Density Models for default risk

Monique Jeanblanc, Université d’Évry; Institut Europlace de Finance

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In many models for credit risk, it is implicitly assumed that the intensity contains all the needed information. Our goal is to present a more general setting.


Related works:
Mathematical Model

A filtered probability space \((\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})\) is given, as well as a random time \(\tau\). The default process is \(H_t = \mathbb{1}_{\tau \leq t}\), the associated filtration is \(\mathbb{H} = (\mathcal{H}_t = \sigma(t \wedge \tau), t \geq 0)\). The filtration \(\mathbb{G}\) is defined as \(\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t\). The \(\mathbb{G}\)-intensity of \(\tau\) is the process \(\lambda^G\) such that

\[
M_t = H_t - \int_0^t \lambda^G_s \, ds
\]

is a \(\mathbb{G}\)-martingale. There exists an \(\mathbb{F}\)-adapted process \(\lambda^F\) such that

\[
M_t = H_t - \int_0^{t \wedge \tau} \lambda^F_s \, ds
\]

Note that \(\lambda^G_t = \mathbb{1}_{\tau \leq t} \lambda^F_t\).

If \(X \in \mathcal{F}_T\), and \(G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)\), then

\[
\mathbb{E}(X \mathbb{1}_{T < \tau} | \mathcal{G}_t) = \mathbb{1}_{t < \tau} \frac{1}{G_t} \mathbb{E}(X G_T | \mathcal{F}_t)
\]

One can think that the knowledge of \(\lambda\) and \(G\) will allow us to have the knowledge of the conditional law of \(\tau\). We shall show that this is not the case.
Mathematical Model

A filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ is given, as well as a random time $\tau$. The default process is $H_t = 1_{\tau \leq t}$, the associated filtration is $\mathbb{H} = (\mathcal{H}_t = \sigma(t \land \tau), t \geq 0)$. The filtration $\mathbb{G}$ is defined as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. The $\mathbb{G}$-intensity of $\tau$ is the process $\lambda^G$ such that

$$M_t = H_t - \int_0^t \lambda^G_s ds$$

is a $\mathbb{G}$-martingale. There exists an $\mathbb{F}$-adapted process $\lambda^F$ such that

$$M_t = H_t - \int_0^{t \land \tau} \lambda^F_s ds$$

Note that $\lambda^G_t = 1_{t < \tau} \lambda^F_t$.

If $X \in \mathcal{F}_T$, and $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$, then

$$\mathbb{E}(X 1_{T < \tau}|\mathbb{G}_t) = 1_{t < \tau} \frac{1}{G_t} \mathbb{E}(X G_T|\mathcal{F}_t)$$

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One can think that the knowledge of \(\lambda\) and \(G\) will allow us to have the knowledge of the conditional law of \(\tau\). We shall show that this is not the case.
If $G_t = \mu_t - A_t$ is the Doob-Meyer decomposition of the survival probability, $A_t = \int_0^t p_u(u)du$ and the intensity is

$$\lambda_t^F dt = \frac{dA_t}{G_t} = \frac{p_t(t)}{G_t} dt$$
Intensity models

Models with a given intensity are constructed as follows. Let $\lambda$ be a given $\mathbb{F}$-adapted positive process and $\Theta$ a random variable independent of $\mathcal{F}_\infty$, with unit exponential law. Then

$$\tau = \inf\{t : \int_0^t \lambda_s ds \geq \Theta\}$$

has intensity equal to $\lambda$.

In that model, $\mathbb{P}(\tau > u | \mathcal{F}_t) = \mathbb{E}(e^{-\Lambda u} | \mathcal{F}_t)$ and immersion property holds:

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty) = e^{\Lambda t}$$

$$\mathbb{E}(X | \mathcal{F}_t) = \mathbb{E}(X | \mathcal{G}_t), \forall X \in \mathcal{F}_\infty$$

Under immersion property, one has

$$p_t(u) du := \mathbb{P}(\tau \in du | \mathcal{F}_t) = \mathbb{E}(\lambda_u e^{-\Lambda u} | \mathcal{F}_t) du$$

and we note that $p_t(u) = p_u(u), \forall t \geq u$. 
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\[ p_t(u)du := \mathbb{P}(\tau \in du|\mathcal{F}_t) = \mathbb{E}(\lambda_u e^{-\Lambda_u}|\mathcal{F}_t)du \]

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Assume that the intensity is known. Then, one can deduce the value of the density process \( p_t(u) \) for \( t \leq u \): by the martingale property of density, for any \( u \geq t \),

\[ p_t(u) = \mathbb{E}[p_u(u)|\mathcal{F}_t]. \]

