Financial market model with influential informed investors

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Introduction

Model for assets prices whose dynamics are modified by the financial strategy of an informed investor.

Two motivations:

1. unrealistic to suppose that the informed agent does not influence the asset prices dynamics when he invests on the market,

2. building a statistical test to detect an informed agent in the market through the assets prices.
Basic model proposed in Cuoco and Cvitanic [4]:
- investor’s influence is modeled with an additive perturbation of the market interest rate and of the asset prices long term return,
- volatility is supposed to be not influenced by investor’s strategy.
Such models in which volatility is perturbed by investor’s strategy are much more tricky.
Purpose: to add a private information, thus, the investor’s strategy is adapted to an enlarged filtration instead of being adapted to the prices filtration.
References


Section 1 introduces model, notations, hypotheses.

Section 2 gives a necessary condition for admissibility and feasibility of financial strategies (actually budget constraint) + a sufficient condition. 
*These conditions allow us to show that there exist market prices which give the optimal strategies*

Sections 3 and 4 studies how these strategies influence the prices on one example (only pressure on price). We then suppose that both asset prices and agent’s strategy are observed.

Section 5 builds statistical tests to decide  
first that there exists (or not) an influential investor optimizing his strategy on the market and (if it is the case)  
second that this investor is informed (or not).
1 Financial model with influential agent

\((W_t, t \in [0, T])\) \(d\) dimensional Brownian motion on \((\Omega, \mathcal{A}, \mathbb{P})\),
\((\mathcal{F}_t, t \in [0, T])\) its completed natural filtration, \(T\) is fixed horizon.
The agents can invest in a non risky assets and in \(d\) risky assets whose prices are continuous \((\mathcal{F}, \mathbb{P})\)–semimartingales.

Two types of agents invest:
- non informed ones know at time \(t\) the \(\sigma\)–field \(\mathcal{F}_t^S\), the assets
prices natural filtration;
- the informed agents know at time \(t\) the \(\sigma\)–field \(\mathcal{Y}_t\):

\[ \mathcal{F}_t \lor \mathcal{F}_t^S \subset \mathcal{Y}_t \ \forall t \leq T. \]

Let \(\mathcal{G}\) be the private initial information of informed agents, \(\mathcal{G} \subset \mathcal{A}\)
and let \(\mathcal{Y}_t = \cap_{s>t}(\mathcal{F}_s \lor \mathcal{G}), t \in [0, T].\)
We suppose that an influential informed agent optimizes his strategy in the set of $\mathcal{Y}$–adapted processes. Basic hypothesis in most financial models with asymmetric information:

**Hypothesis** ($H_3$): there exists $Q$ equivalent to $\mathbb{P}$ under which $\forall t \leq T$, $\mathcal{F}_t$ is independent of $\mathcal{G}$.

Under this hypothesis,

- [1] shows that actually $\mathcal{Y}_t = \mathcal{F}_t \vee \mathcal{G}$, $t \in [0, T]$
- [6] that ($\mathcal{Y}, Q$) admits the martingale representation property, [7] proves there is no arbitrage strategy for the informed agent when hypothesis ($H_3$) is verified.

Hypothesis ($H_3$) will be in force in the whole following.
Remark 1 When the information $G$ is generated by a random variable $L$, we denote $(H_J)$ Jacod’s criterium:

the conditional probability law of $L$ given $F_t$ is equivalent to the probability law of $L$, $\forall t \leq T$, with density $q(t, \cdot)$.
In that case, hypotheses $(H_3)$ and $(H_J)$ are equivalent.

Moreover, under $(H_J)$, we can choose the probability $Q$ as follows: $Q = \frac{1}{q(T,L)}P$ on $\mathcal{Y}_T$.

In such a case, remark that $Q_{|F_t} = P_{|F_t}$, $\forall t \leq T$.
We now use this probability $Q$. 
$d$ risky assets prices evolve according to:

(1) \[ S_t^i = S_0^i + \int_0^t S_s^i (b_s^i \, ds + (\sigma_s)_j^i \, dW_s^j), \quad i = 1, \cdots, d, \quad S_0 \in \mathbb{R}^d, \quad 0 \leq t \leq T, \]

non risky assets is described by

(2) \[ S_t^0 = 1 + \int_0^t S_s^0 r_s \, ds, \]

where $r$ takes its values in $\mathbb{R}$.

Hypothesis (i): $\sigma$ is an $\mathcal{F}$–adapted $d \times d$ matrix process and the matrix $\sigma_t \sigma_t^*$ is uniformly elliptic (hence $\sigma$ is invertible).
Cuoco and Cvitanic’s model: additive perturbation of $r$ and $b$, depending on $\pi$ with values in $\mathbb{R}^{d+1}$ (agent’s strategy).

Hypothesis (ii): for all $(t, \omega) \in [0, T] \times \Omega,$

$$r(t, \omega) = r^1(t, \omega) + r^2(\pi(t, \omega), t, \omega), \quad b(t, \omega) = b^1(t, \omega) + b^2(\pi(t, \omega), t, \omega),$$

$r^1, b^1$ $\mathcal{F}$–adapted, s.t. $\int_0^T (|r^1(s)| + |b^1(s)|)ds < \infty$ and $\exists K > 0$ s.t. $\int_0^T r^1(t) - dt < K$. In particular $\exp[-\int_0^t r^1(s)ds]$ is uniformly bounded.

The maps $b^2$ and $r^2$ are defined on $\mathbb{R}^{d+1} \times [0, T] \times \Omega$ and $\forall \pi \in \mathbb{R}^{d+1}$, $b^2(\pi, ., .)$ and $r^2(\pi, ., .)$ are $\mathcal{F}$–adapted and progressively measurable.

