Modeling Information for Credit Risk

M. L. Bedini

ITN - UBO, Brest

Jena, June 2010
Introduction and Motivation

\((\Omega, \mathcal{F}, \mathbb{P})\) complete probability space. \(B = (B_t, t \geq 0)\) BM, \(\mathbb{F} = \mathbb{F}^B\).

\(\tau \sim \mathcal{E}_\lambda\) r.v. independent of \(B\), \(\left( H_t \triangleq \mathbb{I}_{\{\tau \leq t\}}, t \geq 0 \right)\), \(\mathbb{H} = \mathbb{F}^H\).

---

**CEV Model with Jump to Default (Campi, Polbennikov, Sbuelz ’09)**

\[
dS_t = S_t \mu(S_t) \, dt + \sigma S_t^\rho \, dB_t - S_t \, dH_t, \quad t \geq 0, \ S_0 > 0, \ \rho < 1
\]

\[
d\hat{S}_t = \hat{S}_t \mu(\hat{S}_t) \, dt + \sigma \hat{S}_t^\rho \, dB_t, \quad t \geq 0, \ \hat{S}_0 > 0, \ \rho < 1
\]

**Constant Elasticity of Variance (CEV)**

- Default time for \(\hat{S}\): \(\eta \triangleq \inf \left\{ t > 0 : \hat{S}_t = 0 \right\}\), \(\mathbb{P}\{\eta < +\infty\} = 1\).
- Default time for the company: \(\theta \triangleq \tau \wedge \eta\). \(\mathbb{P}\{\theta < +\infty\} = 1\).
- \(\hat{S}_t = S_t\) for \(t \in [0, \theta]\). \(S_t = 0, \forall t \geq \theta\).
- \(\theta\) is predictable on \(\{\eta \leq \tau\}\) and totally inaccessible on \(\{\eta > \tau\}\) w.r.t. \(\mathbb{F} \vee \mathbb{H}\).
**Objective**

Our approach aims to give a qualitative description of the information on $\tau$ before the default, thus making $\tau$ “a little bit less inaccessible”.

**Information-based Approach (Brody, Hughston, Macrina '07)**

Let $T, \sigma > 0$, $H_T \sim B(1, p)$, $(\beta^T_t, 0 \leq t \leq T)$ Brownian bridge.  

$$\xi_t = \sigma t H_T + \beta^T_t$$

The market filtration $\mathbb{F}^\xi$ is generated by the information process.

In our approach the information will be carried by $\left(\beta_t \triangleq \beta^T_t, t \geq 0\right)$ a Brownian bridge between 0 and 0 on the stochastic interval $[0, \tau]$. 
Introduction and Motivation

In the CEV Model with Jump to Default and in the Hazard Process Approach to Credit-Risk we deal with two filtrations:

- $\mathcal{F}$: the reference filtration that models the common knowledge of financial agents on the market (ex: Brownian filtration). $\tau$ is not an $\mathcal{F}$-stopping time.

- $\mathcal{H}$: the information on the default event generated by the default indicator process $\left( H_t \triangleq \mathbb{I}_{\{\tau \leq t\}}, \, t \geq 0 \right)$.

However, before the default we have some more information on $\tau$ (there are periods in which we are “quite confident” the default will not occur). Thus, in our market model we will deal with:

- $\mathcal{F}$: the reference filtration.

- $\mathcal{F}^{\beta}$: the information on the default event generated by the information process $(\beta_t, \, t \geq 0)$. 
(Ω, F, P) complete probability space, \(\mathcal{N}_P\) the collection of the P-null sets. 
\(W = \{W_t\}_{t \geq 0}\) is a standard BM. \(\tau : \Omega \rightarrow (0, +\infty)\) random variable. 
\(F(t) \triangleq P\{\tau \leq t\} \).