Using the definition of \( G \), and the value of \( \lambda^G \), we obtain

\[ p_t(u) = \mathbb{E}[p_u(u)|\mathcal{F}_t] = \mathbb{E}[p_u(u)\frac{\mathbb{I}_{\{\tau \geq u\}}}{G_u}|\mathcal{F}_t] = \mathbb{E}[\lambda_u^G|\mathcal{F}_t] \]
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\[ p_t(u) = \mathbb{E}\left[p_u(u)|\mathcal{F}_t\right] = \mathbb{E}\left[p_u(u)\frac{1_{\{\tau>u\}}}{G_u}|\mathcal{F}_t\right] = \mathbb{E}[\lambda_u^G|\mathcal{F}_t] \]
We now construct probabilities $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\tau$ has a given intensity $\lambda$, and where immersion does not hold, hence, for $t > u$, the density $p^\mathbb{Q}_t(u)$ is not determined in terms of the intensity.

We start with a model where immersion property is satisfied under $\mathbb{P}$. Let $p_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t)$ and $z(u)$ a family of processes such that

(i) $(z_t(u), t \geq u)$ are positive $(\mathbb{P}, \mathbb{F})$-martingales.

Define, for $z$ positive $\mathbb{F}$-adapted process

$$Z^G_t = z_t \mathbb{1}_{\{\tau > t\}} + z_t(\tau) \mathbb{1}_{\{\tau \leq t\}}$$

and let

$$Z^F_t := \mathbb{E}(Z^G_t|\mathcal{F}_t) = z_t G_t + \int_0^t z_t(u) p_t(u) du$$

be its $\mathbb{F}$-projection. Assume that

(ii) $Z^F$ is a $(\mathbb{P}, \mathbb{F})$-martingale.

Then, $Z^G$ is a positive $G$-martingale.
We now construct probabilities $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\tau$ has a given intensity $\lambda$, and where immersion does not hold, hence, for $t > u$, the density $p_t^\mathbb{Q}(u)$ is not determined in terms of the intensity.

We start with a model where immersion property is satisfied under $\mathbb{P}$.

Let $p_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t)$ and $z(u)$ a family of processes such that

(i) $(z_t(u), t \geq u)$ are positive $(\mathbb{P}, \mathcal{F})$-martingales.

Define, for $z$ positive $\mathcal{F}$-adapted process

$$Z_t^G = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \leq t\}}$$

and let

$$Z_t^\mathbb{F} := \mathbb{E}(Z_t^G|\mathcal{F}_t) = z_t G_t + \int_0^t z_t(u)p_t(u)du$$

be its $\mathcal{F}$-projection (where $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$). Assume that

(ii) $Z^\mathbb{F}$ is a $(\mathbb{P}, \mathcal{F})$-martingale.

Then, $Z^G$ is a positive $(\mathbb{P}, G)$-martingale.
Proof: (we assume here that $G$ is continuous.) Let $s < t$.

\[
E(Z_t^G | G_s) = E(z_t \mathbb{1}_{\tau > t} | G_s) + E(z_t(\tau) \mathbb{1}_{s < \tau \leq t} | G_s) + E(z_t(\tau) \mathbb{1}_{\tau \leq s} | G_s) = I_1 + I_2.
\]

For $I_1$, we apply the standard formula

\[
I_1 = \mathbb{1}_{\tau > s} \frac{1}{G_s} E(Z_t G_t | F_s) + \mathbb{1}_{\tau > s} \frac{1}{G_s} E(z_t(\tau) \mathbb{1}_{s < \tau \leq t} | F_s),
\]

For $I_2$, we obtain

\[
I_2 = E(z_t(\tau) \mathbb{1}_{\tau \leq s} | G_s) = \mathbb{1}_{\tau \leq s} E(z_t(u) | F_s)_{u=\tau} = \mathbb{1}_{\tau \leq s} (z_s(u))_{u=\tau} = \mathbb{1}_{\tau \leq s} z_s(\tau),
\]

where the first equality holds under the immersion hypothesis and the second follows from (i). It thus suffices to show that $I_1 = Z_s \mathbb{1}_{\tau > s}$. 
Proof: (we assume here that $G$ is continuous.) Let $s < t$.

$$
\mathbb{E}(Z_t^G | G_s) = \mathbb{E}(z_t \mathbb{1}_{\tau > t} | G_s) + \mathbb{E}(z_t(\tau) \mathbb{1}_{s < \tau \leq t} | G_s) + \mathbb{E}(z_t(\tau) \mathbb{1}_{\tau \leq s} | G_s) \equiv I_1 + I_2.
$$