$b^2$ and $r^2$ are supposed exogenously given

$b^2(0, ., .) = 0$ in $\mathbb{R}^d$ and $r^2(0, ., .) = 0$ a.s. meaning insider doesn’t act.
For $0 \leq t \leq T$ equations (1) and (2) are now given by

$$S_t = S_0 + \int_0^t S_s(b^1(s) + b^2(\pi, s))ds + \sigma_s dW_s,$$

(3) $$S_0^t = 1 + \int_0^t S_s^0(r^1(s) + r^2(\pi, s))ds.$$

Let be $\kappa_0 = -\sigma^{-1}(b^1 - r^1)$, Doléans’ exponential $L^0 = \mathcal{E}(\kappa_0.W)$ is a $(\mathcal{F}, \mathbb{P})$-martingale thus $L_0^T.Q$ is a risk neutral probability measure on $(\Omega, \mathcal{Y})$ when informed agent doesn’t act on the market (cf. [6]).
Financial agent’s strategy \((c, \alpha, \theta)\):

- \(c\) is the consumption rate, \(\mathcal{Y}\)-adapted positive process
- \(\alpha\) wealth, invested in the non risky assets;
- \(\theta\) wealth invested in the risky assets;
- \(\pi = (\alpha, \theta)\), the portfolio process, also \(\mathcal{Y}\)-adapted.
Informed agent’s initial information $\mathcal{G}$ verifies hypothesis $(H_3)$, initial wealth $X_0 \in \mathcal{A}_0^+$ where

$$\mathcal{A}_0^+ = \{\mathcal{Y}_0\text{–measurable and } > 0 \text{ random variables}\}$$

and an endowment: positive bounded $\mathcal{Y}$–adapted rate $y$.

Wealth at time $t$, $X_t = \sum_{i=0}^{d} \pi^i_t$; consumption is $\int_0^t c_s ds$.

The agent’s strategy is self-financing, the wealth process satisfies

$$dX^\pi_t = \sum_{i=0}^{d} \frac{\pi^i_t}{S^i_t} dS^i_t - c_t dt + y_t dt,$$

precisely: consumption is supported by (strategy, endowments).

$X^\pi_t = \alpha_t + \sum_{i=1}^{d} \theta^i_t$ is solution of a SDE, a priori an anticipating one:

the informed agent uses all his information so processes $\pi$ and $X^\pi$ are $\mathcal{Y}$-adapted, he anticipates the Brownian motion $W$. 

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Enlargement of filtration is used to give sense to that equation \([3, 11]\), under Hypothesis \((H_3)\), \(W\) is a \((\mathcal{Y}, Q)\)-Brownian motion \([7]\):

\[
dX_t^\pi = (X_t^\pi r_t - c_t + y_t)dt + \sum_{i=1}^{d} \pi_t^i (b_t^i - r_t)dt + \sum_{i=1}^{d} \sum_{j=1}^{d} \pi_t^i \sigma_j^i(t).dW_t^j
\]

**Definition 1** A strategy \(\pi = (\alpha, \theta)\) is admissible if almost surely

\[
\int_0^T (|\alpha_t r_t + \langle \theta_t, b_t \rangle| + \|\tilde{\sigma}_t \theta_t\|^2)dt < \infty \ \mathbb{P} \ a.s.
\]

\(\langle ., . \rangle\) denotes the scalar product on \(\mathbb{R}^d\) and on \(\mathbb{R}^{d+1}\).

Let \(\Theta\) be the set of admissible strategies.
To optimize strategies, let an utility function given as

\[ U(c) = E_\mathbb{P}\left(\int_0^T u(c_s, s)ds/Y_0\right) \]

where \( u \) is measurable and satisfies

**Hypothesis (iii) \( \forall t \in [0, T] \)**

\[ x \mapsto u(x, t) \text{ is increasing, strictly concave, } C^2 \text{ on } \mathbb{R}^*_+ \]

\[ \lim_{x \to 0} \partial_x u(x, t) = +\infty \; ; \; \lim_{x \to \infty} \partial_x u(x, t) = 0. \]

*We could add utility of final wealth in our model, without any difficulty.*
Admissible consumption processes set:

\[ C_+^* = \{ c \geq 0, \mathcal{Y} - \text{adapted}, \exists p > 2 : \int_0^T c_s ds \in L^p(\Omega, L^0_T, \mathcal{Q}), \ E_P[\int_0^T u^-(c_s, s)ds / \mathcal{Y}_0] < \infty \}. \]

Below 1 denotes the d-vector with all coordinates equal to 1 :

**Definition 2** A consumption process \( c \in C_+^* \) is feasible if

- \( \exists \pi = (\alpha, \theta) \in \Theta \) s.t. \( \forall t \leq T \) \( X_t = \alpha_t + \langle \theta_t, 1 \rangle \) verifies \( X_0 > 0 \) and

\[ dX_t = [\alpha_t (r^1(t) + r^2(\pi_t, t)) + \langle \theta_t, b_1^1 + b_2^2(\pi_t, t) \rangle]dt + \langle \theta(t), \sigma_t dW_t \rangle - (c_t - y_t)dt \]

meaning that \( \pi \) is self-financing,

- \( X_t \geq -K \exp(\int_0^t r^1(s)ds), \ dt \otimes dQ \ a.s.; X_T \geq 0 \ dQ \ a.s. \)

We say that \( \pi \) finances \( c \).
The optimization problem is then to obtain in

\[ \Theta \times \{ c \in C^*_+ / c \text{ is feasible} \} \]

the maximum of

\((\mathcal{P})\) \quad (\pi, c) \mapsto J(\pi, c) = E_\mathbb{P}[\int_0^T u(c_t, t) dt / \mathcal{Y}_0]. \]

The agent’s influence is summarized in the following:

**Definition 3** Let \( g \) be the function defined on \( \mathbb{R}^{d+1} \times [0, T] \times \Omega: \)

\[ g(\pi, t, \omega) = \alpha r^2(\pi, t, \omega) + \langle \theta, b^2(\pi, t, \omega) \rangle. \]
Hypothesis (iv) : \( \forall(t, \omega), \pi \mapsto g(\pi, t, \omega) \) is concave on \( \mathbb{R}^{d+1} \).

