $Q$, the martingales $M^u \in iM_Z$, $0 \leq u < \infty$, are necessarily solutions of the SDE $(\natural_u)$.\[\blacksquare\]

We can now say that the SDE and the SDF are just two facades of the same thing.
6 Conclusion

We have shown in this talk that $iM_Z$ families combined with SDE provide an efficient tool to deal with credit risk modelling $\mathcal{M}$-problem. We were really amazed by this efficiency and we had tried to understand why. Now we can say that our approach is a continuation of the idea initiated in Yor [12], where the martingale

$$\mathcal{M}_t(du) = Q[\tau \in du|\mathcal{F}_t]$$

of the conditional law of $\tau$ is considered. It has been explained in [12] that the enlargement of filtration problems will have solutions, if (speaking in a very simplistic way) $\mathcal{M}_t(du) = Q[\tau \in du|\mathcal{F}_t]$ satisfies an equation:

$$d\mathcal{M}_t(du) = K_t(du)dM_t$$

where the kernel $K_t(du)$ is absolutely continuous with respect to $\mathcal{M}_t(du)$:

$$K_t(du) = p_t(u) \cdot \mathcal{M}_t(du).$$

In this case a semimartingale decomposition formula can be written down explicitly using $M_t$ and $p_t(u)$. To see the link with our study, we can write formally the SDE($\mathcal{z}$) in the form

$$d_t(d_uM^u_t) = (d_uM^u_t)(-\frac{\alpha}{1-Zt} dN_t)$$

$$+ ((f(M^u_t - (1 - Z_t) + M^u_t f'(M^u_t - (1 - Z_t)))d_uM^u_t) dY_t$$

and remark that $d_uM^u_t = \mathcal{M}_t(du)$ (at least on $u \in [0,t]$). See Ankirchner and al.[1] and Mikeghbali [10] for other applications of the idea.

Nowaday everybody use change of probability of Girsonov to solve enlargement of filtration problem. Probably further we should use more the absolute continuous kernel method of Yor[12].
Références