For $I_1$, we apply the standard formula

$$
I_1 = \mathbb{1}_{\tau > s} \frac{1}{G_s} \mathbb{E}(Z_t G_t | F_s) + \mathbb{1}_{\tau > s} \frac{1}{G_s} \mathbb{E}(z_t(\tau) \mathbb{1}_{s < \tau \leq t} | F_s),
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where the first equality holds under the immersion hypothesis and the second follows from (i). It thus suffices to show that $I_1 = z_s \mathbb{1}_{\tau > s}$.
Proof: (we assume here that $G$ is continuous.) Let $s < t$.

$$\mathbb{E}(Z^G_t | G_s) = \mathbb{E}(z_t 1_{\tau > t} | G_s) + \mathbb{E}(z_t(\tau) 1_{s < \tau \leq t} | G_s) + \mathbb{E}(z_t(\tau) 1_{\tau \leq s} | G_s) = I_1 + I_2.$$  

For $I_1$, we apply the standard formula

$$I_1 = 1_{\tau > s} \frac{1}{G_s} \mathbb{E}(Z_t G_t | F_s) + 1_{\tau > s} \frac{1}{G_s} \mathbb{E}(z_t(\tau) 1_{s < \tau \leq t} | F_s),$$

For $I_2$, we obtain

$$I_2 = \mathbb{E}(z_t(\tau) 1_{\tau \leq s} | G_s) = 1_{\tau \leq s} \mathbb{E}(z_t(u) | F_s)_{u=\tau} = 1_{\tau \leq s} (z_s(u))_{u=\tau} = 1_{\tau \leq s} z_s(\tau),$$

where the first equality holds under the immersion hypothesis and the second follows from (i). It thus suffices to show that $I_1 = z_s 1_{\tau > s}$. 


It thus suffices to show that $I_1 = z_s \mathbb{1}_{\tau > s}$ where

$$I_1 = \mathbb{1}_{\tau > s} \frac{1}{G_s} \mathbb{E}(z_t G_t | \mathcal{F}_s) + \mathbb{1}_{\tau > s} \frac{1}{G_s} \mathbb{E}(z_t(\tau) \mathbb{1}_{s < \tau \leq t} | \mathcal{F}_s),$$

Condition (ii) yields

$$\mathbb{E}(z_t G_t | \mathcal{F}_s) + \mathbb{E}(z_t(\tau) \mathbb{1}_{\tau \leq t} | \mathcal{F}_s) - \mathbb{E}(z_s(\tau) \mathbb{1}_{\tau \leq s} | \mathcal{F}_s) = z_s G_s.$$

Therefore,

$$I_1 = \mathbb{1}_{\tau > s} \frac{1}{G_s} \left( z_s G_s + \mathbb{E}(z_s(\tau) - z_t(\tau)) \mathbb{1}_{\tau \leq s} | \mathcal{F}_s \right) = z_s \mathbb{1}_{\tau > s},$$

where the last equality holds since

$$\mathbb{E}(z_s(\tau) - z_t(\tau)) \mathbb{1}_{\tau \leq s} | \mathcal{F}_s) = \mathbb{1}_{\tau \leq s} \mathbb{E}(z_s(u) - z_t(u) | \mathcal{F}_s)_{u=\tau} = 0.$$

For the last equality in the formula above, we have again used condition (i).
We assume (w.l.g.) that $Z^G_0 = 1$.

Let $\mathcal{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathcal{Q} = Z^G_t d\mathcal{P}$.

We assume that $z_t(t) = z_t$ (so that the RN density has no jump at time $\tau$).

Then, for $t \geq \theta$,

$$p^Q_t(\theta) = p_t(\theta) \frac{z_t(\theta)}{Z^P_t},$$

and the $\mathcal{Q}$-conditional survival process is defined by

$$\mathcal{Q}(\tau > t | \mathcal{F}_t) = e^{-\Lambda_t} \frac{z_t}{Z^P_t} = N^Q_t e^{-\Lambda_t}$$

(in particular, the $\mathcal{Q}$-intensity and the $\mathcal{P}$-intensity are the same.)
We assume (w.l.g.) that $Z_0^G = 1$.

Let $\mathbb{Q}$ be the probability measure defined on $\mathbb{G}_t$ by $d\mathbb{Q} = Z_t^G d\mathbb{P}$.

We assume that $z_t(t) = z_t$ (so that the RN density has no jump at time $\tau$).