This hypothesis gives a stabilizing effect to the perturbation on prices (see examples below in Section 4), indeed as \( r^2(0,.,.) = 0 \) and \( b^2(0,.,.) = 0 \) then under Hypothesis (iv) it is not difficult to prove that

\[ \forall(t, \omega), \ g(., t, \omega) \text{ is maximum and null at } 0. \]
Let us define now on $\mathbb{R}^{d+1} \times [0, T] \times \Omega$ the Legendre transform of $g$:

$$
\tilde{g}(\mu, t, \omega) = \sup_{\pi \in \mathbb{R}^{d+1}} (\alpha r^2(\pi, t, \omega) + \langle \theta, b^2(\pi, t, \omega) \rangle - \langle \pi, \mu \rangle).
$$

Because $0 \in \mathbb{R}^{d+1}$ and $g(0, ., .) = 0$, $\forall (t, \omega)$, $\tilde{g}(., t, \omega) \geq 0$ on $\mathbb{R}^{d+1}$; moreover it is lower semi-continuous as a supremum of continuous functions and convex as a supremum of linear functions.

We denote $\mathcal{N}_t(\omega) = \{ \mu \in \mathbb{R}^{d+1} : \tilde{g}(\mu, t, \omega) < \infty \}$.

**Proposition 1** $\forall (t, \omega)$, $\mathcal{N}_t(\omega)$ non empty convex subset of $\mathbb{R}^{d+1}$.

**Proof**: It is non empty because $0 \in \mathcal{N}_t(\omega)$, indeed $g(., t, \omega) \leq 0$. Convexity of $\tilde{g}$ gives that the set is convex. $\diamond$
$\mathcal{N} := \{Y - \text{adapted } \mathbb{R}^{d+1} \text{ valued processes } \mu, \text{ s.t. } \mu_t(\omega) \in \mathcal{N}_t(\omega)\}$.

Below, processes of $\mathcal{N}$ are denoted $(\nu^0, \nu)$, $\nu^0(\omega, t) \in \mathbb{R}$ and $\nu(\omega, t) \in \mathbb{R}^d$.

Hypothesis (v): $dt \otimes d\mathbb{P}$ almost surely, $\mathcal{N}_t(\omega)$ is a closed subset of $\mathbb{R}^{d+1}$, $\tilde{g}(., t, \omega)$ is essentially bounded on $\mathcal{N}_t(\omega)$ and there exists a bounded subset $K \subset \mathbb{R}^{d+1}$ such that $dt \otimes d\mathbb{P}$ almost surely, $\mathcal{N}_t(\omega) \subset K$.

This (technical) hypothesis seems rather strong. Nevertheless, we gave some examples where this hypothesis holds in the paper, one is given in Section 4.
Proposition 2  \( dt \otimes d\mathbb{P} \) almost surely, \( \tilde{g}(., t, \omega) \) is lower semi-continuous, for all \( (\pi, t, \omega) \in \mathbb{R}^{d+1} \times [0, T] \times \Omega \),
\[
g(\pi, t, \omega) = \text{ess inf}_{\mu \in \mathcal{N}_t(\omega)} (\tilde{g}(\mu, t, \omega) + \langle \pi, \mu \rangle).
\]
Moreover, \( \mathcal{N}_t(\omega) \) is a compact set, hence \( \forall \pi \in \mathbb{R}^{d+1}, \exists \mu \in \mathcal{N}_t(\omega) \)
such that \( g(\pi, t, \omega) = \tilde{g}(\mu, t, \omega) + \langle \pi, \mu \rangle \).

The first point is a standard result in convex analysis (cf. [13]) and the second point is due to Hypothesis (v).
2 A sufficient condition for feasibility

We solve the optimization problem with the duality method developed by I. Karatzas, J. Lehoczky and S. Shreve [10].

The financial market seems to be incomplete since the drifts of the prices depend on the agent’s strategy and then cannot be considered as known.

But (below) Corollary 4 shows that the informed agent can hedge any claim in $\mathcal{V}_T$ thanks to the predictable representation property. Nevertheless the key of the solution is the duality method similarly to the incomplete market case.
2.1 Budget constraint

∀μ = (ν₀, ν) ∈ N let be processes \( R^\mu \) and \( L^\mu \), solution of:

(5) \[ dR^\mu_t = -R^\mu_t (r^1_t + ν^0_t)dt, \quad R^\mu_0 = 1, \quad R^\mu_t = \exp(- \int_0^t (r^1_s + ν^0_s)ds), \]

(6) \[ dL^\mu_t = L^\mu_t \kappa^\mu_t . dW_t, \quad L^\mu_0 = 1 \text{ where } \kappa^\mu_t = -\sigma_t^{-1} (b^1_t + ν_t - (r^1_t + ν^0_t)1). \]

Note that \( R^\mu \) is uniformly bounded.

For any \( p > 0 \) and \( \mu \in N \), the process \( L^\mu \) is a \( L^p(Q) \)-martingale, thanks to the hypotheses (i), (ii) and (v).

We define \( Q^\mu = L^\mu_T . Q \) on \( \mathcal{Y}_T \):

in case \( \mu = (r^2, b^2) \), \( Q^\mu \) is a risk neutral probability measure.
Proposition 3  For all $\mu \in \mathcal{N}$, the process

$$W_t^\mu = W_t + \int_0^t \sigma_s^{-1}(b_s^1 + \nu_s - (r_s^1 + \nu_s^0)1)ds$$

is a $(\mathcal{Y}, Q^\mu)$ Brownian motion.