**Assumption**
\(\tau\) is independent of \(W\).

**Definition**
The process \(\beta = \{\beta_t\}_{t \geq 0}\) is called *Information process* :

\[
\beta_t \triangleq W_t - \frac{t}{\tau \vee t} W_{\tau \vee t}
\]  (1)
Proposition

- $\tau$ is an $F^\beta$-stopping time but, in general, it is not an $\mathcal{O}^\beta$-stopping time.
- For all $t > 0$, $\{\beta_t = 0\} = \{\tau \leq t\}$, $P$-a.s.
- $\beta$ is an $F^\beta$-Markov process.
Properties

**Theorem**

Let \( t > 0, \ g : \mathbb{R}^+ \rightarrow \mathbb{R} \) a Borel function such that \( \mathbb{E} \left[ |g(\tau)| \right] < +\infty \). Then, \( \mathbb{P} \)-almost surely on \( \{\tau > t\} \)

\[
\mathbb{E} \left[ g(\tau) \mathbb{1}_{\{\tau > t\}} \mid \mathcal{F}_t^\beta \right] = \frac{\int_t^{+\infty} g(r) \varphi_t(r, \beta_t) \, dF(r)}{\int_t^{+\infty} \varphi_t(r, \beta_t) \, dF(r)} \mathbb{1}_{\{\tau > t\}}
\]

(2)

where \( \varphi_t(r, x), \ r > t > 0, \ x \in \mathbb{R} \) denotes the density of

\[
\beta_t^r \sim \mathcal{N} \left( 0, \frac{t(r - t)}{r} \right)
\]
Define the Brownian bridge \( b = \{ b_s \}_{0 \leq s \leq 1} \) between 0 and 0 on the time interval \([0, 1]\) by

\[
b_s \triangleq \frac{1}{\sqrt{\tau}} \beta_{s \tau}
\]

The local time \( L(a) = \{ L_s(a) \}_{0 \leq s \leq 1} \) at \( a \in \mathbb{R} \) of the process \( b \) is well defined. Then

\[
(\beta_t, 0 \leq t \leq \tau) = \left( \sqrt{\tau b_{\frac{t}{\tau}}, 0 \leq t \leq \tau} \right)
\]

The local time \( l^x = \{ l^x_t \}_{t \geq 0} \) of the information process \( \beta \) at level \( x \) is then defined as

\[
l^x_t \triangleq \sqrt{\tau} L_{\frac{t}{\tau}} \left( \frac{x}{\sqrt{\tau}} \right)
\]

If \( x = 0 \) we will write \( l_t \) instead of \( l^0_t \).
Suppose $F(t)$ admits a continuous density with respect to the Lebesgue measure: $dF(t) = f(t)dt$. Then $\tau$ is a totally inaccessible stopping time with respect to $\mathbb{F}^\beta$ and the compensator $K = \{K_t\}_{t \geq 0}$ of $(\mathbb{1}_{\{\tau \leq t\}}, \ t \geq 0)$ is given by

$$K_t = \int_0^{\tau \wedge t} \int_{r=0}^{+\infty} \frac{f(r)}{\int_r^{+\infty} \varphi_r(v,0) f(v)dv} dl_r$$

(3)

where $l_t$ is the local time at 0 of the process $\beta$ at time $t$. 

where $l_t$ is the local time at 0 of the process $\beta$ at time $t$. 

M. L. Bedini (ITN - UBO, Brest)  
Modeling Information for Credit Risk  
Jena, June 2010
Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the usual condition. $F_t(dr) \triangleq \mathbf{P} \{\tau \in dr|\mathcal{F}_t\}$

**Assumption**

$\mathcal{F}_{+\infty} \vee \sigma \{\tau\}$ is independent of $\mathcal{W}$.

**Definition**

The filtration $\mathbb{G} = \left( G_t \triangleq \mathcal{F}_t \vee \mathcal{F}_t^\beta \right)_{t \geq 0}$ is called *enlarged filtration*.
Lemma

The process $\beta$ satisfies the following

$$P \left\{ \beta_{t+h} \in \Gamma | G_t \right\} = P \left\{ \beta_{t+h} \in \Gamma | \mathcal{F}_t \vee \sigma \{ \beta_t \} \right\}, \text{ } P - a.s.$$ 

for all $t$, $h \geq 0$ and $\Gamma \in \mathcal{B}(\mathbb{R})$.