Then, for $t \geq \theta$,

$$p_t^\mathbb{Q}(\theta) = p_t(\theta) \frac{z_t(\theta)}{Z_t^\mathbb{F}},$$

and the $\mathbb{Q}$-conditional survival process is defined by

$$\mathbb{Q}(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t} \frac{z_t}{Z_t^\mathbb{F}} = N_t^\mathbb{Q} e^{-\Lambda_t}$$

(in particular, the $\mathbb{Q}$-intensity and the $\mathbb{P}$-intensity are the same.)
Proof: For \( t > u \),

\[
Q(\tau > u | \mathcal{F}_t) = \frac{1}{\mathbb{E}(Z_t^G | \mathcal{F}_t)} \mathbb{E}(Z_t^G \mathbb{1}_{u < \tau} | \mathcal{F}_t)
\]

\[
\mathbb{E}(Z_t^G \mathbb{1}_{u < \tau} | \mathcal{F}_t) = \mathbb{E}(Z_t^G \mathbb{1}_{t < \tau} | \mathcal{F}_t) + \mathbb{E}(Z_t^G \mathbb{1}_{u < \tau \leq t} | \mathcal{F}_t) = z_t G_t + \mathbb{E}(z_t(\tau) \mathbb{1}_{u < \tau \leq t} | \mathcal{F}_t)
\]

\[
= z_t G_t + \int_u^t z_t(v) p_t(v) dv
\]

and the density follows by differentiation. The form of the intensity \( (\lambda_t^G = \frac{p_t^G(t)}{G_t^G}) \) follows. Indeed, if \( G_t = \mu_t - A_t \) is the Doob-Meyer decomposition of \( G \),

\[
A_t = \int_0^t p_u(u) du
\]

and the intensity is \( \lambda_t dt = \frac{dA_t}{G_t} \).
Construction of a random time with given conditional law

Let \( p(u) \) a family of positive \( \mathbb{F} \)-martingales such that

\[
\int_{0}^{\infty} p_t(u) du = 1, \quad \forall t
\]

One can construct (on an extended space) a probability \( Q \) and a random time \( \tau \) such that

\[
Q|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}
\]

\[
Q(\tau \in du|\mathcal{F}_t) = p_t(u) du
\]

as follows:

- Construct \( Q^* \) and \( \tau \) such that \( \tau \) is independent from \( \mathcal{F}_\infty \) and \( Q(\tau \in du) = p_0(u) du \)
- Set \( dQ|_{\mathcal{F}_t \vee \sigma(\tau)} = (p_t(\tau))^{-1} dQ^* \)
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Construction of a random time with given Conditional Survival Probability

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a given probability space. Construct (on an extended space) a probability \(\mathbb{Q}\) and a random time \(\tau\) such that

\[
\mathbb{Q}\big|_{\mathcal{F}_t} = \mathbb{P}\big|_{\mathcal{F}_t}
\]

\[
\mathbb{Q}(\tau > t|\mathcal{F}_t) = G_t
\]

where \(G\) is a **given** \(\mathcal{F}\)-supermartingale. One recall that any supermartingale admits a multiplicative decomposition as \(G_t = N_t D_t = N_t e^{-\Lambda_t}\) where \(D\) (resp. \(\Lambda\)) is decreasing (resp. increasing) In what follows, we assume that \(G\) is continuous, and \(0 \leq G_t < 1\) for \(t > 0\).
Construction of a random time with given Conditional Survival Probability

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a given probability space. Construct (on an extended space) a probability \(\mathbb{Q}\) and a random time \(\tau\) such that

\[
\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t} \\
\mathbb{Q}(\tau > t|\mathcal{F}_t) = G_t
\]

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In what follows, we assume that \(G\) is continuous, and \(0 \leq G_t < 1\) for \(t > 0\).
Construction of a random time with given Conditional Survival Probability

Let us start with a model in which \( P(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t} \), where \( \Lambda_t = \int_0^t \lambda_s ds \) and let \( N \) be an \( \mathbb{F} \)-local martingale such that \( 0 \leq N_t e^{-\Lambda_t} \leq 1 \).

There exists a \( \mathbb{G} \)-martingale \( L \) such that, setting \( d\mathbb{Q} = Ld\mathbb{P} \)

(i) \( \mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty} \)

(ii) \( \mathbb{Q}(\tau > t|\mathcal{F}_t) = N_t e^{-\Lambda_t} \)

The \( \mathbb{G} \)-adapted process \( L \)

\[
L_t = \ell_t 1_{t<\tau} + \ell_t(\tau) 1_{\tau \leq t}
\]

is a martingale if for any \( u, (\ell_t(u), t \geq u) \) is a martingale and if \( \mathbb{E}(L_t|\mathcal{F}_t) \) is a \( \mathbb{F} \)-martingale. Then, (i) is satisfied if

\[
1 = \mathbb{E}(L_t|\mathcal{F}_t) = \ell_t G_t + \int_0^t \ell_t(u) \lambda_u e^{-\Lambda_u} du
\]

and (ii) implies that \( \ell = N \) and \( \ell_t(t) = \ell_t \).
Conditional Survival Probability

Let us start with a model in which $\mathbb{P}(\tau > t | \mathcal{F}_t) = e^{-\Lambda_t}$, where $\Lambda_t = \int_0^t \lambda_s ds$ and let $N$ be an $\mathbb{F}$-local martingale such that $0 \leq N_t e^{-\Lambda_t} \leq 1$.