Hypotheses (i), (ii) and (v) and Girsanov Theorem yield the result.

Note that we only need:

$L^\mu$ is a $(\mathcal{Y}, Q)$-martingale

and we stress $L^\mu = L^0.\mathcal{E}[-\sigma^{-1}(\nu - \nu^0 1).W^0]$.

Corollary 4  The $(\mathcal{Y}, Q^\mu)$-Brownian motion $W^\mu$ has the predictable representation property for $(\mathcal{Y}, Q^\mu)$-martingales.
The process $R^\mu$ is a deflator coefficient; with a self-financing strategy $\pi = (\alpha, \theta)$ the agent gets wealth:

\begin{equation}
\begin{split}
dR^\mu_t X_t &= R^\mu_t \left[ (\alpha_t (r^2_t - \nu^0_t) + \langle \theta_t, b^2_t - \nu_t \rangle) dt + \langle \theta_t, \sigma_t dW^\mu_t \rangle - (c_t - y_t) dt \right].
\end{split}
\end{equation}

It follows that usually the probability law $Q^\mu$ is not a risk neutral probability measure; but it is as soon as $\mu = (r^2(\pi), b^2(\pi))$.

Reciprocally, if $\pi$ is such that $(r^2(\pi), b^2(\pi)) \in \mathcal{N}$ then there exists a risk neutral probability law and the enlarged financial market has no arbitrage opportunity.
Corollary 4 allows us to extend Cuoco-Cvitanic’s results to enlarged filtration concerning a **budget constraint** on every admissible/feasible consumption.

**Proposition 5** Let $c \in C^*_+$ feasible. Then for all $\mu \in \mathcal{N}$:

$$
E_Q^\mu \left[ \int_0^T R_s^\mu (c_s - y_s) ds / \mathcal{Y}_0 \right] \leq X_0 + E_Q^\mu \left[ \int_0^T R_s^\mu \tilde{g}(\mu_s, s) ds / \mathcal{Y}_0 \right].
$$
2.2 Sufficient condition

Let $c$ be an admissible consumption and $M^\mu$ be the martingale

$$M_t^\mu = EQ^\mu \left[ \int_0^T R_s^\mu (c_s - y_s - \tilde{g}(\mu_s, s))ds / \mathcal{Y} \right].$$

$c$ is admissible $\Rightarrow \exists p > 2$ such that this martingale is in $L^p(Q^\mu)$. Process $W^\mu$ has the representation property, $M^\mu \in L^p$, thus

$\exists$ predictable $\phi^c \in L^{p/2}([L^2([0, T], dt)], dQ^\mu)$ s.t.

$M_t^\mu = M_0^\mu + \int_0^t \phi_s^c dW_s^\mu$. We associate $X^c$:

$$M_t^\mu = R_t^\mu X_t^c + \int_0^t R_s^\mu (c_s - y_s - \tilde{g}(\mu_s, s))ds.$$

(note that $X_T^c = 0$) and let $\pi = (\alpha, \theta)$ be the strategy defined by

$$R_s^\mu \sigma_s^t \theta_s = \phi_s^c; \quad \alpha_s = X_s^c - \langle \theta_s, 1 \rangle.$$

**Theorem 6** Let $c \in C_+^*$ be a consumption process. We suppose \( \exists \mu^* \in \mathcal{N} \) such that for all $\mu \in \mathcal{N}$

\[
(11) \quad E_{Q^\mu} \left[ \int_0^T R_t^\mu (c_t - y_t - \tilde{g}(\mu_t, t)) dt \right] \leq \frac{Y_0}{E_{Q^\mu^*} \left[ \int_0^T R_t^\mu^* (c_t - y_t - \tilde{g}(\mu_t^*, t)) dt \right] = X_0,}
\]

then $c$ feasible, wealth process associated to $(c, \alpha, \theta)$ (10) is

\[
X_t^{c, \alpha, \theta} = (R_t^{\mu^*})^{-1} E_{Q^\mu^*} \left[ \int_t^T R_s^{\mu^*} (c_s - y_s - \tilde{g}(\mu_s^*, s)) ds \right] \mathcal{Y}_t).
\]
3 Optimisation

We are looking for an optimal consumption which should be feasible and should satisfy the budget constraint in Proposition 5. We use the Lagrangian on $C^*_+ \times A^+_0 \times \mathcal{N}$:

$$
\mathcal{L} : (c, \lambda, \mu) \mapsto U(c) - \lambda \left( E_{Q^\mu} \left[ \int_0^T R_s^{\mu}(c_s - y_s - \tilde{g}(\mu_s, s))ds / \mathcal{Y}_0 \right] - X_0 \right),
$$

where $X_0 > 0$ is $\mathcal{Y}_0$-measurable and $U(c)$ is to be maximized in $C^*_+ : U(c) = E_P[\int_0^T u(c_s, s)ds / \mathcal{Y}_0]$. 
Let us recall (cf. Hypothesis \((H_3)\)):

\[
(12) \quad t \mapsto q_t = \frac{d\mathbb{P}}{dQ|\mathcal{Y}_t} \text{ is a } (\mathcal{Y}, Q)\text{-martingale.}
\]

So Lagrangian \(\mathcal{L}\) is

\[
\mathcal{L}(c, \lambda, \mu) = E_P \left[ \int_0^T u(c_s, s) \, ds - \lambda \int_0^T q_s^{-1} L_s^\mu R_s^\mu (c_s - y_s - \tilde{g}(\mu_s, s)) \, ds / \mathcal{Y}_0 \right].
\]