Lemma

Denoting by $S_{t,v} \triangleq \int_t^{+\infty} \varphi_t (r, \beta_t) F_v (dr)$

$$P \left\{ \tau \in du | \mathcal{F}_v \vee \mathcal{F}_t^\beta \right\} \mathbb{I} \{ \tau > t \} = \varphi_t (u, \beta_t) F_v (du) S_{t,v}^{-1} \mathbb{I} \{ \tau > t \}, \text{ } P - a.s.$$
Lemma

Let $T > 0$ be a constant and $Y$ be an $\mathcal{F}_T$-measurable r.v. Let $g$ be a bounded Borel function s.t. $E[|g(\tau, Y)|] < +\infty$. Then

$$E[g(\tau, Y) | G_t] \mathbb{1}_{\{\tau > t\}} = E\left[ \frac{\int_t^{+\infty} g(r, Y) \varphi_t(r, \beta_t) F_T(dr)}{\int_t^{+\infty} \varphi_t(r, \beta_t) F_T(dr)} | G_t \right] \mathbb{1}_{\{\tau > t\}}, P-a.s.$$
A Credit Default Swap (CDS) is a financial contract between a buyer and a seller:

- The *buyer* wants to insure the risk of default. *Protection leg*:
  \[ \mathbb{I}_{\{t < \tau \leq T\}} \delta(\tau) \]

- The *seller* is paid by the buyer to provide such insurance. *Fee leg*:
  \[ \mathbb{I}_{\{\tau > t\}} k [ (\tau \land T) - t ] \]

The price \( S_t (k, \delta, T) \) of the CDS is equal to:

\[
S_t (k, \delta, T) = \mathbb{E} \left[ \mathbb{I}_{\{t < \tau \leq T\}} \delta(\tau) | \hat{F}_t \right] - \mathbb{E} \left[ \mathbb{I}_{\{\tau > t\}} k [ (\tau \land T) - t ] | \hat{F}_t \right]
\]

where \( \hat{F} = (\hat{F}_t)_{t \geq 0} \) is the (yet unspecified) market filtration.
Lemma

If the market filtration is $\mathcal{G}$, for $t \in [s, T]$ we have

$$S_t(k, \delta, T) = \mathbb{I}_{\{\tau > t\}} \left[ -\int_t^T \delta(r) d\Psi_t(r) - k \int_t^T \Psi_t(r) dr \right]$$

Where $\Psi_t(r) \triangleq \mathbb{P}\{\tau > r|\mathcal{G}_t\}$.

Lemma

If the market filtration is $\mathcal{H} = \left( \mathcal{H}_t \triangleq \sigma\{t \wedge \tau\} \right)_{t \geq 0}$, for $t \in [s, T]$ we have

$$S_t(k, \delta, T) = \mathbb{I}_{\{\tau > t\}} \left[ -\int_t^T \delta(r) dG(r) - k \int_t^T G(r) dr \right]$$

Where $G(r) \triangleq \mathbb{P}\{\tau > r\}$.
In an arbitrage-free incomplete market we observe the price process of the share of a company:

\[
dS_t = S_t \mu(S_t) \, dt + \sigma S_t^\rho \, dB_t - S_t \, dH_t, \quad t \geq 0, \ S_0 > 0, \ \rho < 1
\]

\[
d\hat{S}_t = \hat{S}_t \mu(\hat{S}_t) \, dt + \sigma \hat{S}_t^\rho \, dB_t
\]

- \( \tau \in L^0(\Omega, \mathcal{F}, P) \), \( H_t \triangleq \mathbb{I}_{\{\tau \leq t\}} \).
- Default time for \( \hat{S} \): \( \eta \triangleq \inf \left\{ t > 0 : \hat{S}_t = 0 \right\} \).
- Default time: \( \theta \triangleq \tau \wedge \eta \).

This model is used to describe the positive link between default and equity volatility.
Application to a CEV Model with Jump to Default

Note that \( \hat{S}_t = S_t \) for \( t \in [0, \theta] \). Furthermore we have that \( \mathbb{P}\{\theta < +\infty\} = 1 \) and \( S_t = 0, \forall t \geq \theta \).

Let \((W_t, t \geq 0)\) be a BM.

**Assumption**

\( \tau, B \) and \( W \) are mutually independent.

- \( \mathcal{F} = \mathcal{F}^B \) is the filtration generated by the BM \((B_t, t \geq 0)\).
- \( \mathcal{F}^\beta \) is the filtration generated by the information process \( \beta_t = W_t - \frac{t}{\tau \lor t} W_{\tau \lor t} \).
- \( \mathcal{G} = \mathcal{F} \lor \mathcal{F}^\beta \) is the market filtration.
- \( \theta \) is \( \mathcal{G} \)-predictable on \( \{\eta \leq \tau\} \) and \( \mathcal{G} \)-totally inaccessible on \( \{\eta > \tau\} \).
Application to a CEV Model with Jump to Default