There exists a $\mathbb{G}$-martingale $L$ such that, setting $dQ = Ld\mathbb{P}$

$$Q|\mathcal{F}_\infty = \mathbb{P}|\mathcal{F}_\infty$$

$$Q(\tau > t|\mathcal{F}_t) = N_t e^{-\Lambda_t}$$

The $\mathbb{G}$-adapted process $L$

$$L_t = \ell_t 1_{t<\tau} + \ell_t(\tau) 1_{\tau \leq t}$$

is a martingale if for any $u$, $(\ell_t(u), t \geq u)$ is a martingale and if $\mathbb{E}(L_t|\mathcal{F}_t)$ is a $\mathbb{F}$-martingale. Then, (i) is satisfied if

$$1 = \mathbb{E}(L_t|\mathcal{F}_t) = \ell_t G_t + \int_0^t \ell_t(u) \lambda_u e^{-\Lambda_u} du$$

and (ii) implies that $\ell = N$ and $\ell_t(t) = \ell_t$. 


Conditional Survival Probability

Let us start with a model in which \( \mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t} \), where \( \Lambda_t = \int_0^t \lambda_s ds \) and let \( N \) be an \( \mathbb{F} \)-local martingale such that \( 0 \leq N_t e^{-\Lambda_t} \leq 1 \).

There exists a \( \mathbb{G} \)-martingale \( L \) such that, setting \( d\mathbb{Q} = Ld\mathbb{P} \)

(i) \( \mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty} \)

(ii) \( \mathbb{Q}(\tau > t|\mathcal{F}_t) = N_t e^{-\Lambda_t} \)

The \( \mathbb{G} \)-adapted process \( L \)

\[ L_t = \ell_t \mathbb{1}_{t<\tau} + \ell_t(\tau) \mathbb{1}_{\tau \leq t} \]

is a martingale if for any \( u \), \( (\ell_t(u), t \geq u) \) is a martingale and if \( \mathbb{E}(L_t|\mathcal{F}_t) \) is a \( \mathbb{F} \)-martingale. Then, (i) is satisfied if

\[ 1 = \mathbb{E}(L_t|\mathcal{F}_t) = \ell_t e^{-\Lambda_t} + \int_0^t \ell_t(u) \lambda_u e^{-\Lambda_u} du \]

and (ii) implies that \( \ell = N \) and \( \ell_t(t) = \ell_t \).
It remains to find a family of martingales $\ell(u)$ such that

$$
\ell_u(u) = N_u
$$

$$
1 = N_t e^{-\Lambda_t} + \int_0^t \ell_t(u) \lambda_u e^{-\Lambda_u} du
$$

We choose

$$
\ell_t(u) = \frac{N_u}{1-G_u} (1-G_t) \exp \left( - \int_u^t \frac{G_s}{1-G_s} \lambda_s ds \right)
$$

Then, $\mathbb{Q}[\tau \leq u | \mathcal{F}_t] = M^u_t$ for $0 \leq u \leq t \leq \infty$ where

$$
M^u_t = (1-G_t) \exp \left( - \int_u^t \frac{G_s}{1-G_s} \lambda_s ds \right) \quad 0 \leq u \leq t \leq \infty
$$

One can also construct other martingales $M^u_t$ which give a solution (i.e., families of $[0, 1]$-valued martingales such that $u \rightarrow M^u_t$ is decreasing and $M^t_t = 1 - G_t$).
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One can also construct other martingales $M^u_t$ which give a solution (i.e., families of $[0, 1]$-valued martingales such that $u \rightarrow M_t^u$ is decreasing and $M_t^t = 1 - G_t$).
Extension of Cox processes

Let $\lambda$ be a strictly positive $\mathbb{F}$-adapted process, and $\Lambda_t = \int_0^t \lambda_s ds$.

Let $\Theta$ be a strictly positive random variable whose conditional distribution w.r.t. $\mathbb{F}$ admits a density w.r.t. the Lebesgue measure, i.e., there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions $\gamma_t(u)$ such that $\mathbb{P}(\Theta > \theta | \mathcal{F}_t) = \int_\theta^\infty \gamma_t(u) du$.

Let $\tau = \inf\{t > 0 : \Lambda_t \geq \Theta\}$.