Consider the convex conjugate of \(u\):

\[
\tilde{u}(y, t) = \max_{c \geq 0} [u(c, t) - yc] = u(I(y, t), t) - yI(y, t)
\]

where \(I(., t) = (u')^{-1}(., t), y \in \mathbb{R}^+\).
Well known results:

\[ \tilde{u}(., t) \text{ is strictly convex decreasing on } \mathbb{R}^+ \]

\[
\begin{align*}
\partial_y \tilde{u} &= -I \\
\tilde{u}(0, t) &= u(\infty, t) ; \quad \tilde{u}(\infty, t) = u(0, t).
\end{align*}
\] (13)

For \((\lambda, \mu)\) fixed in \(A_0^+ \times \mathcal{N}\), the Lagrangian is maximum for

\[
c_t^{\lambda, \mu} = I(\lambda M_t^\mu)
\]

where \(M_t^\mu = q_t^{-1} L_t^\mu R_t^\mu\) and the associated optimal value is

\[
J(\lambda, \mu) = E_P[\int_0^T (\tilde{u}(\lambda M_s^\mu) + \lambda M_s^\mu(y_s + \tilde{g}(\mu_s, s)))ds/\mathcal{Y}_0] + \lambda X_0.
\]

**Proposition 7** If there exist \(\hat{\lambda} \in A_0^+ \) and \(p > 2\) such that :

\[
E_{Q^\mu}(\int_0^T R_s^\mu I(\hat{\lambda} M_s^\mu)ds)^p/\mathcal{Y}_0 < \infty,
\]

then the consumption \(c_t^{\lambda, \mu}\) is admissible for all \(\lambda \geq \hat{\lambda} Q \text{ a.s..} \)
Next theorem (generalization of [4] Theorem 2) shows that the existence of a solution to dual problem $\mathcal{P}^*$ implies the existence of a consumption $c \in \mathcal{C}^*$ which is feasible and optimal for problem $\mathcal{P}$.

**Theorem 8** If there exists $(\hat{\lambda}, \hat{\mu}) \in \mathcal{A}_0^+ \times \mathcal{N}$ such that

$$\forall (\lambda, \mu) \in \mathcal{A}_0^+ \times \mathcal{N}, \quad J(\hat{\lambda}, \hat{\mu}) \leq J(\lambda, \mu)$$

where $R^\mu_s I(\hat{\lambda} M^\hat{\mu}_s)$ is $\mathcal{Y}_s$-measurable and

$$E_{Q^\mu} [(\int_0^T R^\mu_s I(\hat{\lambda} M^\hat{\mu}_s) ds)^p / \mathcal{Y}_0] < \infty \text{ for a } p > 2,$$

then consumption $\hat{c} = I(\hat{\lambda} M^\hat{\mu})$ is optimal in the feasible consumptions subset of $\mathcal{C}^*_+$. 
4 Examples where the hypotheses are in force

Some examples where parameters $r^1$, $b^1$ and $\sigma$ satisfy measurability, boundness and integrability Hypotheses (i) and (ii) and where $r^2$ and $b^2$ are such that Hypotheses (iv) and (v) are satisfied.
4.1 Pressure on prices

Example in [4], 1-dimensional case:

\[ r^2 = 0, \quad b^2(\pi, t, \omega) = -a_t(\omega) \frac{\theta}{|\theta|} \]

a positive uniformly bounded \( F \)-adapted process.

H(iv): \( g(\pi, t, \omega) = -a_t(\omega)|\theta| \), concave in \( \theta \), uniformly bounded by 0.

H(v):

\[ \tilde{g}(\mu, t, \omega) = \sup_{\alpha, \theta} [-a_t(\omega)|\theta| - \langle \theta, \nu \rangle - \alpha \nu^0] \]

is null if \( \nu^0_t(\omega) = 0 \) and \( |\nu_t(\omega)| \leq a_t(\omega) \), otherwise it is \( \infty \).

Then \( \mathcal{N}_t(\omega) = \{0\} \times [-a_t(\omega), a_t(\omega)] \)

is a uniformly bounded closed set in \( \mathbb{R}^2 \) on which \( \tilde{g} \) is zero, \( \Rightarrow \mathcal{N} \) is the set of \( \mathcal{Y} \)-adapted processes taking their values in \( \{0\} \times [-a, +a] \) where \( a = ess sup\{|a_t(\omega)|, (t, \omega)\} \).
4.2 $d$–dimensional case

$r^2 = 0, \ b^2(\pi, t, \omega) = -\frac{1}{\|\theta\|} A_t(\omega)\theta 1_{\{\theta \neq 0\}}, \ A$ bounded $\mathcal{F}$-adapted process, in the set of positive symmetric matrices. Then

$$\theta \mapsto g(\pi, t, \omega) = -\frac{1}{\|\theta\|} \langle \theta, A_t(\omega)\theta \rangle$$

is a concave uniformly bounded by 0 function (Hyp (iv)):

use spherical coordinates: $\theta = \rho.u$, with $\rho \in \mathbb{R}^+$ and $\|u\| = 1$, $g(\rho, \pi, \omega) = -\rho \tilde{u} A_t(\omega)u$ and its Hessian is non positive. Now

$$\tilde{g}(\mu, t, \omega) = \sup_{\alpha, \rho, u} [-\rho \tilde{u} A_t(\omega)u - \rho \langle u, \nu \rangle - \alpha \nu^0].$$

This supremum is finite (and in fact zero)

$\Leftrightarrow [\nu^0 = 0 \text{ and } \tilde{u} A_t(\omega)u + \langle u, \nu \rangle \geq 0 \text{ for all } u, \|u\| = 1].$

Tricky proof but this condition $\Leftrightarrow \|A_t^{-1}(\omega)\nu\| \leq 1$. So