In the following we will assume for convenience that:

- $\tau$ is exponentially distributed with parameter $\lambda > 0$
- $\mu(x) = \frac{1}{2} \rho \sigma^2 x^{2\rho-1}$, $x > 0$

So that, up to the predictable stopping time $\eta > 0$

$$\hat{S}_t = \left( S_0^{1-\rho} + (1 - \rho) \sigma B_t \right)^{1/(1-\rho)}$$

Then we have that

$$\mathbb{P}\{\eta \in du | \mathcal{F}_t\} = F^\eta_t(du) \triangleq \mathbb{I}_{\{\eta \leq t\}} \delta_\eta(du) + \mathbb{I}_{\{\eta > t\}} h(u - t, B_t) \, du$$

where

$$h(u, x) = \frac{1}{u^{3/2}} \left( x + \frac{S_0^{1-\rho}}{(1 - \rho)\sigma} \right) \varphi \left( \frac{1}{\sqrt{u}} \left( x + \frac{S_0^{1-\rho}}{(1 - \rho)\sigma} \right) \right) \mathbb{I}_{\{u > 0\}}$$
Let $g$ be a bounded Borel function such that $\text{E} \left[ |g(\theta)| \right] < +\infty$. Then

$$\text{E} \left[ g(\theta) \mathbb{I}_{\{\theta > t\}} \mid \mathcal{G}_t \right] = \int_t^{+\infty} \int_t^{+\infty} \frac{g(r \land u) \varphi_t(r, \beta_t) \lambda e^{-\lambda r} dr}{\int_t^{+\infty} \varphi_t(r, \beta_t) \lambda e^{-\lambda r} dr} F_t^\eta(du) \mathbb{I}_{\{\theta > t\}}$$

In particular

$$\text{P} \left\{ t < \theta \leq T \mid \mathcal{G}_t \right\} = \text{E} \left[ \mathbb{I}_{\{t < \theta \leq T\}} \mid \mathcal{G}_t \right] = \left[ \text{P} \left\{ \tau \leq T \mid \mathcal{F}_t^\beta \right\} \text{P} \left\{ \eta > T \mid \mathcal{F}_t \right\} + \text{P} \left\{ \eta \leq T \mid \mathcal{F}_t \right\} \right] \mathbb{I}_{\{\theta > t\}}$$
The $\mathbb{G}$-compensator $(K^G_t, t \geq 0)$ of $\mathbb{I}_{\{\theta \leq t\}}$ is given by:

$$K^G_t = \int_0^t \left( \left( \mathbb{E} \left[ K_{+\infty} | \mathcal{F}_{r^-}^\beta \right] - K_r \right) \mathbb{I}_{[\eta, +\infty)}(dr) + \mathbb{I}_{\{\eta > r\}} dK_r \right), \quad t \geq 0$$
If the default protection is equal to \( \delta > 0 \), the market price \( S_t(k, \delta, T) \) of a CDS is equal to

\[
S_t(k, \delta, T) = \mathbb{I}_{\{\theta > t\}} \delta \left[ \Phi_t(T) (1 - F^\eta_t(T)) + F^\eta_t(T) \right] + \\
+ \mathbb{I}_{\{\theta > t\}} k \left[ t - \int_t^T \left( \int_t^u r d\Phi_t(r) + u (1 - \Phi_t(u)) \right) F^\eta_t(du) \right] + \\
- \mathbb{I}_{\{\theta > t\}} k \left( \int_t^T r d\Phi_t(r) + T (1 - \Phi_t(T)) \right) (1 - F^\eta_t(T))
\]

(where \( \Phi_t(r) = \mathbb{P} \left\{ \tau \leq r | \mathcal{F}_t^\beta \right\} \)).
Conclusion and Further Development

- Modeling the information regarding a default time $\tau$ with a Brownian bridge on the stochastic interval $[0, \tau]$, allows to reconcile the Information-based approach to Credit-Risk with the reduced-form models.

- Explicit formulas can be obtained and they appear to be an intuitive generalization of some simple models already present in literature. We can obtain pricing formulas (Credit Default Swap, Zero-Coupon Bond).

- The information process $\beta_t$ can substitute the default indicator process $\mathbb{1}_{\{\tau \leq t\}}$ in many models providing interesting insight on the role of information on financial markets.

- Under opportune condition we can compute explicitly the compensator of the default indicator process in the enlarged filtration.