Then $\tau$ admits the density

$$p_t(\theta) = \lambda_\theta \gamma_t(\Lambda_\theta) \text{ if } t \geq \theta \quad \text{ and } \quad p_t(\theta) = \mathbb{E}[\lambda_\theta \gamma_\theta(\Lambda_\theta) | \mathcal{F}_t] \text{ if } t < \theta.$$
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Proof: By definition and by the fact that $\Lambda$ is strictly increasing and absolutely continuous, we have for $t \geq \theta$,

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \mathbb{P}(\Theta > \Lambda_\theta | \mathcal{F}_t) = \int_{\Lambda_\theta}^{\infty} \gamma_t(u)du = \int_{\theta}^{\infty} \gamma_t(\Lambda_u)d\Lambda_u = \int_{\theta}^{\infty} \gamma_t(\Lambda_u)\lambda_u du,$$

which implies $p_t(\theta) = \lambda_\theta \gamma_t(\Lambda_\theta)$. The martingale property of $p$ gives the whole density.
Conversely, if we are given a density $p$, and hence an associated process
\[ \Lambda_t = \int_0^t \lambda_s \, ds \] with \( \lambda_s = \frac{p_s(s)}{G_s} \), then it is possible to find a threshold \( \Theta \) such that \( \tau \) has \( p \) as density.

We denote by \( \Lambda^{-1} \) the inverse of the strictly increasing process \( \lambda \).

We let \( \Lambda_t = \int_0^t \frac{p_s(s)}{G_s} \, ds \) and \( \Theta = \Lambda_{\tau} \). Then \( \Theta \) has a density \( \gamma \) with respect to \( \mathbb{F} \) given by

\[
\gamma_t(\theta) = \mathbb{E}\left[p_{t \vee \Lambda_\theta^{-1}}(\Lambda_\theta^{-1}) \frac{1}{\lambda_{\Lambda_\theta^{-1}}} | \mathcal{F}_t \right].
\]
Proof: We set $\Theta = \Lambda_\tau$ and compute the density of $\Theta$ w.r.t. $\mathbb{F}$

$$
\mathbb{P}(\Theta > \theta | \mathcal{F}_t) = \mathbb{P}(\Lambda_\tau > \theta | \mathcal{F}_t) = \mathbb{P}(\tau > t, \Lambda_\tau > \theta | \mathcal{F}_t) + \mathbb{P}(\tau \leq t, \Lambda_\tau > \theta | \mathcal{F}_t)
$$

$$
= \mathbb{E}[- \int_t^\infty \mathbb{1}_{\{\Lambda_u > \theta\}} dG_u | \mathcal{F}_t] + \int_0^t \mathbb{1}_{\{\Lambda_u > \theta\}} p_t(u) du
$$

$$
= \mathbb{E} \left[ \int_t^\infty \mathbb{1}_{\{\Lambda_u > \theta\}} p_u(u) du | \mathcal{F}_t \right] + \int_0^t \mathbb{1}_{\{\Lambda_u > \theta\}} p_t(u) du
$$

where the last equality comes from the fact that $(G_t + \int_0^t p_u(u) du, t \geq 0)$ is an $\mathbb{F}$-martingale. Note that since the process $\Lambda$ is continuous and strictly increasing, also is its inverse. Hence

$$
\mathbb{E} \left[ \int_\theta^\infty p_{\Lambda^{-1}_s \vee t}(\Lambda^{-1}_s) \frac{1}{\lambda_{\Lambda^{-1}_s}} ds | \mathcal{F}_t \right] = \mathbb{E} \left[ \int_\Lambda^{-1}_\theta \vee t(s) \frac{1}{\lambda_s} d\Lambda_s | \mathcal{F}_t \right]
$$

$$
= \mathbb{E} \left[ \int_0^\infty \mathbb{1}_{\{s > \Lambda^{-1}_\theta\}} p_s \vee t(s) ds | \mathcal{F}_t \right] = \mathbb{E} \left[ \int_0^\infty \mathbb{1}_{\{\Lambda_s > \theta\}} p_s \vee t(s) ds | \mathcal{F}_t \right],
$$

which equals $\mathbb{P}(\Theta > \theta | \mathcal{F}_t)$. 

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Example

Let us give an (explicit?) example of density processes.

**Gaussian Processes: enlargement with** $\int_0^\infty f(s)dB_s$ **and change of variable**

In a first step, we set $L = \int_0^\infty f(s)dB_s$ and we compute the conditional law of $L$ with respect to $\mathcal{F}_t$.

Then, we set $\tilde{L} = \psi(L)$ where $\psi$ is a strictly increasing function and we compute $\mathbb{P}(\tilde{L} \in dx|\mathcal{F}_t) =: G_t(u)$.