$\mathcal{N}_t(\omega) = \{0\} \times \{\nu \in \mathbb{R}^d : \|A_t^{-1}(\omega)\nu\| \leq 1\}$ is a uniformly bounded closed set in $\mathbb{R}^{d+1}$ (hypothesis (v)) on which $\tilde{g}$ is zero.
4.3 Pressure on rates

\[ b^2 = 0, \quad r^2(\pi, t, \omega) = (R_t(\omega) - r^1(t, \omega))1_{\{\alpha<0\}} \]

where \( R \) is an \( \mathcal{F} \)-adapted process with values in the interval 
\([r^1(t, \omega), r^1(t, \omega) + C]\) where \( C \) is a strictly positive constant.

\[ g(\pi, t, \omega) = (R_t(\omega) - r^1(t, \omega))\alpha1_{\{\alpha<0\}} \]

is a concave function in \( \alpha \), uniformly bounded by 0 (Hyp. (iv)).

\[ \tilde{g}(\mu, t, \omega) = \sup_{\alpha, \theta}[(R_t(\omega) - r^1(t, \omega))\alpha1_{\alpha<0} - \langle \theta, \nu \rangle - \alpha \nu^0], \]

this supremum is finite (here it is zero) \( \Leftrightarrow \nu_t = 0 \) and 
\( \nu^0_t(\omega) \in [0, R_t(\omega) - r^1(t, \omega)] \). Then

\[ \mathcal{N}_t(\omega) = [0, R_t(\omega) - r^1(t, \omega)] \times \{0\} \]

is a uniformly bounded closed set in \( \mathbb{R}^2 \) (Hyp.(v)). \( \tilde{g} \) is zero on \( \mathcal{N} \).
4.4 Utility and optimization

Here the utility function is $u(c, t) = \exp(-\rho t) \log c$, $\rho > 0$, no endowments $y$.

The conjugate function of $u$ is

$$
\tilde{u}(y, t) = \max_{c \geq 0} [e^{-\rho t} \log c - yc] = -\exp(-\rho t)(1 + \rho t + \log y).
$$

\(\tilde{g}\) is zero in set $N_t(\omega)$ and the dual problem corresponds to the study of function $J$ defined on $A_0^+ \times N$ by:

$$
J(\lambda, \mu) = \mathbb{E}_\mathbb{P}[-\int_0^T e^{-\rho s}(1 + \rho s + \log(\lambda M_s^{\mu}))ds / \mathcal{Y}_0] + \lambda X_0
$$

where $M^{\mu} = q^{-1}R^{\mu}L^{\mu}$, $dq^{-1}_s = -q^{-1}_s l_s dB_s$, $B = W - \int_0 \cdot l_s ds$

$q^{-1}L^{\mu}$ ($\mathcal{Y}, \mathbb{P}$)-martingale

$d(q^{-1}L^{\mu})_s = (q^{-1}L^{\mu})_s[-\sigma^{-1}(b^1 + \nu - (r^1 + \nu^0)1)(s) - l_s]dB_s$
Remark 2 In the case $\mathcal{G} = \sigma(L)$, where $L$ is a random variable in Sobolev space $\mathbb{D}^{1,2}(\mathcal{F}_T)$, process $l$ can be computed (cf. [11] p.12) :

$$l_s = E[D_sL \int_s^T 1_{(D_uL)^2du>0}(\int_s^T (D_uL)^2du)^{-1}D_uLdW_u/\mathcal{Y}_s],$$

where $D$ denotes the stochastic gradient with respect to the Brownian motion $W$. 
\[ \log R_t^\mu = - \int_0^t (r^1(s) + \nu^0(s)) ds, \] and dual function \( J \) is

\[ J(\lambda, \mu) = D_t + \lambda X_0 - \log \lambda \frac{1 - \exp(-\rho T)}{\rho} - E_P[\int_0^T \exp(-\rho t) \int_0^t \left( \frac{1}{2} \| \kappa^\mu(s) - l_s \|^2 + \nu^0(s) \right) ds dt / \mathcal{Y}_0], \]

\( D_t \) not depending on \( \lambda \) and \( \mu \).

To apply Theorem 8 (optimality sufficient condition) we look for a couple in \( \mathcal{A}_0^+ \times \mathcal{N} \) achieving the minimum of \( J \).

Straightforward: the optimal \( \lambda \) is \( \hat{\lambda} = \frac{1 - \exp(-\rho T)}{\rho X_0} \in \mathcal{A}_0^+ \).

\( \kappa^\mu \) being \( -\sigma^{-1}(b^1 + \nu - (r^1 + \nu^0)1) \), we have to minimize in \( \mathcal{N} \)

\[ \mu \rightarrow \frac{1}{2} \| \kappa^\mu_s - l_s \|^2 + \nu^0, \]

strictly convex on the compact set \( \mathcal{N} \), so \( \hat{\mu} \) exists and is \( \mathcal{Y} \)-adapted:

\[ \hat{\mu} = (0, a \wedge r^1 - b^1 - \sigma l \vee (-a)). \]
\( p \)-integrability hypothesis of Theorem 8 is verified in this example, thus optimal consumption and wealth are

\[
\hat{c}_t = X_0 \frac{\rho \exp(-\rho t)}{1 - \exp(-\rho T)} (M_t^{\hat{\mu}})^{-1},
\]

\[
R_t^{\hat{\mu}} \hat{X}_t + \int_0^t R_s^{\hat{\mu}} \hat{c}_s ds = E_{Q^{\hat{\mu}}}[\int_0^T R_s^{\hat{\mu}} \hat{c}_s ds / \mathcal{Y}_t].
\]

More precisely we get

\[
\hat{X}_t = X_0 \frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho T}} (M_t^{\hat{\mu}})^{-1},
\]

and the optimal portfolio

\[
\hat{\theta}_t = \hat{X}_t \tilde{\sigma}_t^{-1} (l_t - \kappa_t^{\hat{\mu}}), \quad \hat{\alpha}_t = \hat{X}_t - \langle \hat{\theta}_t, 1 \rangle.
\]
5 Tests in pressure on prices case

Assumption: if there exists an informed agent in the market, he/she is also influential. So the aim is
1. to detect an influence.
2. if it exists, to detect if the suspected agent is an informed agent.

Observations: assets prices $S_t$ for all times $t \in [0; T]$ (actually, on a finite number of times).
During this period, a specified optimizing agent is observed.
We do not know if that agent influences or not the assets prices, and we do not know if he has a private information or not.
But we do know, if he is influential, what kind of influence he could have (e.g. we know his investments make some pressure on assets prices).
Basic market parameters, $b^1$, $r^1$, $\sigma$ are known (meaning usual market without influential or informed agent is well known).

On a first set of observations, we estimate prices trends and detect changes in these trends and times of changes. Using our initial knowledge ( $b^1$, $r^1$, $\sigma$) and these trend estimates we can construct a test to decide if this agent is influential.

With a second set of observations (including the agent’s portfolio), we test the suspected agent’s information and decide whether or not that agent is an informed agent.
Remember \( \hat{\mu} = (0, a \land r^1 - b^1 - \sigma l \lor (-a)) \),
\[
\frac{\theta}{|\theta|} = -\text{sign}(r^1 - b^1 - \sigma l).
\]

**Proposition 9** The estimates depend on three events as following, where we also specify the optimal portfolio \( \hat{\theta} \) sign:

- On \( \{(t, \omega) : a_t(\omega) < r_t^1 - b_t^1 - \sigma_t l_t\} \), \( \hat{\nu}(t, \omega) = a_t(\omega), \hat{\theta}(t, \omega) < 0 \),
- On \( \{(t, \omega) : a_t(\omega) \geq |r_t^1 - b_t^1 - \sigma_t l_t|\} \), \( \hat{\nu}(t, \omega) = r_t^1 - b_t^1 - \sigma_t l_t, \hat{\theta}(t, \omega) = 0 \),
- On \( \{(t, \omega) : -a_t(\omega) > r_t^1 - b_t^1 - \sigma_t l_t\} \), \( \hat{\nu}(t, \omega) = -a_t(\omega), \hat{\theta}(t, \omega) > 0 \).

\( b^1 \) supposed relatively smooth, the agent’s influence will impulse a new trend on time \( t \) such that: \( |b_t^1 - r_t^1 + \sigma_t l_t| > a_t \),
\( l = 0 \) means there is no private information.
1/ **First test** we only observe the assets prices.

No influential agent = ($H_0$) : $b_t = b_t^1$.

Alternative hypothesis, there is an influential agent

$$(H_1) : b_t = b_t^1 + a_t 1_{\{b_t^1 - r_t^1 + \sigma t l_t < -a_t\}} - a_t 1_{\{b_t^1 - r_t^1 + \sigma t l_t > a_t\}},$$

A trend rupture detected on time $t$ yields agent’s influence on the market is suspected.

At time $t$, $|\hat{b}_t - \hat{b}_{t^-}|$ estimate $a_t$ and $\hat{b}_{t^-}$ estimate $b_t^1$.

$\alpha$-level test of influential agent’s presence, we look for $K$:

$$\alpha = \mathbb{P}_{H_0}(\{(\omega) : |\hat{b}(t, \omega) - \hat{b}(t^-, \omega)| > K\}).$$
A local trend estimator at \( t \) and \( t^- \) could be

\[
\hat{b}_t = \frac{\log S(t) - \log S(t - n)}{n} ; \quad \hat{b}_{t^-} = \frac{\log S(t) - \log S(t - 2n)}{2n}.
\]

Black-Scholes model, under \( H_0 : \hat{b}_t - \hat{b}_{t^-} \sim \mathcal{N}(0, \frac{\sigma^2}{2n}) \).

Level of significance \( \alpha = 0.05 \), critical threshold is \( K = 1.96 \frac{\sigma}{\sqrt{2n}} \).

Under hypothesis \( (H_1) \), \( \hat{b}_t - \hat{b}_{t^-} \sim \mathcal{N}(-a_t \text{sign}(\hat{\theta}_t), \frac{\sigma^2}{2n}) \) so the power function is

\[
1 - \Phi[-1.96 + a_t \text{sign}(\hat{\theta}_t) \frac{\sqrt{2n}}{\sigma} ; 1.96 + a_t \text{sign}(\hat{\theta}_t) \frac{\sqrt{2n}}{\sigma}]
\].
2/ Second test suspected agent’s portfolio $\theta$ is now observed, a supposed to be known: Test $l_t = 0$ against $l_t \neq 0$. Conclude that $l_t \neq 0$ ($\exists$ a private information) in 4 cases:

(i) $\hat{\theta}_t > 0$ and $\hat{b}_t < r^1_t$; $\hat{\theta}_t > 0 \iff b^1_t - r^1_t + \sigma_t l_t > a_t$

i.e. $l_t > \sigma_t^{-1}(r^1_t - b_t)$ suspected to be $> 0$ if $\hat{b}_t < r^1_t$.

(ii) $\hat{\theta}_t < 0$ and $\hat{b}_t > r^1_t$ is the symmetric case.

Now, if $\hat{\theta}_t = 0$ (thus $\hat{b}_t = b^1_t$) and $|\hat{b}_t - r^1_t| > a_t$ (an influential not informed agent should invest but this one does not):

$\hat{\theta}_t = 0 \iff |b^1_t - r^1_t + \sigma_t l_t| \leq a_t \iff -a_t - b^1_t + r^1_t \leq \sigma_t l_t \leq a_t - b^1_t + r^1_t$.