Then, we construct $\tau$ (on an enlarged space) with given conditional law.

$$G_t(u) = \mathcal{N}\left(\frac{m_t - \psi^{-1}(u)}{\sigma(t)}\right)$$

where $\psi$ is a strictly increasing function, $m_t = \int_0^t f(s)dB_s$, $\sigma(t) = \int_0^\infty f^2(s)ds$ then

$$p_t(u) = \frac{1}{\sigma(t)\sqrt{2\pi}} \frac{1}{\psi'(\psi^{-1}(u))} \exp - \frac{(m_t - \psi^{-1}(u))^2}{2\sigma^2(t)}$$
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\]
Other examples can be constructed as

\[ G_t(u) = \mathbb{E}(e^{-uX} | \mathcal{F}_t) \]

where \( X \) is a non-negative r.v.
Defaultable Zero-Coupon Bonds

A defaultable zero-coupon with maturity $T$ associated with the default time $\tau$ is an asset which pays one monetary unit at time $T$ if (and only if) the default has not occurred before $T$. We assume that $\mathbb{P}$ is the pricing measure.

\[
D(t, T) := \mathbb{P}(\tau > T \mid G_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{P}(\tau > T \mid \mathcal{F}_t)}{G_t} = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_\mathbb{P}(N_Te^{-\Lambda T} \mid \mathcal{F}_t)}{G_t}
\]

However, using a change of probability, one can get rid of the martingale part $N$, assuming that there exists $p$ such that

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$$\mathbb{P}(\tau > \theta \mid \mathcal{F}_t) = \int_\theta^\infty p_t(u)du$$
Let $\mathbb{P}^*$ be defined as

$$\left. d\mathbb{P}^* \right|_{G_t} = Z_t^* d\mathbb{P} \mid_{G_t}$$

where $Z^*$ is the $(\mathbb{P}, G)$-martingale defined as

$$Z_t^* = \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{t\geq\tau\}} \lambda_{\tau} e^{-\Lambda_{\tau}} \frac{N_t}{p_t(\tau)}$$

Then,

(a) Immersion property holds under $\mathbb{P}^*$,
(b) $d\mathbb{P}^* \mid_{\mathcal{F}_t} = N_t d\mathbb{P} \mid_{\mathcal{F}_t}$
(c) $\mathbb{P}^*$ and $\mathbb{P}$ coincide on $G_\tau$.

However, $\mathbb{P}^*$ and $\mathbb{P}$ do not coincide on $\mathcal{F}_\infty$. 
Let $\mathbb{P}^*$ be defined as

$$d\mathbb{P}^*|_{\mathcal{G}_t} = Z^*_t d\mathbb{P}|_{\mathcal{G}_t}$$

where $Z^*$ is the $(\mathbb{P}, \mathcal{G})$-martingale defined as

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Then,

(a) Immersion property holds under $\mathbb{P}^*$,

(b) $d\mathbb{P}^*|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t}$

(c) $\mathbb{P}^*$ and $\mathbb{P}$ coincide on $\mathcal{G}_\tau$.

However, $\mathbb{P}^*$ and $\mathbb{P}$ do not coincide on $\mathcal{F}_\infty$.
Proof: We prove first that $d\mathbb{P}^*|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t}$

$$
\mathbb{E}_\mathbb{P}(Z^*_t|\mathcal{F}_t) = G_t + \int_0^t \lambda_u e^{-\Lambda u} \frac{N_t}{p_t(u)} p_t(u) du = N_t e^{-\Lambda_t} + N_t (1 - e^{-\Lambda_t}) = N_t
$$

We compute the $\mathbb{P}^*$ conditional law of $\tau$. For $t > \theta$,

$$
\mathbb{P}^*(\theta < \tau|\mathcal{F}_t) = \frac{1}{N_t} \mathbb{E}_\mathbb{P}(Z^*_t \mathbb{1}_{\{\theta < \tau\}}|\mathcal{F}_t) = \frac{1}{N_t} \mathbb{E}_\mathbb{P}(\mathbb{1}_{\{t < \tau\}} + \mathbb{1}_{\{t \geq \tau > \theta\}} \lambda_\tau e^{-\Lambda_\tau} \frac{N_t}{p_t(\tau)}|\mathcal{F}_t)
$$

$$
= \frac{1}{N_t} \left( N_t e^{-\Lambda_t} + \int_\theta^t \lambda_u e^{-\Lambda u} \frac{N_t}{p_t(u)} p_t(u) du \right) = e^{-\Lambda_\theta}
$$

which proves that immersion holds true under $\mathbb{P}^*$, and the intensity of $\tau$ is the same under $\mathbb{P}$ and $\mathbb{P}^*$. It follows that

$$
\mathbb{E}^*(X \mathbb{1}_{\{T < \tau\}}|\mathcal{G}_t) = \mathbb{1}_{\{t < \tau\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*(e^{-\Lambda T} X|\mathcal{F}_t) = \mathbb{E}_\mathbb{P}(X \mathbb{1}_{\{T < \tau\}}|\mathcal{G}_t)
$$