There are two more cases:

(iii) $l_t \leq \sigma_t^{-1}(a_t - b_t + r^1_t)$, suspected to be $< 0$ when $\hat{b}_t - r^1_t > a_t$,

(iv) $l_t \geq \sigma_t^{-1}(-a_t - b_t + r^1_t)$, suspected to be $> 0$ when $\hat{b}_t - r^1_t < -a_t$. 
Thus following critical region, union of four events:

\[
R := \{\hat{\theta}_t > 0, \hat{b}_t \leq r^1_t - c_1\} \cup \{\hat{\theta}_t < 0, \hat{b}_t \geq r^1_t + c_2\}
\]
\[
\cup \{\hat{\theta}_t = 0, \hat{b}_t \geq r^1_t + a_t + c_3\} \cup \{\hat{\theta}_t = 0, \hat{b}_t \leq r^1_t - a_t - c_4\}.
\]

For instance, let 0.05 as the test level so \(c_i\) s.t. \(P_{H_0}(R) \leq 0.05\):

\[
P_{\{t=0\}}\{\hat{\theta}_t > 0, \hat{b}_t \leq r^1_t - c_1\} = P\{\hat{\theta}_t > 0, b_t + \frac{\sigma_t}{\sqrt{n}}(W_t - W_{t-n}) \leq r^1_t - c_1\} \leq
\]

\[
P\left\{\frac{W_t - W_{t-n}}{\sqrt{n}} \leq \frac{\sqrt{n}}{\sigma_t}(r^1_t - b^1_t + a_t - c_1)\right\} = 0.0125
\]

as soon as \(c_1 = 2.24 \frac{\sigma_t}{\sqrt{n}} + r^1_t - b^1_t + a_t\)

(remember under \(l = 0\), \(\hat{\theta}_t > 0 \iff r^1_t - b^1_t + a_t < 0\)).
Symmetrically

\[
P_{\{l=0\}}\{\hat{b}_t \geq r_t^1 + c_2, \hat{\theta}_t < 0\} \leq P\left\{\frac{W_t-W_{t-n}}{\sqrt{n}} \geq \frac{\sqrt{n}}{\sigma_t}(r_t^1 - b_t^1 - a_t + c_2)\right\} = 0.0125
\]

as soon as \(c_2 = 2.24 \frac{\sigma_t}{\sqrt{n}} - r_t^1 + b_t^1 + a_t\)

(recall that when \(l = 0\), \(\hat{\theta}_t < 0\) is equivalent to \(r_t^1 - b_t^1 - a_t > 0\))

Finally, choose \(c_3 = 2.24 \frac{\sigma_t}{\sqrt{n}} - r_t^1 + b_t^1 - a_t\) and \(c_4 = 2.24 \frac{\sigma_n}{\sqrt{n}} + r_t^1 - b_t^1 - a_t\)

and we get the first type error \(P_{\{l=0\}}(R) = 0.05\).
Under \((H_1)\), \(l \neq 0\), recall that
\[
\hat{\theta}_t > 0 \iff r^1_t - b^1_t + a_t - \sigma_l l_t < 0 \iff \sigma_l l_t > r^1_t - b^1_t + a_t.
\]
The second type error for the first part in critical region \(R\) is
\[
\mathbb{P}_{\{l \neq 0\}} \{\hat{\theta}_t > 0, \hat{b}_t > r^1_t - c_1\} = \mathbb{P}_{\{l \neq 0\}} \{\sigma_l l_t > r^1_t - b^1_t + a_t, \hat{b}_t > r^1_t - c_1\} =
\[
\mathbb{P} \{r^1_t - b^1_t + a_t < \inf (\sigma_l l_t, \frac{\sigma_t}{n}(W_t - W_{t-n}) + c_1)\}.
\]
and so on for the other parts:
\[
\mathbb{P}_{\{l \neq 0\}}(\{\hat{\theta}_t > 0, \hat{b}_t > r^1_t - c_1\} \cup \{\hat{\theta}_t < 0, \hat{b}_t < r^1_t + c_2\}) +
\]
\[
\mathbb{P}_{\{l \neq 0\}}(\{\hat{\theta}_t = 0, \hat{b}_t < r^1_t + a_t + c_3\} \cup \{\hat{\theta}_t = 0, \hat{b}_t > r^1_t - a_t - c_4\}).
\]
As usually, such an error is not easy to compute since \(l\) is unknown in the most of cases, we can not compute this second type error nor introduce a power function: this is not a parametric test.
6 Some comments and other developments

• Possibly terminal wealth instead of consumption, same results: roughly speaking, enough to define “feasible” terminal wealth, and to replace $\int_0^T R_s^\mu (c_s - y_s)ds$ by $R_T^\mu X_T$ in Prop. 5, Theorems 6, 8.

• Logarithm utility: optimal wealth is $\hat{X}_T = X_0 q_T (R_T^{\hat{\mu}} L_T^{\hat{\mu}})^{-1}$ and the portfolio $\theta$ is the same so the tests are identical.

• Some limits of this model and the tests sequence:
  - in pressure on prices example, the first test strongly depends on how long time the model is stationary (at least, on 20 minutes length...)
  - obviously, we definitely need that the informed agent acts.
• Nevertheless, this model seems to present some coherence between the situations (i)(ii) (in second test) and the intuition about an informed agent’s behaviour:
  - in situation (i) he buys and the trend decreases but.... it could be more convenient to say:
    the informed agent knows that the trend soon will decrease, so he buys by anticipation
  (don’t forget that portfolios have to be predictable).
  - in situation (ii), he decides to sell since he guesses that the trend will soon increase....
Thank you for your attention !!