Note that, if the intensity is the same under $\mathbb{P}$ and $\mathbb{P}^*$, its dynamics under $\mathbb{P}^*$ will involve a change of driving process, since $\mathbb{P}$ and $\mathbb{P}^*$ do not coincide on $\mathcal{F}_\infty$. 
Let us now study the pricing of a recovery. Let $Z$ be an $\mathbb{F}$-predictable bounded process.

\[
\mathbb{E}_\mathbb{P}(Z_\tau \mathbb{1}_{\{t<\tau \leq T\}} | \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_\mathbb{P}\left(- \int_t^T Z_u dG_u | \mathcal{F}_t\right)
\]

\[
= \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_\mathbb{P}\left(\int_t^T Z_u N_u \lambda_u e^{-\Lambda u} du | \mathcal{F}_t\right)
\]

\[
= \mathbb{1}_{\{t<\tau\}} \frac{1}{e^{-\Lambda t}} \mathbb{E}^*\left(\int_t^T Z_u \lambda_u e^{-\Lambda u} du | \mathcal{F}_t\right)
\]

\[
= \mathbb{E}^*\left(Z_\tau \mathbb{1}_{\{t<\tau \leq T\}} | \mathcal{G}_t\right)
\]
The problem is different for pricing a recovery paid at maturity, i.e. for $X \in \mathcal{F}_T$

$$
\mathbb{E}_P(X \mathbb{1}_{\tau<T} | \mathcal{G}_t) = \mathbb{E}_P(X | \mathcal{G}_t) - \mathbb{E}_P(X \mathbb{1}_{\tau>T} | \mathcal{G}_t)
$$

$$
= \mathbb{E}_P(X | \mathcal{G}_t) - \mathbb{1}_{\{\tau>t\}} \frac{1}{e^{-\Lambda t}} \mathbb{E}^*(X e^{-\Lambda T} | \mathcal{F}_t)
$$

If both quantities $\mathbb{E}_P(X \mathbb{1}_{\tau<T} | \mathcal{G}_t)$ and $\mathbb{E}^*(X \mathbb{1}_{\tau<T} | \mathcal{G}_t)$ are the same, this would imply that $\mathbb{E}_P(X | \mathcal{G}_t) = \mathbb{E}^*(X | \mathcal{F}_t)$ i.e., immersion holds under $\mathbb{P}$. 


Misspecification of the Information

In this section, we point out that the price of a derivative product written on a default $\tau$ depends strongly on the other default and the hedging strategies have to be constructed with the full observation. Let us study the following toy model.

Two default times $\tau_1, \tau_2$ let $G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)$

We consider two filtrations $\mathbb{H}^1$ an $\mathbb{H} = \mathbb{H}^1 \lor \mathbb{H}^2$

The price of a DZC is

$$\bar{D}^1(t, T) = \mathbb{P}(\tau_1 > T|\mathcal{H}^1_t) = (1 - H^1_t) \frac{G(T, 0)}{G(t, 0)}$$

$$D^1(t, T) = \mathbb{P}(\tau_1 > T|\mathcal{H}_t) = (1 - H^1_t) \left( (1 - H^2_t) \frac{G(T, t)}{G(t, t)} + H^2_t \frac{\partial_2 G(T, \tau_2)}{\partial_2 G(t, \tau_2)} \right)$$
The un-informed agent knows only $\mathbb{H}^1$. He will hedge the contingent claim $C = h(\tau_1) \mathbb{1}_{\tau_1 > T} + k \mathbb{1}_{\tau_1 > T}$ thinking the market is complete, with an initial wealth $x = \mathbb{E}(C)$ buying $\zeta_t \text{DZC}$, so that

$$\hat{X}_T := x + \int_0^T \zeta_s \hat{D}(t, T) = C$$

and he will invest $\zeta_t^0 = X_t - \zeta_t \hat{D}(t, T)$ in the savings account in a self financing way. However, his "real" wealth will be $X_t = \zeta_t^0 + \zeta_t D(t, T)$ and the strategy is not self-financing. The cost of refinancing is

$$dX_t - \zeta_t dD(t, T) = d\zeta_t^0 + D(t, T)d\zeta_t^0 + d\langle \zeta, D(\cdot, T) \rangle_t$$

If he uses a self financing strategy, his terminal wealth will be $X_T^* = x + \int_0^T \zeta_t dD(t, T)$ and the associated cost is $C - X_T^* = \int_0^T \zeta_t (d\hat{D}(t, T) - dD(t, T))$. One has $\mathbb{E}(C - X_T^*) = 0$
Thank you for your attention